Chapter 1
Markov Chains

1.1 Definitions and Examples

The importance of Markov chains comes from two facts: (i) there are a large number of physical, biological, economic, and social phenomena that can be modeled in this way, and (ii) there is a well-developed theory that allows us to do computations. We begin with a famous example, then describe the property that is the defining feature of Markov chains

Example 1.1 (Gambler's Ruin). Consider a gambling game in which on any turn you win \$1 with probability $p = 0.4$ or lose \$1 with probability $1 - p = 0.6$. Suppose further that you adopt the rule that you quit playing if your fortune reaches $\$N$. Of course, if your fortune reaches \$0 the casino makes you stop.

Let X_n be the amount of money you have after n plays. Your fortune, X_n has the "Markov property." In words, this means that given the current state, X_n, any other information about the past is irrelevant for predicting the next state X_{n+1}. To check this for the gambler's ruin chain, we note that if you are still playing at time n, i.e., your fortune $X_n = i$ with $0 < i < N$, then for any possible history of your wealth $i_{n-1}, i_{n-2}, \ldots i_1, i_0$

$$P(X_{n+1} = i + 1 | X_n = i, X_{n-1} = i_{n-1}, \ldots X_0 = i_0) = 0.4$$

since to increase your wealth by one unit you have to win your next bet. Here we have used $P(B|A)$ for the conditional probability of the event B given that A occurs. Recall that this is defined by

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

If you need help with this notion, see Sect. A.1 of the appendix.

R. Durrett, *Essentials of Stochastic Processes*, Springer Texts in Statistics,
DOI 10.1007/978-3-319-45614-0_1

Turning now to the formal definition, we say that X_n is a discrete time **Markov chain** with **transition matrix** $p(i,j)$ if for any $j, i, i_{n-1}, \ldots i_0$

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = p(i,j) \tag{1.1}$$

Here and in what follows, **boldface** indicates a word or phrase that is being defined or explained.

Equation (1.1) explains what we mean when we say that "given the current state X_n, any other information about the past is irrelevant for predicting X_{n+1}." In formulating (1.1) we have restricted our attention to the **temporally homogeneous** case in which the **transition probability**

$$p(i,j) = P(X_{n+1} = j | X_n = i)$$

does not depend on the time n.

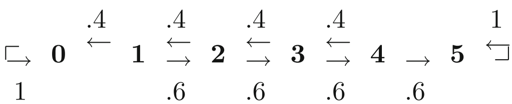
Intuitively, the transition probability gives the rules of the game. It is the basic information needed to describe a Markov chain. In the case of the gambler's ruin chain, the transition probability has

$$p(i, i+1) = 0.4, \quad p(i, i-1) = 0.6, \quad \text{if } 0 < i < N$$
$$p(0,0) = 1 \qquad p(N,N) = 1$$

When $N = 5$ the matrix is

	0	**1**	**2**	**3**	**4**	**5**
0	1.0	0	0	0	0	0
1	0.6	0	0.4	0	0	0
2	0	0.6	0	0.4	0	0
3	0	0	0.6	0	0.4	0
4	0	0	0	0.6	0	0.4
5	0	0	0	0	0	1.0

or the chain be represented pictorially as



Example 1.2 (Ehrenfest Chain). This chain originated in physics as a model for two cubical volumes of air connected by a small hole. In the mathematical version, we have two "urns," i.e., two of the exalted trash cans of probability theory, in which there are a total of N balls. We pick one of the N balls at random and move it to the other urn.

Let X_n be the number of balls in the "left" urn after the nth draw. It should be clear that X_n has the Markov property; i.e., if we want to guess the state at time $n+1$, then the current number of balls in the left urn X_n is the only relevant information from the observed sequence of states $X_n, X_{n-1}, \dots X_1, X_0$. To check this we note that

$$P(X_{n+1} = i + 1 | X_n = i, X_{n-1} = i_{n-1}, \dots X_0 = i_0) = (N - i)/N$$

since to increase the number we have to pick one of the $N - i$ balls in the other urn. The number can also decrease by 1 with probability i/N. In symbols, we have computed that the transition probability is given by

$$p(i, i + 1) = (N - i)/N, \quad p(i, i - 1) = i/N \qquad \text{for } 0 \le i \le N$$

with $p(i, j) = 0$ otherwise. When $N = 4$, for example, the matrix is

	0	**1**	**2**	**3**	**4**
0	0	1	0	0	0
1	1/4	0	3/4	0	0
2	0	2/4	0	2/4	0
3	0	0	3/4	0	1/4
4	0	0	0	1	0

In the first two examples we began with a verbal description and then wrote down the transition probabilities. However, one more commonly describes a Markov chain by writing down a transition probability $p(i, j)$ with

(i) $p(i, j) \ge 0$, since they are probabilities.
(ii) $\sum_j p(i, j) = 1$, since when $X_n = i$, X_{n+1} will be in some state j.

The equation in (ii) is read "sum $p(i, j)$ over all possible values of j." In words the last two conditions say: the entries of the matrix are nonnegative and each ROW of the matrix sums to 1.

Any matrix with properties (i) and (ii) gives rise to a Markov chain, X_n. To construct the chain we can think of playing a board game. When we are in state i, we roll a die (or generate a random number on a computer) to pick the next state, going to j with probability $p(i, j)$.

Example 1.3 (Weather Chain). Let X_n be the weather on day n in Ithaca, NY, which we assume is either: 1 = *rainy*, or 2 = *sunny*. Even though the weather is not exactly a Markov chain, we can propose a Markov chain model for the weather by writing down a transition probability

	1	**2**
1	.6	.4
2	.2	.8

The table says, for example, the probability a rainy day (state 1) is followed by a sunny day (state 2) is $p(1,2) = 0.4$. A typical question of interest is

Q. What is the long-run fraction of days that are sunny?

Example 1.4 (Social Mobility). Let X_n be a family's social class in the nth generation, which we assume is either 1 = *lower*, 2 = *middle*, or 3 = *upper*. In our simple version of sociology, changes of status are a Markov chain with the following transition probability

	1	**2**	**3**
1	.7	.2	.1
2	.3	.5	.2
3	.2	.4	.4

Q. Do the fractions of people in the three classes approach a limit?

Example 1.5 (Brand Preference). Suppose there are three types of laundry detergent, 1, 2, and 3, and let X_n be the brand chosen on the nth purchase. Customers who try these brands are satisfied and choose the same thing again with probabilities 0.8, 0.6, and 0.4, respectively. When they change they pick one of the other two brands at random. The transition probability is

	1	**2**	**3**
1	.8	.1	.1
2	.2	.6	.2
3	.3	.3	.4

Q. Do the market shares of the three product stabilize?

Example 1.6 (Inventory Chain). We will consider the consequences of using an s, S inventory control policy. That is, when the stock on hand at the end of the day falls to s or below, we order enough to bring it back up to S. For simplicity, we suppose happens at the beginning of the next day. Let X_n be the amount of stock on hand at the end of day n and D_{n+1} be the demand on day $n+1$. Introducing notation for the **positive part** of a real number,

$$x^+ = \max\{x, 0\} = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

then we can write the chain in general as

$$X_{n+1} = \begin{cases} (X_n - D_{n+1})^+ & \text{if } X_n > s \\ (S - D_{n+1})^+ & \text{if } X_n \le s \end{cases}$$

In words, if $X_n > s$ we order nothing and begin the day with X_n units. If the demand $D_{n+1} \le X_n$, we end the day with $X_{n+1} = X_n - D_{n+1}$. If the demand $D_{n+1} > X_n$, we end the day with $X_{n+1} = 0$. If $X_n \le s$, then we begin the day with S units, and the reasoning is the same as in the previous case.

Suppose now that an electronics store sells a video game system and uses an inventory policy with $s = 1$, $S = 5$. That is, if at the end of the day, the number of units they have on hand is 1 or 0, they order enough new units so their total on hand at the beginning of the next day is 5. If we assume that

for $k =$	0	1	2	3
$P(D_{n+1} = k)$	.3	.4	.2	.1

then we have the following transition matrix:

	0	**1**	**2**	**3**	**4**	**5**
0	0	0	.1	.2	.4	.3
1	0	0	.1	.2	.4	.3
2	.3	.4	.3	0	0	0
3	.1	.2	.4	.3	0	0
4	0	.1	.2	.4	.3	0
5	0	0	.1	.2	.4	.3

To explain the entries, we note that when $X_n \ge 3$ then $X_n - D_{n+1} \ge 0$. When $X_{n+1} = 2$ this is almost true but $p(2,0) = P(D_{n+1} = 2 \text{ or } 3)$. When $X_n = 1$ or 0 we start the day with five units so the end result is the same as when $X_n = 5$.

In this context we might be interested in:

Q. Suppose we make \$12 profit on each unit sold but it costs \$2 a day to store items. What is the long-run profit per day of this inventory policy? How do we choose s and S to maximize profit?

Example 1.7 (Repair Chain). A machine has three critical parts that are subject to failure, but can function as long as two of these parts are working. When two are broken, they are replaced and the machine is back to working order the next day. To formulate a Markov chain model we declare its state space to be the parts that are broken $\{0, 1, 2, 3, 12, 13, 23\}$. If we assume that parts 1, 2, and 3 fail with probabilities .01, .02, and .04, but no two parts fail on the same day, then we arrive at the following transition matrix:

	0	**1**	**2**	**3**	**12**	**13**	**23**
0	.93	.01	.02	.04	0	0	0
1	0	.94	0	0	.02	.04	0
2	0	0	.95	0	.01	0	.04
3	0	0	0	.97	0	.01	.02
12	1	0	0	0	0	0	0
13	1	0	0	0	0	0	0
23	1	0	0	0	0	0	0

If we own a machine like this, then it is natural to ask about the long-run cost per day to operate it. For example, we might ask:

Q. If we are going to operate the machine for 1800 days (about 5 years), then how many parts of types 1, 2, and 3 will we use?

Example 1.8 (Branching Processes). These processes arose from Francis Galton's statistical investigation of the extinction of family names. Consider a population in which each individual in the nth generation independently gives birth, producing k children (who are members of generation $n+1$) with probability p_k. In Galton's application only male children count since only they carry on the family name.

To define the Markov chain, note that the number of individuals in generation n, X_n, can be any nonnegative integer, so the state space is $\{0,1,2,\ldots\}$. If we let $Y_1, Y_2, \ldots$ be independent random variables with $P(Y_m = k) = p_k$, then we can write the transition probability as

$$p(i,j) = P(Y_1 + \cdots + Y_i = j) \quad \text{for } i > 0 \text{ and } j \geq 0$$

When there are no living members of the population, no new ones can be born, so $p(0,0) = 1$.

Galton's question, originally posed in the Educational Times of 1873, is

Q. What is the probability that the line of a man becomes extinct?, i.e., the branching process becomes absorbed at 0?

Reverend Henry William Watson replied with a solution. Together, they then wrote an 1874 paper entitled *On the probability of extinction of families*. For this reason, these chains are often called Galton–Watson processes.

Example 1.9 (Wright–Fisher Model). Thinking of a population of $N/2$ diploid individuals who have two copies of each of their chromosomes, or of N haploid individuals who have one copy, we consider a fixed population of N genes that can be one of two types: A or a. In the simplest version of this model the population at time $n+1$ is obtained by drawing with replacement from the population at time n.

In this case, if we let X_n be the number of A alleles at time n, then X_n is a Markov chain with transition probability

$$p(i,j) = \binom{N}{j} \left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{N-j}$$

since the right-hand side is the binomial distribution for N independent trials with success probability i/N.

In this model the states $x = 0$ and N that correspond to fixation of the population in the all a or all A states are **absorbing states**, that is, $p(x,x) = 1$. So it is natural to ask:

Q1. Starting from i of the A alleles and $N - i$ of the a alleles, what is the probability that the population fixates in the all A state?

