Internal Diffusion-Limited Aggregation

By Author 1, Author 2, Author 3.

Background. Consider the two dimensional lattice (grid), represented as

$$\mathbb{Z}^2 = \{(x, y) \mid x, y \text{ are integers } \}.$$

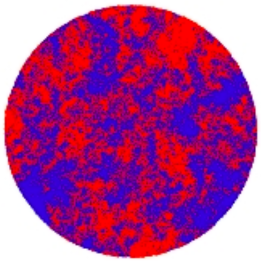
Consider a sequence of (random) sets $P_0, P_1, P_2, \ldots \subset \mathbb{Z}^2$ generated as follows:

(1) P_0 consists of a single element, say, the point $(0, 0)$.
(2) Given P_n, initiate a random walk at $(0, 0)$, the random walk stops the first time it reaches a location (x_0, y_0) which is not part of P_n, then we set

$$P_{n+1} = P_n \cup \{(x_0, y_0)\}$$

Then, each set P_n is made out of a connected set of the grid with $n+1$ points (the first point being the origin $(0, 0)$). As the consecutive locations where the random walks stop are random, the sets P_n themselves are random.

This *internal Diffusion-Limited Aggregation* model was introduced by Lawler et al [1]. The authors were inspired by a more complex shape-growth model proposed by Wittern and Sander [2], known simply as Diffusion-Limited Aggregation. DLA was originally meant to describe the growth of crystals, but has been found to model other phenomena such a flow through porous media. Internal DLA was found to be related to various mathematical objects, such as the Abelian sandpile and algebraic operations on sets. Computer simulations have produced examples such as



This suggests that for large n, and at least looking from a distance, the resulting sets P_n are close to a disc (this at least happening with high probability). Natural questions raised in the literature include the following: What is the radius of this approximate disc? By how much does P_n deviate from it? Does the deviation decrease with some rate as $n \to \infty$?.

Goals. First, to understand the mathematical construction of internal DLA, and the tools used to analyze its behavior. Secondly, to run a computer simulation illustrating the long time behavior of the Internal DLA which reflects the emergent circular shape, and review some of the mathematics illustrating why this pattern emerges. Lastly, run another computer simulation using a honeycomb (a.k.a hexagonal) lattice instead of a square lattice, and see if this results in a qualitative different shape.

References

[1] Lawler, Gregory F and Bramson, Maury and Griffeath, David, *Internal diffusion limited aggregation*, The Annals of Probability (1992), 2117–2140.
[2] Witten Jr, T. A., and Leonard M. Sander. *Diffusion-limited aggregation, a kinetic critical phenomenon.* Physical review letters 47.19 (1981): 1400.