To make this simple model more realistic we can introduce the possibility of mutations: an A that is drawn ends up being an a in the next generation with probability u, while an a that is drawn ends up being an A in the next generation with probability v. In this case the probability an A is produced by a given draw is

$$\rho_i = \frac{i}{N}(1-u) + \frac{N-i}{N}v$$

but the transition probability still has the binomial form

$$p(i,j) = \binom{N}{j}(\rho_i)^j(1-\rho_i)^{N-j}$$

If u and v are both positive, then 0 and N are no longer absorbing states, so we ask:

Q2. Does the genetic composition settle down to an equilibrium distribution as time $t \to \infty$?

As the next example shows it is easy to extend the notion of a Markov chain to cover situations in which the future evolution is independent of the past when we know the last two states.

Example 1.10 (Two-Stage Markov Chains). In a Markov chain the distribution of X_{n+1} only depends on X_n. This can easily be generalized to case in which the distribution of X_{n+1} only depends on (X_n, X_{n-1}). For a concrete example consider a basketball player who makes a shot with the following probabilities:

1/2 if he has missed the last two times
2/3 if he has hit one of his last two shots
3/4 if he has hit both of his last two shots

To formulate a Markov chain to model his shooting, we let the states of the process be the outcomes of his last two shots: $\{HH, HM, MH, MM\}$ where M is short for miss and H for hit. The transition probability is

	HH	**HM**	**MH**	**MM**
HH	3/4	1/4	0	0
HM	0	0	2/3	1/3
MH	2/3	1/3	0	0
MM	0	0	1/2	1/2

To explain suppose the state is HM, i.e., $X_{n-1} = H$ and $X_n = M$. In this case the next outcome will be H with probability 2/3. When this occurs the next state will be $(X_n, X_{n+1}) = (M, H)$ with probability 2/3. If he misses an event of probability 1/3, $(X_n, X_{n+1}) = (M, M)$.

The Hot Hand is a phenomenon known to most people who play or watch basketball. After making a couple of shots, players are thought to "get into a groove" so that subsequent successes are more likely. Purvis Short of the Golden State Warriors describes this more poetically as

> You're in a world all your own. It's hard to describe. But the basket seems to be so wide. No matter what you do, you know the ball is going to go in.

Unfortunately for basketball players, data collected by Gliovich et al. (1985) shows that this is a misconception. The next table gives data for the conditional probability of hitting a shot after missing the last three, missing the last two, … hitting the last three, for nine players of the Philadelphia 76ers: Darryl Dawkins (403), Maurice Cheeks (339), Steve Mix (351), Bobby Jones (433), Clint Richardson (248), Julius Erving (884), Andrew Toney (451), Caldwell Jones (272), and Lionel Hollins (419). The numbers in parentheses are the number of shots for each player.

$P(H\|3M)$	$P(H\|2M)$	$P(H\|1M)$	$P(H\|1H)$	$P(H\|2H)$	$P(H\|3H)$
.88	.73	.71	.57	.58	.51
.77	.60	.60	.55	.54	.59
.70	.56	.52	.51	.48	.36
.61	.58	.58	.53	.47	.53
.52	.51	.51	.53	.52	.48
.50	.47	.56	.49	.50	.48
.50	.48	.47	.45	.43	.27
.52	.53	.51	.43	.40	.34
.50	.49	.46	.46	.46	.32

In fact, the data supports the opposite assertion: after missing a player will hit more frequently.

1.2 Multistep Transition Probabilities

The transition probability $p(i,j) = P(X_{n+1} = j|X_n = i)$ gives the probability of going from i to j in one step. Our goal in this section is to compute the probability of going from i to j in $m > 1$ steps:

$$p^m(i,j) = P(X_{n+m} = j|X_n = i)$$

As the notation may already suggest, p^m will turn out to be the mth power of the transition matrix, see Theorem 1.1.

To warm up, we recall the transition probability of the social mobility chain:

	1	**2**	**3**
1	.7	.2	.1
2	.3	.5	.2
3	.2	.4	.4

and consider the following concrete question:

Q1. *Your parents were middle class (state 2). What is the probability that you are in the upper class (state 3) but your children are lower class (state 1)?*

Solution. Intuitively, the Markov property implies that starting from state 2 the probability of jumping to 3 and then to 1 is given by

$$p(2,3)p(3,1)$$

To get this conclusion from the definitions, we note that using the definition of conditional probability,

$$\begin{aligned} P(X_2 = 1, X_1 = 3|X_0 = 2) &= \frac{P(X_2 = 1, X_1 = 3, X_0 = 2)}{P(X_0 = 2)} \\ &= \frac{P(X_2 = 1, X_1 = 3, X_0 = 2)}{P(X_1 = 3, X_0 = 2)} \cdot \frac{P(X_1 = 3, X_0 = 2)}{P(X_0 = 2)} \\ &= P(X_2 = 1|X_1 = 3, X_0 = 2) \cdot P(X_1 = 3|X_0 = 2) \end{aligned}$$

By the Markov property (1.1) the last expression is

$$P(X_2 = 1|X_1 = 3) \cdot P(X_1 = 3|X_0 = 2) = p(2,3)p(3,1)$$

Moving on to the real question:

Q2. *What is the probability your children are lower class (1) given your parents were middle class (2)?*

Solution. To do this we simply have to consider the three possible states for your class and use the solution of the previous problem.

$$P(X_2 = 1|X_0 = 2) = \sum_{k=1}^{3} P(X_2 = 1, X_1 = k|X_0 = 2) = \sum_{k=1}^{3} p(2,k)p(k,1)$$
$$= (.3)(.7) + (.5)(.3) + (.2)(.2) = .21 + .15 + .04 = .21$$

There is nothing special here about the states 2 and 1 here. By the same reasoning,

$$P(X_2 = j|X_0 = i) = \sum_{k=1}^{3} p(i,k)\, p(k,j)$$

The right-hand side of the last equation gives the (i,j)th entry of the matrix p is multiplied by itself.

To explain this, we note that to compute $p^2(2,1)$ we multiplied the entries of the second row by those in the first column:

$$\begin{pmatrix} . & . & . \\ .3 & .5 & .2 \\ . & . & . \end{pmatrix} \begin{pmatrix} .7 & . & . \\ .3 & . & . \\ .2 & . & . \end{pmatrix} = \begin{pmatrix} . & . & . \\ .40 & . & . \\ . & . & . \end{pmatrix}$$

If we wanted $p^2(1,3)$ we would multiply the first row by the third column:

$$\begin{pmatrix} .7 & .2 & .1 \\ . & . & . \\ . & . & . \end{pmatrix} \begin{pmatrix} . & . & .1 \\ . & . & .2 \\ . & . & .4 \end{pmatrix} = \begin{pmatrix} . & . & .15 \\ . & . & . \\ . & . & . \end{pmatrix}$$

When all of the computations are done we have

$$\begin{pmatrix} .7 & .2 & .1 \\ .3 & .5 & .2 \\ .2 & .4 & .4 \end{pmatrix} \begin{pmatrix} .7 & .2 & .1 \\ .3 & .5 & .2 \\ .2 & .4 & .4 \end{pmatrix} = \begin{pmatrix} .57 & .28 & .15 \\ .40 & .39 & .21 \\ .34 & .40 & .26 \end{pmatrix}$$

All of this becomes much easier if we use a scientific calculator like the T1-83. Using 2nd-MATRIX we can access a screen with NAMES, MATH, EDIT at the top. Selecting EDIT we can enter the matrix into the computer as say [A]. Then selecting the NAMES we can enter $[A] \wedge 2$ on the computation line to get A^2. If we use this procedure to compute A^{20}, we get a matrix with three rows that agree in the first six decimal places with

$$.468085 \quad .340425 \quad .191489$$

Later we will see that as $n \to \infty$, p^n converges to a matrix with all three rows equal to $(22/47, 16/47, 9/47)$.

To explain our interest in p^m we will now prove:

Theorem 1.1. *The m step transition probability* $P(X_{n+m} = j | X_n = i)$ *is the mth power of the transition matrix p.*

The key ingredient in proving this is the **Chapman–Kolmogorov equation**

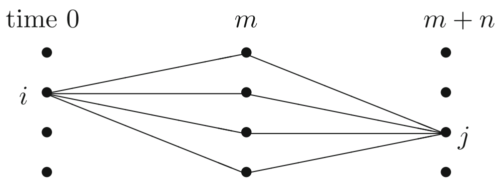
$$p^{m+n}(i,j) = \sum_k p^m(i,k)\, p^n(k,j) \tag{1.2}$$

Once this is proved, Theorem 1.1 follows, since taking $n = 1$ in (1.2), we see that

$$p^{m+1}(i,j) = \sum_k p^m(i,k)\, p(k,j)$$

That is, the $m + 1$ step transition probability is the m step transition probability times p. Theorem 1.1 now follows.

Why is (1.2) true? To go from i to j in $m + n$ steps, we have to go from i to some state k in m steps and then from k to j in n steps. The Markov property implies that the two parts of our journey are independent. □



Proof of (1.2). We do this by combining the solutions of Q1 and Q2. Breaking things down according to the state at time m,

$$P(X_{m+n} = j | X_0 = i) = \sum_k P(X_{m+n} = j, X_m = k | X_0 = i)$$

Using the definition of conditional probability as in the solution of Q1,

$$\begin{aligned}
P(X_{m+n} = j, X_m = k | X_0 = i) &= \frac{P(X_{m+n} = j, X_m = k, X_0 = i)}{P(X_0 = i)} \\
&= \frac{P(X_{m+n} = j, X_m = k, X_0 = i)}{P(X_m = k, X_0 = i)} \cdot \frac{P(X_m = k, X_0 = i)}{P(X_0 = i)} \\
&= P(X_{m+n} = j | X_m = k, X_0 = i) \cdot P(X_m = k | X_0 = i)
\end{aligned}$$

By the Markov property (1.1) the last expression is

$$= P(X_{m+n} = j|X_m = k) \cdot P(X_m = k|X_0 = i) = p^m(i,k)p^n(k,j)$$

and we have proved (1.2). □

Having established (1.2), we now return to computations.

Example 1.11 (Gambler's Ruin). Suppose for simplicity that $N = 4$ in Example 1.1, so that the transition probability is

	0	**1**	**2**	**3**	**4**
0	1.0	0	0	0	0
1	0.6	0	0.4	0	0
2	0	0.6	0	0.4	0
3	0	0	0.6	0	0.4
4	0	0	0	0	1.0

To compute p^2 one row at a time we note:

$p^2(0,0) = 1$ and $p^2(4,4) = 1$, since these are absorbing states.
$p^2(1,3) = (.4)^2 = 0.16$, since the chain has to go up twice.
$p^2(1,1) = (.4)(.6) = 0.24$. The chain must go from 1 to 2 to 1.
$p^2(1,0) = 0.6$. To be at 0 at time 2, the first jump must be to 0.

Leaving the cases $i = 2, 3$ to the reader, we have

$$p^2 = \begin{pmatrix} 1.0 & 0 & 0 & 0 & 0 \\ .6 & .24 & 0 & .16 & 0 \\ .36 & 0 & .48 & 0 & .16 \\ 0 & .36 & 0 & .24 & .4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Using a calculator one can easily compute

$$p^{20} = \begin{pmatrix} 1.0 & 0 & 0 & 0 & 0 \\ .87655 & .00032 & 0 & .00022 & .12291 \\ .69186 & 0 & .00065 & 0 & .30749 \\ .41842 & .00049 & 0 & .00032 & .58437 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

0 and 4 are absorbing states. Here we see that the probability of avoiding absorption for 20 steps is 0.00054 from state 1, 0.00065 from state 2, and 0.00081 from state 3.

Later we will see that

$$\lim_{n\to\infty} p^n = \begin{pmatrix} 1.0 & 0\,0\,0 & 0 \\ 57/65 & 0\,0\,0 & 8/65 \\ 45/65 & 0\,0\,0 & 20/65 \\ 27/65 & 0\,0\,0 & 38/65 \\ 0 & 0\,0\,0 & 1 \end{pmatrix}$$

1.3 Classification of States

We begin with some important notation. We are often interested in the behavior of the chain for a fixed initial state, so we will introduce the shorthand

$$P_x(A) = P(A|X_0 = x)$$

Later we will have to consider expected values for this probability and we will denote them by E_x.

Let $T_y = \min\{n \geq 1 : X_n = y\}$ be the **time of the first return to** y (i.e., being there at time 0 doesn't count), and let

$$\rho_{yy} = P_y(T_y < \infty)$$

be the probability X_n returns to y when it starts at y. Note that if we didn't exclude $n = 0$ this probability would always be 1.

Intuitively, the Markov property implies that the probability X_n will return to y at least twice is ρ_{yy}^2, since after the first return, the chain is at y, and the probability of a second return following the first is again ρ_{yy}.

To show that the reasoning in the last paragraph is valid, we have to introduce a definition and state a theorem. We say that T is a **stopping time** if the occurrence (or nonoccurrence) of the event "we stop at time n," $\{T = n\}$, can be determined by looking at the values of the process up to that time: $X_0, \ldots, X_n$. To see that T_y is a stopping time note that

$$\{T_y = n\} = \{X_1 \neq y, \ldots, X_{n-1} \neq y, X_n = y\}$$

and that the right-hand side can be determined from $X_0, \ldots, X_n$.

Since stopping at time n depends only on the values $X_0, \ldots, X_n$, and in a Markov chain the distribution of the future only depends on the past through the current state, it should not be hard to believe that the Markov property holds at stopping times. This fact can be stated formally as

Theorem 1.2 (Strong Markov Property). *Suppose T is a stopping time. Given that $T = n$ and $X_T = y$, any other information about $X_0, \ldots X_T$ is irrelevant for* *predicting the future, and* X_{T+k}, $k \geq 0$ *behaves like the Markov chain with initial state* y.

Why is this true? To keep things as simple as possible we will show only that

$$P(X_{T+1} = z | X_T = y, T = n) = p(y, z)$$

Let V_n be the set of vectors $(x_0, \ldots, x_n)$ so that if $X_0 = x_0, \ldots, X_n = x_n$, then $T = n$ and $X_T = y$. Breaking things down according to the values of $X_0, \ldots, X_n$ gives

$$P(X_{T+1} = z, X_T = y, T = n) = \sum_{x \in V_n} P(X_{n+1} = z, X_n = x_n, \ldots, X_0 = x_0)$$

$$= \sum_{x \in V_n} P(X_{n+1} = z | X_n = x_n, \ldots, X_0 = x_0) P(X_n = x_n, \ldots, X_0 = x_0)$$

where in the second step we have used the **multiplication rule**

$$P(A \cap B) = P(B|A)P(A)$$

For any $(x_0, \ldots, x_n) \in V_n$ we have $T = n$ and $X_T = y$ so $x_n = y$. Using the Markov property, (1.1), and recalling the definition of V_n shows the above

$$P(X_{T+1} = z, T = n, X_T = y) = p(y, z) \sum_{x \in V_n} P(X_n = x_n, \ldots, X_0 = x_0)$$

$$= p(y, z) P(T = n, X_T = y)$$

Dividing both sides by $P(T = n, X_T = y)$ gives the desired result. □

Let $T_y^1 = T_y$ and for $k \geq 2$ let

$$T_y^k = \min\{n > T_y^{k-1} : X_n = y\} \tag{1.3}$$

be the **time of the** ***k*****th return to** y. The strong Markov property implies that the conditional probability we will return one more time given that we have returned $k - 1$ times is ρ_{yy}. This and induction imply that

$$P_y(T_y^k < \infty) = \rho_{yy}^k \tag{1.4}$$

At this point, there are two possibilities:

(i) $\rho_{yy} < 1$: The probability of returning k times is $\rho_{yy}^k \to 0$ as $k \to \infty$. Thus, eventually the Markov chain does not find its way back to y. In this case the state y is called **transient**, since after some point it is never visited by the Markov chain.

(ii) $\rho_{yy} = 1$: The probability of returning k times $\rho_{yy}^k = 1$, so the chain returns to y infinitely many times. In this case, the state y is called **recurrent**, it continually recurs in the Markov chain.

To understand these notions, we turn to our examples, beginning with

Example 1.12 (Gambler's Ruin). Consider, for concreteness, the case $N = 4$.

	0	**1**	**2**	**3**	**4**
0	1	0	0	0	0
1	.6	0	.4	0	0
2	0	.6	0	.4	0
3	0	0	.6	0	.4
4	0	0	0	0	1

We will show that eventually the chain gets stuck in either the bankrupt (0) or happy winner (4) state. In the terms of our recent definitions, we will show that states $0 < y < 4$ are transient, while the states 0 and 4 are recurrent.

It is easy to check that 0 and 4 are recurrent. Since $p(0,0) = 1$, the chain comes back on the next step with probability one, i.e.,

$$P_0(T_0 = 1) = 1$$

and hence $\rho_{00} = 1$. A similar argument shows that 4 is recurrent. In general if y is an absorbing state, i.e., if $p(y,y) = 1$, then y is a very strongly recurrent state—the chain always stays there.

To check the transience of the interior states, 1, 2, 3, we note that starting from 1, if the chain goes to 0, it will never return to 1, so the probability of never returning to 1,

$$P_1(T_1 = \infty) \geq p(1,0) = 0.6 > 0$$

Similarly, starting from 2, the chain can go to 1 and then to 0, so

$$P_2(T_2 = \infty) \geq p(2,1)p(1,0) = 0.36 > 0$$

Finally, for starting from 3, we note that the chain can go immediately to 4 and never return with probability 0.4, so

$$P_3(T_3 = \infty) \geq p(3,4) = 0.4 > 0$$

In some cases it is easy to identify recurrent states.

Example 1.13 (Social Mobility). Recall that the transition probability is

	1	**2**	**3**
1	.7	.2	.1
2	.3	.5	.2
3	.2	.4	.4

To begin we note that no matter where X_n is, there is a probability of at least 0.1 of hitting 3 on the next step so

$$P_3(T_3 > n) \le (0.9)^n \to 0 \text{ as } n \to \infty$$

i.e., we will return to 3 with probability 1. The last argument applies even more strongly to states 1 and 2, since the probability of jumping to them on the next step is always at least 0.2. Thus all three states are recurrent.

The last argument generalizes to give the following useful fact.

Lemma 1.3. *Suppose $P_x(T_y \le k) \ge \alpha > 0$ for all x in the state space S. Then*

$$P_x(T_y > nk) \le (1-\alpha)^n$$

Generalizing from our experience with the last two examples, we will introduce some general results that will help us identify transient and recurrent states.

Definition 1.1. We say that x **communicates with** y and write $x \to y$ if there is a positive probability of reaching y starting from x, that is, the probability

$$\rho_{xy} = P_x(T_y < \infty) > 0$$

Note that the last probability includes not only the possibility of jumping from x to y in one step but also going from x to y after visiting several other states in between. The following property is simple but useful. Here and in what follows, lemmas are a means to prove the more important conclusions called theorems. The two are numbered in the same sequence to make results easier to find.

Lemma 1.4. *If $x \to y$ and $y \to z$, then $x \to z$.*

Proof. Since $x \to y$ there is an m so that $p^m(x,y) > 0$. Similarly there is an n so that $p^n(y,z) > 0$. Since $p^{m+n}(x,z) \ge p^m(x,y)p^n(y,z)$ it follows that $x \to z$. □

Theorem 1.5. *If $\rho_{xy} > 0$, but $\rho_{yx} < 1$, then x is transient.*

Proof. Let $K = \min\{k : p^k(x,y) > 0\}$ be the smallest number of steps we can take to get from x to y. Since $p^K(x,y) > 0$ there must be a sequence $y_1, \dots y_{K-1}$ so that

$$p(x,y_1)p(y_1,y_2)\cdots p(y_{K-1},y) > 0$$

Since K is minimal all the $y_i \neq x$ (or there would be a shorter path), and we have

$$P_x(T_x = \infty) \geq p(x, y_1)p(y_1, y_2) \cdots p(y_{K-1}, y)(1 - \rho_{yx}) > 0$$

so x is transient. □

We will see later that Theorem 1.5 allows us to identify all the transient states when the state space is finite. An immediate consequence of Theorem 1.5 is

Lemma 1.6. *If x is recurrent and $\rho_{xy} > 0$, then $\rho_{yx} = 1$.*

Proof. If $\rho_{yx} < 1$, then Lemma 1.5 would imply x is transient. □

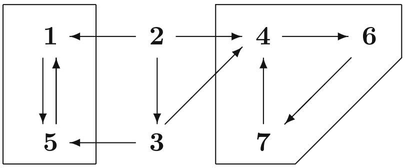
To be able to analyze any finite state Markov chain we need some theory. To motivate the developments consider

Example 1.14 (A Seven-State Chain). Consider the transition probability:

	1	**2**	**3**	**4**	**5**	**6**	**7**
1	.7	0	0	0	.3	0	0
2	.1	.2	.3	.4	0	0	0
3	0	0	.5	.3	.2	0	0
4	0	0	0	.5	0	.5	0
5	.6	0	0	0	.4	0	0
6	0	0	0	0	0	.2	.8
7	0	0	0	1	0	0	0

To identify the states that are recurrent and those that are transient, we begin by drawing a graph that will contain an arc from i to j if $p(i,j) > 0$ and $i \neq j$. We do not worry about drawing the self-loops corresponding to states with $p(i, i) > 0$ since such transitions cannot help the chain get somewhere new.

In the case under consideration the graph is



The state 2 communicates with 1, which does not communicate with it, so Theorem 1.5 implies that 2 is transient. Likewise 3 communicates with 4, which doesn't communicate with it, so 3 is transient. To conclude that all the remaining states are recurrent we will introduce two definitions and a fact.

A set A is **closed** if it is impossible to get out, i.e., if $i \in A$ and $j \notin A$ then $p(i,j) = 0$. In Example 1.14, $\{1, 5\}$ and $\{4, 6, 7\}$ are closed sets. Their union

$\{1,4,5,6,7\}$ is also closed. One can add 3 to get another closed set $\{1,3,4,5,6,7\}$. Finally, the whole state space $\{1,2,3,4,5,6,7\}$ is always a closed set.

Among the closed sets in the last example, some are obviously too big. To rule them out, we need a definition. A set B is called **irreducible** if whenever $i, j \in B$, i communicates with j. The irreducible closed sets in the Example 1.14 are $\{1,5\}$ and $\{4,6,7\}$. The next result explains our interest in irreducible closed sets.

Theorem 1.7. *If C is a finite closed and irreducible set, then all states in C are recurrent.*

Before entering into an explanation of this result, we note that Theorem 1.7 tells us that 1, 5, 4, 6, and 7 are recurrent, completing our study of the Example 1.14 with the results we had claimed earlier.

In fact, the combination of Theorems 1.5 and 1.7 is sufficient to classify the states in any finite state Markov chain. An algorithm will be explained in the proof of the following result.

Theorem 1.8. *If the state space S is finite, then S can be written as a disjoint union $T \cup R_1 \cup \cdots \cup R_k$, where T is a set of transient states and the R_i, $1 \le i \le k$, are closed irreducible sets of recurrent states.*

Proof. Let T be the set of x for which there is a y so that $x \to y$ but $y \not\to x$. The states in T are transient by Theorem 1.5. Our next step is to show that all the remaining states, $S - T$, are recurrent.

Pick an $x \in S - T$ and let $C_x = \{y : x \to y\}$. Since $x \notin T$ it has the property if $x \to y$, then $y \to x$. To check that C_x is closed note that if $y \in C_x$ and $y \to z$, then Lemma 1.4 implies $x \to z$ so $z \in C_x$. To check irreducibility, note that if $y, z \in C_x$, then by our first observation $y \to x$ and we have $x \to z$ by definition, so Lemma 1.4 implies $y \to z$. C_x is closed and irreducible so all states in C_x are recurrent. Let $R_1 = C_x$. If $S - T - R_1 = \emptyset$, we are done. If not, pick a site $w \in S - T - R_1$ and repeat the procedure. □

The rest of this section is devoted to the proof of Theorem 1.7. To do this, it is enough to prove the following two results.

Lemma 1.9. *If x is recurrent and $x \to y$, then y is recurrent.*

Lemma 1.10. *In a finite closed set there has to be at least one recurrent state.*

To prove these results we need to introduce a little more theory. Recall the time of the kth visit to y defined by

$$T_y^k = \min\{n > T_y^{k-1} : X_n = y\}$$

and $\rho_{xy} = P_x(T_y < \infty)$ the probability we ever visit y at some time $n \ge 1$ when we start from x. Using the strong Markov property as in the proof of (1.4) gives

$$P_x(T_y^k < \infty) = \rho_{xy}\rho_{yy}^{k-1}. \tag{1.5}$$

Let $N(y)$ be the number of visits to y at times $n \ge 1$. Using (1.5) we can compute $EN(y)$.

Lemma 1.11. $E_x N(y) = \rho_{xy}/(1 - \rho_{yy})$

Proof. Accept for the moment the fact that for any nonnegative integer valued random variable X, the expected value of X can be computed by

$$EX = \sum_{k=1}^{\infty} P(X \geq k) \tag{1.6}$$

We will prove this after we complete the proof of Lemma 1.11. Now the probability of returning at least k times, $\{N(y) \geq k\}$, is the same as the event that the kth return occurs, i.e., $\{T_y^k < \infty\}$, so using (1.5) we have

$$E_x N(y) = \sum_{k=1}^{\infty} P(N(y) \geq k) = \rho_{xy} \sum_{k=1}^{\infty} \rho_{yy}^{k-1} = \frac{\rho_{xy}}{1 - \rho_{yy}}$$

since $\sum_{n=0}^{\infty} \theta^n = 1/(1-\theta)$ whenever $|\theta| < 1$. □

Proof of (1.6). Let $1_{\{X \geq k\}}$ denote the random variable that is 1 if $X \geq k$ and 0 otherwise. It is easy to see that

$$X = \sum_{k=1}^{\infty} 1_{\{X \geq k\}}.$$

Taking expected values and noticing $E1_{\{X \geq k\}} = P(X \geq k)$ gives

$$EX = \sum_{k=1}^{\infty} P(X \geq k)$$

□

Our next step is to compute the expected number of returns to y in a different way.

Lemma 1.12. $E_x N(y) = \sum_{n=1}^{\infty} p^n(x, y)$.

Proof. Let $1_{\{X_n = y\}}$ denote the random variable that is 1 if $X_n = y$, 0 otherwise. Clearly

$$N(y) = \sum_{n=1}^{\infty} 1_{\{X_n = y\}}.$$

Taking expected values now gives

$$E_x N(y) = \sum_{n=1}^{\infty} P_x(X_n = y)$$

□

With the two lemmas established we can now state our next main result.

Theorem 1.13. *y is recurrent if and only if*

$$\sum_{n=1}^{\infty} p^n(y,y) = E_y N(y) = \infty$$

Proof. The first equality is Lemma 1.12. From Lemma 1.11 we see that $E_y N(y) = \infty$ if and only if $\rho_{yy} = 1$, which is the definition of recurrence. □

With this established we can easily complete the proofs of our two lemmas .

Proof of Lemma 1.9. Suppose x is recurrent and $\rho_{xy} > 0$. By Lemma 1.6 we must have $\rho_{yx} > 0$. Pick j and ℓ so that $p^j(y,x) > 0$ and $p^\ell(x,y) > 0$. $p^{j+k+\ell}(y,y)$ is probability of going from y to y in $j+k+\ell$ steps while the product $p^j(y,x)p^k(x,x)p^\ell(x,y)$ is the probability of doing this and being at x at times j and $j+k$. Thus we must have

$$\sum_{k=0}^{\infty} p^{j+k+\ell}(y,y) \ge p^j(y,x)\left(\sum_{k=0}^{\infty} p^k(x,x)\right) p^\ell(x,y)$$

If x is recurrent, then $\sum_k p^k(x,x) = \infty$, so $\sum_m p^m(y,y) = \infty$ and Theorem 1.13 implies that y is recurrent. □

Proof of Lemma 1.10. If all the states in C are transient, then Lemma 1.11 implies that $E_x N(y) < \infty$ for all x and y in C. Since C is finite, using Lemma 1.12

$$\infty > \sum_{y \in C} E_x N(y) = \sum_{y \in C} \sum_{n=1}^{\infty} p^n(x,y)$$
$$= \sum_{n=1}^{\infty} \sum_{y \in C} p^n(x,y) = \sum_{n=1}^{\infty} 1 = \infty$$

where in the next to last equality we have used that C is closed. This contradiction proves the desired result. □

1.4 Stationary Distributions

In the next section we will see that if we impose an additional assumption called aperiodicity an irreducible finite state Markov chain converges to a stationary distribution

$$p^n(x,y) \to \pi(y)$$

To prepare for that this section introduces stationary distributions and shows how to compute them. Our first step is to consider

What happens in a Markov chain when the initial state is random? Breaking things down according to the value of the initial state and using the definition of conditional probability

$$\begin{aligned} P(X_n = j) &= \sum_i P(X_0 = i, X_n = j) \\ &= \sum_i P(X_0 = i)P(X_n = j|X_0 = i) \end{aligned}$$

If we introduce $q(i) = P(X_0 = i)$, then the last equation can be written as

$$P(X_n = j) = \sum_i q(i)p^n(i,j) \tag{1.7}$$

In words, we multiply the transition matrix on the left by the vector q of initial probabilities. If there are k states, then $p^n(x,y)$ is a $k \times k$ matrix. So to make the matrix multiplication work out right, we should take q as a $1 \times k$ matrix or a "row vector."

Example 1.15. Consider the weather chain (Example 1.3) and suppose that the initial distribution is $q(1) = 0.3$ and $q(2) = 0.7$. In this case

$$\begin{pmatrix} .3 & .7 \end{pmatrix} \begin{pmatrix} .6 & .4 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} .32 & .68 \end{pmatrix}$$

$$\text{since} \qquad \begin{aligned} .3(.6) + .7(.2) &= .32 \\ .3(.4) + .7(.8) &= .68 \end{aligned}$$

Example 1.16. Consider the social mobility chain (Example 1.4) and suppose that the initial distribution: $q(1) = .5$, $q(2) = .2$, and $q(3) = .3$. Multiplying the vector q by the transition probability gives the vector of probabilities at time 1.

$$\begin{pmatrix} .5 & .2 & .3 \end{pmatrix} \begin{pmatrix} .7 & .2 & .1 \\ .3 & .5 & .2 \\ .2 & .4 & .4 \end{pmatrix} = \begin{pmatrix} .47 & .32 & .21 \end{pmatrix}$$

To check the arithmetic note that the three entries on the right-hand side are

$$.5(.7) + .2(.3) + .3(.2) = .35 + .06 + .06 = .47$$
$$.5(.2) + .2(.5) + .3(.4) = .10 + .10 + .12 = .32$$
$$.5(.1) + .2(.2) + .3(.4) = .05 + .04 + .12 = .21$$

If $qp = q$, then q is called a **stationary distribution**. If the distribution at time 0 is the same as the distribution at time 1, then by the Markov property it will be the distribution at all times $n \geq 1$.

Stationary distributions have a special importance in the theory of Markov chains, so we will use a special letter π to denote solutions of the equation

$$\pi p = \pi.$$

To have a mental picture of what happens to the distribution of probability when one step of the Markov chain is taken, it is useful to think that we have $q(i)$ pounds of sand at state i, with the total amount of sand $\sum_i q(i)$ being one pound. When a step is taken in the Markov chain, a fraction $p(i,j)$ of the sand at i is moved to j. The distribution of sand when this has been done is

$$qp = \sum_i q(i)p(i,j)$$

If the distribution of sand is not changed by this procedure q is a stationary distribution.

Example 1.17 (Weather Chain). To compute the stationary distribution we want to solve

$$\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} .6 & .4 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}$$

Multiplying gives two equations:

$$.6\pi_1 + .2\pi_2 = \pi_1$$
$$.4\pi_1 + .8\pi_2 = \pi_2$$

Both equations reduce to $.4\pi_1 = .2\pi_2$. Since we want $\pi_1 + \pi_2 = 1$, we must have $.4\pi_1 = .2 - .2\pi_1$, and hence

$$\pi_1 = \frac{.2}{.2 + .4} = \frac{1}{3} \qquad \pi_2 = \frac{.4}{.2 + .4} = \frac{2}{3}$$

To check this we note that

$$(1/3\ 2/3) \begin{pmatrix} .6 & .4 \\ .2 & .8 \end{pmatrix} = \left(\frac{.6}{3} + \frac{.4}{3}, \frac{.4}{3} + \frac{1.6}{3} \right)$$

General two state transition probability.

	1	**2**
1	$1-a$	a
2	b	$1-b$

We have written the chain in this way so the stationary distribution has a simple formula

$$\pi_1 = \frac{b}{a+b} \qquad \pi_2 = \frac{a}{a+b} \tag{1.8}$$

As a first check on this formula we note that in the weather chain $a = 0.4$ and $b = 0.2$ which gives $(1/3, 2/3)$ as we found before. We can prove this works in general by drawing a picture:

$$\frac{b}{a+b} \overset{\mathbf{1}}{\bullet} \;\underset{b}{\overset{a}{\rightleftarrows}}\; \overset{\mathbf{2}}{\bullet} \frac{a}{a+b}$$

In words, the amount of sand that flows from 1 to 2 is the same as the amount that flows from 2 to 1 so the amount of sand at each site stays constant. To check algebraically that $\pi p = \pi$:

$$\begin{aligned}
\frac{b}{a+b}(1-a) + \frac{a}{a+b}b &= \frac{b - ba + ab}{a+b} = \frac{b}{a+b} \\
\frac{b}{a+b}a + \frac{a}{a+b}(1-b) &= \frac{ba + a - ab}{a+b} = \frac{a}{a+b}
\end{aligned} \tag{1.9}$$

Formula (1.8) gives the stationary distribution for any two state chain, so we progress now to the three state case and consider the

Example 1.18 (Social Mobility (Continuation of 1.4)).

	1	**2**	**3**
1	.7	.2	.1
2	.3	.5	.2
3	.2	.4	.4

The equation $\pi p = \pi$ says

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3\end{pmatrix} \begin{pmatrix} .7 & .2 & .1 \\ .3 & .5 & .2 \\ .2 & .4 & .4 \end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2 & \pi_3\end{pmatrix}$$

which translates into three equations

$$.7\pi_1 + .3\pi_2 + .2\pi_3 = \pi_1$$
$$.2\pi_1 + .5\pi_2 + .4\pi_3 = \pi_2$$
$$.1\pi_1 + .2\pi_2 + .4\pi_3 = \pi_3$$

Note that the columns of the matrix give the numbers in the rows of the equations. The third equation is redundant since if we add up the three equations we get

$$\pi_1 + \pi_2 + \pi_3 = \pi_1 + \pi_2 + \pi_3$$

If we replace the third equation by $\pi_1 + \pi_2 + \pi_3 = 1$ and subtract π_1 from each side of the first equation and π_2 from each side of the second equation we get

$$\begin{aligned} -.3\pi_1 + .3\pi_2 + .2\pi_3 &= 0 \\ .2\pi_1 - .5\pi_2 + .4\pi_3 &= 0 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned} \tag{1.10}$$

At this point we can solve the equations by hand or using a calculator.

By Hand We note that the third equation implies $\pi_3 = 1 - \pi_1 - \pi_2$ and substituting this in the first two gives

$$.2 = .5\pi_1 - .1\pi_2$$
$$.4 = .2\pi_1 + .9\pi_2$$

Multiplying the first equation by .9 and adding .1 times the second gives

$$2.2 = (0.45 + 0.02)\pi_1 \qquad \text{or} \qquad \pi_1 = 22/47$$

Multiplying the first equation by .2 and adding $-.5$ times the second gives

$$-0.16 = (-.02 - 0.45)\pi_2 \qquad \text{or} \qquad \pi_2 = 16/47$$

Since the three probabilities add up to 1, $\pi_3 = 9/47$.

Using the TI83 calculator is easier. To begin we write (1.10) in matrix form as

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3\end{pmatrix} \begin{pmatrix} -.3 & .2 & 1 \\ .3 & -.5 & 1 \\ .2 & .4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

If we let A be the 3×3 matrix in the middle this can be written as $\pi A = (0, 0, 1)$. Multiplying on each side by A^{-1} we see that

$$\pi = (0, 0, 1)A^{-1}$$

which is the third row of A^{-1}. To compute A^{-1}, we enter A into our calculator (using the MATRX menu and its EDIT submenu), use the MATRIX menu to put $[A]$ on the computation line, press x^{-1}, and then ENTER. Reading the third row we find that the stationary distribution is

$$(0.468085,\ 0.340425,\ 0.191489)$$

Converting the answer to fractions using the first entry in the MATH menu gives

$$(22/47,\ 16/47,\ 9/47)$$

Example 1.19 (Brand Preference (Continuation of 1.5)).

	1	**2**	**3**
1	.8	.1	.1
2	.2	.6	.2
3	.3	.3	.4

Using the first two equations and the fact that the sum of the π's is 1

$$.8\pi_1 + .2\pi_2 + .3\pi_3 = \pi_1$$
$$.1\pi_1 + .6\pi_2 + .3\pi_3 = \pi_2$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$

Subtracting π_1 from both sides of the first equation and π_2 from both sides of the second, this translates into $\pi A = (0, 0, 1)$ with

$$A = \begin{pmatrix} -.2 & .1 & 1 \\ .2 & -.4 & 1 \\ .3 & .3 & 1 \end{pmatrix}$$

Note that here and in the previous example the first two columns of A consist of the first two columns of the transition probability with 1 subtracted from the diagonal entries, and the final column is all 1's. Computing the inverse and reading the last row gives

$$(0.545454,\ 0.272727,\ 0.181818)$$

Converting the answer to fractions using the first entry in the MATH menu gives

$$(6/11,\ 3/11,\ 2/11)$$

To check this we note that

$$(6/11\ 3/11\ 2/11)\begin{pmatrix} .8 & .1 & .1 \\ .2 & .6 & .2 \\ .3 & .3 & .4 \end{pmatrix}$$

$$= \left(\frac{4.8 + .6 + .6}{11} \quad \frac{.6 + 1.8 + .6}{11} \quad \frac{.6 + .6 + .8}{11} \right)$$

Example 1.20 (Hot Hand (Continuation of 1.10)). To find the stationary matrix in this case we can follow the same procedure. A consists of the first three columns of the transition matrix with 1 subtracted from the diagonal, and a final column of all 1's.

$$\begin{matrix} -1/4 & 1/4 & 0 & 1 \\ 0 & -1 & 2/3 & 1 \\ 2/3 & 1/3 & -1 & 1 \\ 0 & 0 & 1/2 & 1 \end{matrix}$$

The answer is given by the fourth row of A^{-1}:

$$(0.5,\ 0.1875,\ 0.1875,\ 0.125) = (1/2,\ 3/16,\ 3/16,\ 1/8)$$

Thus the long run fraction of time the player hits a shot is

$$\pi(HH) + \pi(MH) = 0.6875 = 11/36.$$

1.4.1 Doubly Stochastic Chains

Definition 1.2. A transition matrix p is said to be **doubly stochastic** if its COLUMNS sum to 1, or in symbols $\sum_x p(x,y) = 1$.

The adjective "doubly" refers to the fact that by its definition a transition probability matrix has ROWS that sum to 1, i.e., $\sum_y p(x,y) = 1$. The stationary distribution is easy to guess in this case:

Theorem 1.14. *If p is a doubly stochastic transition probability for a Markov chain with N states, then the uniform distribution, $\pi(x) = 1/N$ for all x, is a stationary distribution.*

Proof. To check this claim we note that if $\pi(x) = 1/N$ then

$$\sum_x \pi(x)p(x,y) = \frac{1}{N}\sum_x p(x,y) = \frac{1}{N} = \pi(y)$$

Looking at the second equality we see that conversely, if $\pi(x) = 1/N$ then p is doubly stochastic. □

Example 1.21 (Symmetric Reflecting Random Walk on the Line). The state space is $\{0, 1, 2 \ldots, L\}$. The chain goes to the right or left at each step with probability 1/2, subject to the rules that if it tries to go to the left from 0 or to the right from L it stays put. For example, when $L = 4$ the transition probability is

	0	**1**	**2**	**3**	**4**
0	0.5	0.5	0	0	0
1	0.5	0	0.5	0	0
2	0	0.5	0	0.5	0
3	0	0	0.5	0	0.5
4	0	0	0	0.5	0.5

It is clear in the example $L = 4$ that each column adds up to 1. With a little thought one sees that this is true for any L, so the stationary distribution is uniform, $\pi(i) = 1/(L + 1)$.

Example 1.22 (Tiny Board Game). Consider a circular board game with only six spaces $\{0, 1, 2, 3, 4, 5\}$. On each turn we roll a die with 1 on three sides, 2 on two sides, and 3 on one side to decide how far to move. Here we consider 5 to be adjacent to 0, so if we are there and we roll a 2 then the result is $5 + 2 \bmod 6 = 1$, where $i + k \bmod 6$ is the remainder when $i + k$ is divided by 6. In this case the transition probability is

	0	**1**	**2**	**3**	**4**	**5**
0	0	1/3	1/3	1/6	0	0
1	0	0	1/2	1/3	1/6	0
2	0	0	0	1/2	1/3	1/6
3	1/6	0	0	0	1/2	1/3
4	1/3	1/6	0	0	0	1/2
5	1/2	1/3	1/6	0	0	0

It is clear that the columns add to one, so the stationary distribution is uniform. To check the hypothesis of the convergence theorem, we note that after 3 turns we will have moved between three and nine spaces so $p^3(i, j) > 0$ for all i and j.

Example 1.23 (Mathematician's Monopoly). The game Monopoly is played on a game board that has 40 spaces arranged around the outside of a square. The squares have names like *Reading Railroad* and *Park Place* but we will number the squares 0 (*Go*), 1 (*Baltic Avenue*), ... 39 (*Boardwalk*). In Monopoly you roll two dice and move forward a number of spaces equal to the sum. For the moment, we will ignore things like *Go to Jail*, *Chance*, and other squares that make the game more interesting and formulate the dynamics as following. Let r_k be the probability that the sum of two dice is k ($r_2 = 1/36$, $r_3 = 2/36$, … $r_7 = 6/36$, …, $r_{12} = 1/36$) and let

$$p(i,j) = r_k \qquad \text{if } j = i + k \bmod 40$$

where $i+k$ mod 40 is the remainder when $i+k$ is divided by 40. To explain suppose that we are sitting on *Park Place* $i = 37$ and roll $k = 6$. $37 + 6 = 43$ but when we divide by 40 the remainder is 3, so $p(37, 3) = r_6 = 5/36$.

This example is larger but has the same structure as the previous example. Each row has the same entries but shifts one unit to the right each time with the number that goes off the right edge emerging in the 0 column. This structure implies that each entry in the row appears once in each column and hence the sum of the entries in the column is 1, and the stationary distribution is uniform. To check the hypothesis of the convergence theorem note that in four rolls you can move forward by 8–48 squares, so $p^4(i,j) > 0$ for all i and j.

Example 1.24. **Real Monopoly** has two complications:

- Square 30 is "Go to Jail," which sends you to square 10. You can buy your way out of jail but in the results we report below, we assume that you are cheap. If you roll a double, then you get out for free. If you don't get doubles in three tries, you have to pay.
- There are three *Chance* squares at 7, 12, and 36 (diamonds on the graph), and three *Community Chest* squares at 2, 17, 33 (squares on the graph), where you draw a card, which can send you to another square.

The graph gives the long run frequencies of being in different squares on the Monopoly board at the end of your turn, as computed by simulation (Fig. 1.1). We have removed the 9.46 % chance of being *In Jail* to make the probabilities easier to see. The value reported for square 10 is the 2.14 % probability of *Just Visiting Jail*, i.e., being brought there by the roll of the dice. Square 30, *Go to Jail*, has probability 0 for the obvious reasons. The other three lowest values occur for *Chance* squares. Due to the transition from 30 to 10, frequencies for squares near 20 are increased relative to the average of 2.5 % while those after 30 or before 10 are decreased. Squares 0 (*Go*) and 5 (*Reading Railroad*) are exceptions to this trend since there are Chance cards that instruct you to go there.

1.5 Detailed Balance Condition

π is said to satisfy the **detailed balance condition** if

$$\pi(x)p(x,y) = \pi(y)p(y,x) \tag{1.11}$$

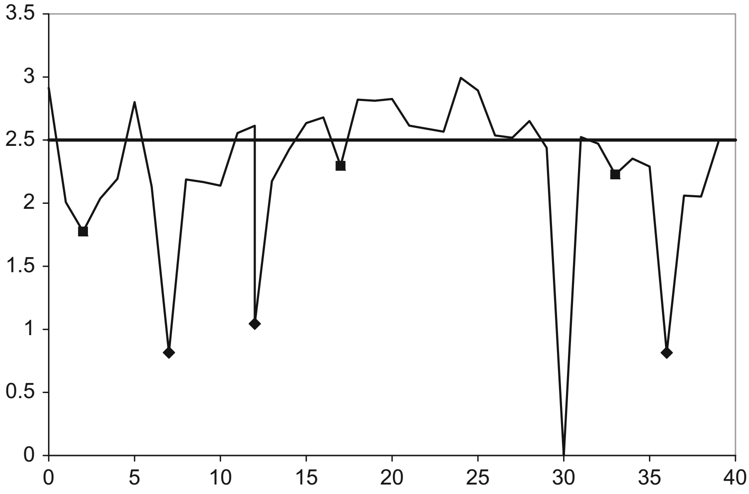


Fig. 1.1 Stationary distribution for monopoly

To see that this is a stronger condition than $\pi p = \pi$, we sum over x on each side to get

$$\sum_x \pi(x)p(x,y) = \pi(y)\sum_x p(y,x) = \pi(y)$$

As in our earlier discussion of stationary distributions, we think of $\pi(x)$ as giving the amount of sand at x, and one transition of the chain as sending a fraction $p(x,y)$ of the sand at x to y. In this case the detailed balance condition says that the amount of sand going from x to y in one step is exactly balanced by the amount going back from y to x. In contrast the condition $\pi p = \pi$ says that after all the transfers are made, the amount of sand that ends up at each site is the same as the amount that starts there.

Many chains do not have stationary distributions that satisfy the detailed balance condition.

Example 1.25. Consider

	1	**2**	**3**
1	.5	.5	0
2	.3	.1	.6
3	.2	.4	.4

There is no stationary distribution with detailed balance since $\pi(1)p(1,3) = 0$ but $p(1,3) > 0$ so we would have to have $\pi(3) = 0$ and using $\pi(3)p(3,i) = \pi(i)p(i,3)$ we conclude all the $\pi(i) = 0$. This chain is doubly stochastic so $(1/3, 1/3, 1/3)$ is a stationary distribution.

Example 1.26. **Birth and death chains** are defined by the property that the state space is some sequence of integers $\ell, \ell+1, \dots r-1, r$ and it is impossible to jump by more than one:

$$p(x, y) = 0 \quad \text{when } |x - y| > 1$$

Suppose that the transition probability has

$$\begin{aligned} p(x, x+1) &= p_x & &\text{for } x < r \\ p(x, x-1) &= q_x & &\text{for } x > \ell \\ p(x, x) &= 1 - p_x - q_x & &\text{for } \ell \le x \le r \end{aligned}$$

while the other $p(x, y) = 0$. If $x < r$ detailed balance between x and $x+1$ implies $\pi(x)p_x = \pi(x+1)q_{x+1}$, so

$$\pi(x+1) = \frac{p_x}{q_{x+1}} \cdot \pi(x) \tag{1.12}$$

Using this with $x = \ell$ gives $\pi(\ell+1) = \pi(\ell)p_\ell/q_{\ell+1}$. Taking $x = \ell+1$

$$\pi(\ell+2) = \frac{p_{\ell+1}}{q_{\ell+2}} \cdot \pi(\ell+1) = \frac{p_{\ell+1} \cdot p_\ell}{q_{\ell+2} \cdot q_{\ell+1}} \cdot \pi(\ell)$$

Extrapolating from the first two results we see that in general

$$\pi(\ell+i) = \pi(\ell) \cdot \frac{p_{\ell+i-1} \cdot p_{\ell+i-2} \cdots p_{\ell+1} \cdot p_\ell}{q_{\ell+i} \cdot q_{\ell+i-1} \cdots q_{\ell+2} \cdot q_{\ell+1}}$$

To keep the indexing straight note that: (i) there are i terms in the numerator and in the denominator, (ii) the indices decrease by 1 each time, (iii) the answer will not depend on q_ℓ (which is 0) or $p_{\ell+i}$.

For a concrete example to illustrate the use of this formula consider

Example 1.27 (Ehrenfest Chain). For concreteness, suppose there are three balls. In this case the transition probability is

	0	**1**	**2**	**3**
0	0	3/3	0	0
1	1/3	0	2/3	0
2	0	2/3	0	1/3
3	0	0	3/3	0

Setting $\pi(0) = c$ and using (1.12) we have

$$\pi(1) = 3c, \qquad \pi(2) = \pi(1) = 3c \qquad \pi(3) = \pi(2)/3 = c.$$

The sum of the π's is $8c$, so we pick $c = 1/8$ to get

$$\pi(0) = 1/8, \qquad \pi(1) = 3/8, \qquad \pi(2) = 3/8, \qquad \pi(3) = 1/8$$

Knowing the answer, one can look at the last equation and see that π represents the distribution of the number of Heads when we flip three coins, then guess and verify that in general that the binomial distribution with $p = 1/2$ is the stationary distribution:

$$\pi(x) = 2^{-n}\binom{n}{x}$$

Here $m! = 1 \cdot 2 \cdots (m-1) \cdot m$, with $0! = 1$, and

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

is the binomial coefficient which gives the number of ways of choosing x objects out of a set of n.

To check that our guess satisfies the detailed balance condition, we note that

$$\begin{aligned}\pi(x)p(x, x+1) &= 2^{-n}\frac{n!}{x!(n-x)!} \cdot \frac{n-x}{n} \\ &= 2^{-n}\frac{n!}{(x+1)!(n-x-1)!} \cdot \frac{x+1}{n} = \pi(x+1)p(x+1, x)\end{aligned}$$

However the following proof in words is simpler. Create X_0 by flipping coins, with heads = "in the left urn." The transition from X_0 to X_1 is done by picking a coin at random and then flipping it over. It should be clear that all 2^n outcomes of the coin tosses at time 1 are equally likely, so X_1 has the binomial distribution.

Example 1.28 (Three Machines, One Repairman). Suppose that an office has three machines that each break with probability .1 each day, but when there is at least one broken, then with probability 0.5 the repairman can fix one of them for use the next day. If we ignore the possibility of two machines breaking on the same day, then the number of working machines can be modeled as a birth and death chain with the following transition matrix:

	0	**1**	**2**	**3**
0	.5	.5	0	0
1	.05	.5	.45	0
2	0	.1	.5	.4
3	0	0	.3	.7

Rows 0 and 3 are easy to see. To explain row 1, we note that the state will only decrease by 1 if one machine breaks and the repairman fails to repair the one he is working on, an event of probability (.1)(.5), while the state can only increase by 1 if he succeeds and there is no new failure, an event of probability .5(.9). Similar reasoning shows $p(2,1) = (.2)(.5)$ and $p(2,3) = .5(.8)$.

To find the stationary distribution we use the recursive formula (1.12) to conclude that if $\pi(0) = c$ then

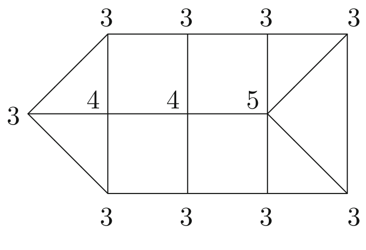
$$\pi(1) = \pi(0) \cdot \frac{p_0}{q_1} = c \cdot \frac{0.5}{0.05} = 10c$$
$$\pi(2) = \pi(1) \cdot \frac{p_1}{q_2} = 10c \cdot \frac{0.45}{0.1} = 45c$$
$$\pi(3) = \pi(2) \cdot \frac{p_2}{q_3} = 45c \cdot \frac{0.4}{0.3} = 60c$$

The sum of the π's is $116c$, so if we let $c = 1/116$ then we get

$$\pi(3) = \frac{60}{116}, \quad \pi(2) = \frac{45}{116}, \quad \pi(1) = \frac{10}{116}, \quad \pi(0) = \frac{1}{116}$$

There are many other Markov chains that are not birth and death chains but have stationary distributions that satisfy the detailed balance condition. A large number of possibilities are provided by

Example 1.29 (Random Walks on Graphs). A graph is described by giving two things: (i) a set of vertices V (which we suppose is a finite set) and (ii) an adjacency matrix $A(u,v)$, which is 1 if there is an edge connecting u and v and 0 otherwise. By convention we set $A(v,v) = 0$ for all $v \in V$.



The degree of a vertex u is equal to the number of neighbors it has. In symbols,

$$d(u) = \sum_v A(u,v)$$

since each neighbor of u contributes 1 to the sum. To help explain the concept, we have indicated the degrees on our example. We write the degree this way to make it clear that

$$p(u,v) = \frac{A(u,v)}{d(u)} \tag{$*$}$$

defines a transition probability. In words, if $X_n = u$, we jump to a randomly chosen neighbor of u at time $n+1$.

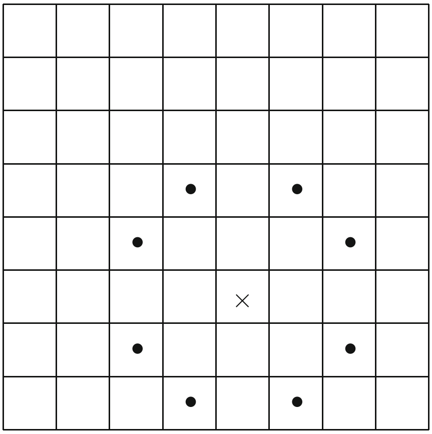
It is immediate from ($*$) that if c is a positive constant then $\pi(u) = cd(u)$ satisfies the detailed balance condition:

$$\pi(u)p(u,v) = cA(u,v) = cA(v,u) = \pi(v)p(u,v)$$

Thus, if we take $c = 1/\sum_u d(u)$, we have a stationary probability distribution. In the example $c = 1/40$.

For a concrete example, consider

Example 1.30 (Random Walk of a Knight on a Chess Board). A chess board is an 8 by 8 grid of squares. A knight moves by walking two steps in one direction and then one step in a perpendicular direction.



By patiently examining all of the possibilities, one sees that the degrees of the vertices are given by the following table. Lines have been drawn to make the symmetries more apparent.

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

The sum of the degrees is $4 \cdot 2 + 8 \cdot 3 + 20 \cdot 4 + 16 \cdot 6 + 16 \cdot 8 = 336$, so the stationary probabilities are the degrees divided by 336.

1.5.1 Reversibility

Let $p(i,j)$ be a transition probability with stationary distribution $\pi(i)$. Let X_n be a realization of the Markov chain starting from the stationary distribution, i.e., $P(X_0 = i) = \pi(i)$. The next result says that if we watch the process X_m, $0 \le m \le n$, backwards, then it is a Markov chain.

Theorem 1.15. *Fix n and let $Y_m = X_{n-m}$ for $0 \le m \le n$. Then Y_m is a Markov chain with transition probability*

$$\hat{p}(i,j) = P(Y_{m+1} = j | Y_m = i) = \frac{\pi(j)p(j,i)}{\pi(i)} \tag{1.13}$$

Proof. We need to calculate the conditional probability.

$$\begin{aligned} &P(Y_{m+1} = i_{m+1} | Y_m = i_m, Y_{m-1} = i_{m-1} \dots Y_0 = i_0) \\ &\qquad = \frac{P(X_{n-(m+1)} = i_{m+1}, X_{n-m} = i_m, X_{n-m+1} = i_{m-1} \dots X_n = i_0)}{P(X_{n-m} = i_m, X_{n-m+1} = i_{m-1} \dots X_n = i_0)} \end{aligned}$$

Using the Markov property, we see the numerator is equal to

$$\pi(i_{m+1})p(i_{m+1}, i_m)P(X_{n-m+1} = i_{m-1}, \dots X_n = i_0 | X_{n-m} = i_m)$$

Similarly the denominator can be written as

$$\pi(i_m)P(X_{n-m+1} = i_{m-1}, \dots X_n = i_0 | X_{n-m} = i_m)$$

Dividing the last two formulas and noticing that the conditional probabilities cancel we have

$$P(Y_{m+1} = i_{m+1} | Y_m = i_m, \dots Y_0 = i_0) = \frac{\pi(i_{m+1})p(i_{m+1}, i_m)}{\pi(i_m)}$$

This shows Y_m is a Markov chain with the indicated transition probability. □

The formula for the transition probability in (1.13), which is called the **dual transition probability**, may look a little strange, but it is easy to see that it works; i.e., the $\hat{p}(i,j) \geq 0$, and have

$$\sum_j \hat{p}(i,j) = \sum_j \pi(j)p(j,i)\pi(i) = \frac{\pi(i)}{\pi(i)} = 1$$

since $\pi p = \pi$. When π satisfies the detailed balance conditions:

$$\pi(i)p(i,j) = \pi(j)p(j,i)$$

the transition probability for the reversed chain,

$$\hat{p}(i,j) = \frac{\pi(j)p(j,i)}{\pi(i)} = p(i,j)$$

is the same as the original chain. In words, if we make a movie of the Markov chain X_m, $0 \leq m \leq n$ starting from an initial distribution that satisfies the detailed balance condition and watch it backwards (i.e., consider $Y_m = X_{n-m}$ for $0 \leq m \leq n$), then we see a random process with the same distribution m. To help explain the concept,

1.5.2 The Metropolis–Hastings Algorithm

Our next topic is a method for generating samples from a distribution $\pi(x)$. It is named for two of the authors of the fundamental papers on the topic. One written by Nicholas Metropolis and two married couples with last names Rosenbluth and Teller (1953) and the other by Hastings (1970). This is a very useful tool for computing posterior distributions in Bayesian statistics (Tierney 1994), reconstructing images (Geman and Geman 1984), and investigating complicated models in statistical physics (Hammersley and Handscomb 1964). It would take us too far afield to describe these applications, so we will content ourselves to describe the simple idea that is the key to the method.

We begin with a Markov chain $q(x, y)$ that is the proposed jump distribution. A move is accepted with probability

$$r(x, y) = \min \left\{ \frac{\pi(y)q(y, x)}{\pi(x)q(x, y)}, 1 \right\}$$

so the transition probability

$$p(x, y) = q(x, y)r(x, y)$$

To check that π satisfies the detailed balance condition we can suppose that $\pi(y)q(y,x) > \pi(x)q(x,y)$. In this case

$$\pi(x)p(x,y) = \pi(x)q(x,y) \cdot 1$$
$$\pi(y)p(y,x) = \pi(y)q(y,x)\frac{\pi(x)q(x,y)}{\pi(y)q(y,x)} = \pi(x)q(x,y)$$

To generate one sample from $\pi(x)$ we run the chain for a long time so that it reaches equilibrium. To obtain many samples, we output the state at widely separated times. Of course there is an art of knowing how long is long enough to wait between outputting the state to have independent realizations. If we are interested in the expected value of a particular function, then (if the chain is irreducible and the state space is finite) Theorem 1.22 guarantees that

$$\frac{1}{n}\sum_{m=1}^{n} f(X_m) \to \sum_x f(x)\pi(x)$$

The Metropolis–Hastings algorithm is often used when space is continuous, but that requires a more sophisticated Markov chain theory, so we will use discrete examples to illustrate the method.

Example 1.31 (Geometric Distribution). Suppose $\pi(x) = \theta^x(1-\theta)$ for $x = 0, 1, 2, \ldots$. To generate the jumps we will use a symmetric random walk $q(x, x+1) = q(x, x-1) = 1/2$. Since q is symmetric $r(x,y) = \min\{1, \pi(y)/\pi(x)\}$. In this case if $x > 0$, $\pi(x-1) > \pi(x)$ and $\pi(x+1)/\pi(x) = \theta$ so

$$p(x,x-1) = 1/2 \quad p(x,x+1) = \theta/2 \quad p(x,x) = (1-\theta)/2.$$

When $x = 0$, $\pi(-1) = 0$ so

$$p(0,-1) = 0 \quad p(0,1) = \theta/2 \quad p(0,0) = 1-(\theta/2).$$

To check reversibility we note that if $x \geq 0$ then

$$\pi(x)p(x,x+1) = \theta^x(1-\theta)\cdot\frac{\theta}{2} = \pi(x+1)p(x+1,x)$$

Here, as in most applications of the Metropolis–Hastings algorithm the choice of q is important. If θ is close to 1, then we would want to choose $q(x, x+i) = 1/2L+1$ for $-L \leq i \leq L$ where $L = O(1/(1-\theta))$ to make the chain move around the state space faster while not having too many steps rejected.

Example 1.32 (Binomial Distribution). Suppose $\pi(x)$ is Binomial(N,θ). In this case we can let $q(x,y) = 1/(N+1)$ for all $0 \leq x,y \leq N$. Since q is symmetric $r(x,y) = \min\{1, \pi(y)/\pi(x)\}$. This is closely related to the method of **rejection sampling**, in which one generates independent random variables U_i uniform on $\{0, 1, \ldots, N\}$ and keep U_i with probability $\pi(U_i)/\pi^*$ where $\pi^* = \max_{0\leq x\leq n}\pi(x)$.

Example 1.33 (Two Dimensional Ising Model). The Metropolis–Hastings algorithm has its roots in statistical physics. A typical problem is the Ising model of ferromagnetism. Space is represented by a two dimensional grid $\Lambda = \{-L, \ldots L\}^2$. If we made the lattice three dimensional, we could think of the atoms in an iron bar. In reality each atom has a spin which can point in some direction, but we simplify by supposing that each spin can be up $+1$ or down -1. The state of the systems is a function $\xi : \Lambda \to \{-1, 1\}$ i.e., a point in the product space $\{-1, 1\}^\Lambda$. We say that points x and y in Λ are neighbors if y is one of the four points $x + (1, 0)$, $x + (-1, 0)$, $x + (0, 1)$, $x + (0, -1)$. See the picture:

$$\begin{array}{ccccccc}
+ & - & + & + & + & - & - \\
- & - & - & + & + & + & - \\
+ & - & + & + & - & - & + \\
+ & + & - & - & \mathbf{y} & + & - \\
+ & - & + & \mathbf{y} & \mathbf{x} & \mathbf{y} & - \\
- & - & - & + & \mathbf{y} & + & - \\
+ & - & + & + & - & - & +
\end{array}$$

Given an interaction parameter β, which is inversely proportional to the temperature, the equilibrium state is

$$\pi(x) = \frac{1}{Z(\beta)} \exp\left(\beta \sum_{x,y\sim x} \xi_x \xi_y\right)$$

where the sum is over all $x, y \in \Lambda$ with y a neighbor of x, and $Z(\beta)$ is a constant that makes the probabilities sum to one. At the boundaries of the square spins have only three neighbors. There are several options for dealing with this: (i) we consider the spins outside to be 0, or (ii) we could specify a fixed boundary condition such as all spins $+$.

The sum is largest in case (i) when all of the spins agree or in case (ii) when all spins are $+$. This configuration minimizes the energy $H = -\sum_{x,y\sim x} \eta_x \eta_y$ but there many more configurations one with a random mixture of $+$'s and $-$'s. It turns out that as β increases the system undergoes a phase transition from a random state with an almost equal number of $+$'s and $-$'s to one in which more than 1/2 of the spins point in the same direction.

$Z(\beta)$ is difficult to compute so it is fortunate that only the ratio of the probabilities appears in the Metropolis–Hastings recipe. For the proposed jump distribution we let $q(\xi, \xi^x) = 1/(2L + 1)^2$ if the two configurations ξ and ξ^x differ only at x. In this case the transition probability is

$$p(\xi, \xi^x) = q(\xi, \xi^x) \min\left\{\frac{\pi(\xi^x)}{\pi(\xi)}, 1\right\}$$

Note that the ratio $\pi(\xi^x)/\pi(\xi)$ is easy to compute because $Z(\beta)$ cancels out, as do all the terms in the sum that do not involve x and its neighbors. Since $\xi^x(x) = -\xi(x)$.

$$\frac{\pi(\xi^x)}{\pi(\xi)} = \exp\left(-2\beta \sum_{y\sim x} \xi_x \xi_y\right)$$

If x agrees with k of its four neighbors, the ratio is $\exp(-2(4-2k))$. In words $p(x,y)$ can be described by saying that we accept the proposed move with probability 1 if it lowers the energy and with probability $\pi(y)/\pi(x)$ if not.

Example 1.34 (Simulated Annealing). The Metropolis–Hastings algorithm can also be used to minimize complicated functions. Consider, for example, the traveling salesman problem, which is to find the shortest (or least expensive) route that allows one to visit all of the cities on a list. In this case the state space will be lists of cities, x and $\pi(x) = \exp(-\beta\ell(x))$ where $\ell(x)$ is the length of the tour. The proposal kernel q is chosen to modify the list in some way. For example, we might move a city to another place on the list or reverse the order of a sequence of cities. When β is large the stationary distribution will concentrate on optimal and near optimal tours. As in the Ising model, β is thought of as inverse temperature. The name derives from the fact that to force the chain to better solution we increase β (i.e., reduce the temperature) as we run the simulation. One must do this slowly or the process will get stuck in local minima. For more of simulated annealing, see Kirkpatrick et al. (1983)

1.5.3 Kolmogorow Cycle Condition

In this section we will state and prove a necessary and sufficient condition for an irreducible Markov chain to have a stationary distribution that satisfies the detailed balance condition. All irreducible two state chains have stationary distributions that satisfy detailed balance so we begin with the case of three states.

Example 1.35. Consider the chain with transition probability:

	1	**2**	**3**
1	$1-(a+d)$	a	d
2	e	$1-(b+e)$	b
3	c	f	$1-(c+f)$

and suppose that all entries in the matrix are positive. To satisfy detailed balance we must have

$$e\pi(2) = a\pi(1) \qquad f\pi(3) = b\pi(2) \qquad d\pi(1) = c\pi(3)$$

To construct a measure that satisfies detailed balance we take $\pi(1) = k$. From this it follows that

$$\pi(2) = k\frac{a}{e} \qquad \pi(3) = k\frac{ab}{ef} \qquad k = \pi(1) = k\frac{abc}{def}$$

From this we see that there is a stationary distribution satisfying (1.11) if and only if

$$abc = def \quad \text{or} \quad \frac{abc}{def}$$

To lead into the next result we note that this condition is

$$\frac{p(1,2) \cdot p(2,3) \cdot p(3,1)}{p(2,1) \cdot p(3,2) \cdot p(1,3)} = 1$$

i.e., the probabilities around the loop $1, 2, 3$ in either direction are equal.

Kolmogorov Cycle Condition Consider an irreducible Markov chain with state space S. We say that the cycle condition is satisfied if given a cycle of states $x_0, x_1, \ldots, x_n = x_0$ with $p(x_{i-1}, x_i) > 0$ for $1 \le i \le n$, we have

$$\prod_{i=1}^{n} p(x_{i-1}, x_i) = \prod_{i=1}^{n} p(x_i, x_{i-1}) \tag{1.14}$$

Note that if this holds then $p(x_i, x_{i-1}) > 0$ for $1 \le i \le n$.

Theorem 1.16. *There is a stationary distribution that satisfies detailed balance if and only if* (1.14) *holds.*

Proof. We first show that if p has a stationary distribution that satisfies the detailed balance condition, then the cycle condition holds. Detailed balance implies

$$\frac{\pi(x_i)}{\pi(x_{i-1})} = \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})}.$$

Here and in what follows all of the transition probabilities we write down are positive. Taking the product from $i = 1, \ldots n$ and using the fact that $x_0 = x_n$ we have

$$1 = \prod_{i=1}^{n} \frac{\pi(x_i)}{\pi(x_{i-1})} = \prod_{i=1}^{n} \frac{p(x_{i-1}, x_i)}{q(x_i, x_{i-1})} \tag{1.15}$$

To prove the converse, suppose that the cycle condition holds. Let $a \in S$ and set $\pi(a) = c$. For $b \ne a$ in S let $x_0 = a, x_1 \ldots x_k = b$ be a path from a to b with $p(x_{i-1}, x_i) > 0$ for $1 \le i \le k$ (and hence $p(x_i, x_{i-1}) > 0$ for $1 \le i \le k$. By the reasoning used to derive (1.15)

$$\pi(b) = c \prod_{j=1}^{k} \frac{q(x_{i-1}, x_i)}{q(x_i, x_{i-1})}$$

The first step is to show that $\pi(b)$ is well defined, i.e., is independent of the path chosen. Let $x_0 = a, \dots x_k = b$ and $y_0 = a, \dots y_\ell = b$ be two paths from a to b. Combine these to get a loop that begins and ends at a.

$$z_0 = x_0, \dots z_k = x_k = y_\ell = b,$$
$$z_{k+1} = y_{\ell-1}, \dots, z_{k+\ell} = y_0 = a$$

Since $z_{k+j} = y_{\ell-j}$ for $1 \le j \le \ell$, letting $h = \ell - j$ we have

$$1 = \prod_{h=1}^{k+\ell} \frac{p(z_{h-1}, z_h)}{p(z_h, z_{h-1})} = \prod_{i=1}^{k} \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})} \cdot \prod_{j=1}^{\ell} \frac{p(y_{\ell-j+1}, y_{\ell-j})}{p(y_{\ell-j}, y_{\ell-j+1})}$$

Changing variables $m = \ell - j + 1$ we have

$$\prod_{i=1}^{k} \frac{q(x_{i-1}, x_i)}{p(x_i, x_{i-1})} = \prod_{m=1}^{\ell} \frac{p(y_{m-1}, y_m)}{p(y_m, y_{m-1})}$$

This shows that the definition is independent of the path chosen. To check that π satisfies the detailed balance condition suppose $q(c, b) > 0$. Let $x_0 = a, \dots, x_k = b$ be a path from a to b with $q(x_i, x_{i-1}) > 0$ for $1 \le i \le k$. If we let $x_{k+1} = c$, then since the definition is independent of path we have

$$\pi(b) = \prod_{i=1}^{k} \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})} \quad \pi(c) = \prod_{i=1}^{k+1} \frac{q(x_{i-1}, x_i)}{p(x_i, x_{i-1})} = \pi(b) \frac{p(b, c)}{p(c, b)}$$

and the detailed balance condition is satisfied. □

1.6 Limit Behavior

If y is a transient state, then by Lemma 1.11, $\sum_{n=1}^{\infty} p^n(x, y) < \infty$ for any initial state x and hence

$$p^n(x, y) \to 0$$

This means that we can restrict our attention to recurrent states and in view of the decomposition theorem, Theorem 1.8, to chains that consist of a single irreducible class of recurrent states. Our first example shows one problem that can prevent the convergence of $p^n(x, y)$.

Example 1.36 (Ehrenfest Chain (Continuation of 1.2)). For concreteness, suppose there are three balls. In this case the transition probability is

	0	**1**	**2**	**3**
0	0	3/3	0	0
1	1/3	0	2/3	0
2	0	2/3	0	1/3
3	0	0	3/3	0

In the second power of p the zero pattern is shifted:

	0	**1**	**2**	**3**
0	1/3	0	2/3	0
1	0	7/9	0	2/9
2	2/9	0	7/9	0
3	0	2/3	0	1/3

To see that the zeros will persist, note that if we have an odd number of balls in the left urn, then no matter whether we add or subtract one the result will be an even number. Likewise, if the number is even, then it will be odd on the next one step. This alternation between even and odd means that it is impossible to be back where we started after an odd number of steps. In symbols, if n is odd, then $p^n(x,x) = 0$ for all x.

To see that the problem in the last example can occur for multiples of any number N consider:

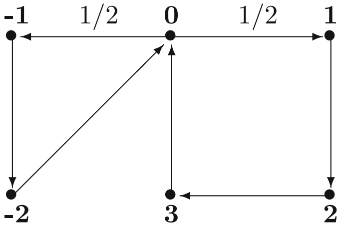
Example 1.37 (Renewal Chain). We will explain the name in Sect. 3.3. For the moment we will use it to illustrate "pathologies." Let f_k be a distribution on the positive integers and let $p(0, k-1) = f_k$. For states $i > 0$ we let $p(i, i-1) = 1$. In words the chain jumps from 0 to $k-1$ with probability f_k and then walks back to 0 one step at a time. If $X_0 = 0$ and the jump is to $k-1$, then it returns to 0 at time k. If say $f_5 = f_{15} = 1/2$, then $p^n(0,0) = 0$ unless n is a multiple of 5.

The **period** of a state is the largest number that will divide all the $n \geq 1$ for which $p^n(x,x) > 0$. That is, it is the greatest common divisor of $I_x = \{n \geq 1 : p^n(x,x) > 0\}$. To check that this definition works correctly, we note that in Example 1.36, $\{n \geq 1 : p^n(x,x) > 0\} = \{2, 4, \dots\}$, so the greatest common divisor is 2. Similarly, in Example 1.37, $\{n \geq 1 : p^n(x,x) > 0\} = \{5, 10, \dots\}$, so the greatest common divisor is 5. As the next example shows, things aren't always so simple.

Example 4.4 (Triangle and Square). Consider the transition matrix:

	-2	**-1**	**0**	**1**	**2**	**3**
-2	0	0	1	0	0	0
-1	1	0	0	0	0	0
0	0	0.5	0	0.5	0	0
1	0	0	0	0	1	0
2	0	0	0	0	0	1
3	0	0	1	0	0	0

In words, from 0 we are equally likely to go to 1 or -1. From -1 we go with probability one to -2 and then back to 0, from 1 we go to 2 then to 3 and back to 0. The name refers to the fact that $0 \to -1 \to -2 \to 0$ is a triangle and $0 \to 1 \to 2 \to 3 \to 0$ is a square.



Clearly, $p^3(0,0) > 0$ and $p^4(0,0) > 0$ so $3, 4 \in I_0$ and hence $d_0 = 1$. To determine the periods of the other states it is useful to know.

Lemma 1.17. *If $\rho_{xy} > 0$ and $\rho_{yx} > 0$, then x and y have the same period.*

Why is this true? The short answer is that if the two states have different periods, then by going from x to y, from y to y in the various possible ways, and then from y to x, we will get a contradiction.

Proof. Suppose that the period of x is c, while the period of y is $d < c$. Let k be such that $p^k(x,y) > 0$ and let m be such that $p^m(y,x) > 0$. Since

$$p^{k+m}(x,x) \geq p^k(x,y)p^m(y,x) > 0$$

we have $k + m \in I_x$. Since x has period c, $k + m$ must be a multiple of c. Now let ℓ be any integer with $p^\ell(y,y) > 0$. Since

$$p^{k+\ell+m}(x,x) \geq p^k(x,y)p^\ell(y,y)p^m(y,x) > 0$$

$k+\ell+m \in I_x$, and $k+\ell+m$ must be a multiple of c. Since $k+m$ is itself a multiple of c, this means that ℓ is a multiple of c. Since $\ell \in I_y$ was arbitrary, we have shown that c is a divisor of every element of I_y, but $d < c$ is the greatest common divisor, so we have a contradiction. □

With Lemma 1.17 in hand, we can show aperiodicity in many examples by using.

Lemma 1.18. *If* $p(x,x) > 0$, *then x has period 1.*

Proof. If $p(x,x) > 0$, then $1 \in I_x$, so the greatest common divisor is 1. □

This result by itself is enough to show that all states in the weather chain (Example 1.3), social mobility (Example 1.4), and brand preference chain (Example 1.5) are aperiodic. Combining it with Lemma 1.17 easily settles the question for the inventory chain (Example 1.6)

	0	**1**	**2**	**3**	**4**	**5**
0	0	0	.1	.2	.4	.3
1	0	0	.1	.2	.4	.3
2	.3	.4	.3	0	0	0
3	.1	.2	.4	.3	0	0
4	0	.1	.2	.4	.3	0
5	0	0	.1	.2	.4	.3

Since $p(x,x) > 0$ for $x = 2,3,4,5$, Lemma 1.18 implies that these states are aperiodic. Since this chain is irreducible it follows from Lemma 1.17 that 0 and 1 are aperiodic.

Consider now the basketball chain (Example 1.10):

	HH	**HM**	**MH**	**MM**
HH	3/4	1/4	0	0
HM	0	0	2/3	1/3
MH	2/3	1/3	0	0
MM	0	0	1/2	1/2

Lemma 1.18 implies that **HH** and **MM** are aperiodic. Since this chain is irreducible it follows from Lemma 1.17 that **HM** and **MH** are aperiodic.

We now come to the main results of the chapter. We first list the assumptions. All of these results hold when S is finite or infinite.

- I : p is irreducible
- A : aperiodic, all states have period 1
- R : all states are recurrent
- S : there is a stationary distribution π

Theorem 1.19 (Convergence Theorem). *Suppose I, A, S. Then as* $n \to \infty$, $p^n(x,y) \to \pi(y)$.

The next result describes the "limiting fraction of time we spend in each state."

Theorem 1.20 (Asymptotic Frequency). *Suppose I and R. If* $N_n(y)$ *be the number of visits to y up to time n, then*

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y}$$

We will see later that when the state space is infinite we may have $E_y T_y = \infty$ in which case the limit is 0. As a corollary we get the following.

Theorem 1.21. *If I and S hold, then*

$$\pi(y) = 1/E_y T_y$$

and hence the stationary distribution is unique.

In the next two examples we will be interested in the long run cost associated with a Markov chain. For this, we will need the following extension of Theorem 1.20.

Theorem 1.22. *Suppose I, S, and* $\sum_x |f(x)|\pi(x) < \infty$ *then*

$$\frac{1}{n}\sum_{m=1}^{n} f(X_m) \to \sum_x f(x)\pi(x)$$

Note that only Theorem 1.19 requires aperiodicity. Taking $f(x) = 1$ if $x = y$ and 0 otherwise in Theorem 1.22 gives Theorem 1.20. If we then take expected value we have

Theorem 1.23. *Suppose I, S.*

$$\frac{1}{n}\sum_{m=1}^{n} p^m(x,y) \to \pi(y)$$

Thus while the sequence $p^m(x,y)$ will not converge in the periodic case, the average of the first n values will.

To illustrate the use of these results, we consider

Example 1.38 (Repair Chain (Continuation of 1.7)). A machine has three critical parts that are subject to failure, but can function as long as two of these parts are working. When two are broken, they are replaced and the machine is back to working order the next day. Declaring the state space to be the parts that are broken $\{0, 1, 2, 3, 12, 13, 23\}$, we arrived at the following transition matrix:

	0	**1**	**2**	**3**	**12**	**13**	**23**
0	.93	.01	.02	.04	0	0	0
1	0	.94	0	0	.02	.04	0
2	0	0	.95	0	.01	0	.04
3	0	0	0	.97	0	.01	.02
12	1	0	0	0	0	0	0
13	1	0	0	0	0	0	0
23	1	0	0	0	0	0	0

and we asked: If we are going to operate the machine for 1800 days (about 5 years), then how many parts of types 1, 2, and 3 will we use?

To find the stationary distribution we look at the last row of

$$\begin{pmatrix} -.07 & .01 & .02 & .04 & 0 & 0 & 1 \\ 0 & -.06 & 0 & 0 & .02 & .04 & 1 \\ 0 & 0 & -.05 & 0 & .01 & 0 & 1 \\ 0 & 0 & 0 & -.03 & 0 & .01 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

where after converting the results to fractions we have

$$\pi(0) = 3000/8910$$

$$\pi(1) = 500/8910 \quad \pi(2) = 1200/8910 \quad \pi(3) = 4000/8910$$

$$\pi(12) = 22/8910 \quad \pi(13) = 60/8910 \quad \pi(23) = 128/8910$$

We use up one part of type 1 on each visit to 12 or to 13, so on the average we use 82/8910 of a part per day. Over 1800 days we will use an average of $1800 \cdot 82/8910 = 16.56$ parts of type 1. Similarly type 2 and type 3 parts are used at the long run rates of 150/8910 and 188/8910 per day, so over 1800 days we will use an average of 30.30 parts of type 2 and 37.98 parts of type 3.

Example 1.39 (Inventory Chain (Continuation of 1.6)). We have an electronics store that sells a videogame system, with the potential for sales of 0, 1, 2, or 3 of these units each day with probabilities .3, .4, .2, and .1. Each night at the close of business new units can be ordered which will be available when the store opens in the morning.

As explained earlier if X_n is the number of units on hand at the end of the day then we have the following transition matrix:

	0	**1**	**2**	**3**	**4**	**5**
0	0	0	.1	.2	.4	.3
1	0	0	.1	.2	.4	.3
2	.3	.4	.3	0	0	0
3	.1	.2	.4	.3	0	0
4	0	.1	.2	.4	.3	0
5	0	0	.1	.2	.4	.3

In the first section we asked the question:

Q. Suppose we make \$12 profit on each unit sold but it costs \$2 a day to store items. What is the long-run profit per day of this inventory policy?

The first thing we have to do is to compute the stationary distribution. The last row of

$$\begin{pmatrix} -1 & 0 & .1 & .2 & .4 & 1 \\ 0 & -1 & .1 & .2 & .4 & 1 \\ .3 & .4 & -.7 & 0 & 0 & 1 \\ .1 & .2 & .4 & -.7 & 0 & 1 \\ 0 & .1 & .2 & .4 & -.7 & 1 \\ 0 & 0 & .1 & .2 & .4 & .1 \end{pmatrix}^{-1}$$

has entries

$$\frac{1}{9740}(885, 1516, 2250, 2100, 1960, 1029)$$

We first compute the average number of units sold per day. To do this we note that the average demand per day is

$$ED = 0.4 \cdot 1 + 0.2 \cdot 2 + 0.1 \cdot 3 = 1.1$$

so we only have to compute the average number of lost sales per day. This only occurs when $X_n = 2$ and $D_{n+1} = 3$ and in this case we lose one sale so the lost sales per day is

$$\pi(2)P(D_{n+1} = 3) = \frac{2250}{9740} \cdot 0.1 = 0.0231$$

Thus the average number of sales per day is $1.1 - 0.023 = 1.077$ for a profit of $1.077 \cdot 12 = 12.94$.

Taking $f(k) = 2k$ in Theorem 1.22 we see that in the long our average holding costs per day are

$$\frac{1}{9740}(1516 \cdot 2 + 2250 \cdot 4 + 2100 \cdot 6 + 1960 \cdot 8 + 1029 \cdot 10) = 5.20.$$

1.7 Returns to a Fixed State

In this section we will prove Theorems 1.20, 1.21, and 1.22. The key is to look at the times the chain returns to a fixed state.

Theorem 1.20. *Suppose p is irreducible and recurrent. Let $N_n(y)$ be the number of visits to y at times $\leq n$. As $n \to \infty$*

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y}$$

Why is this true? Suppose first that we start at y. The times between returns, $t_1, t_2, \ldots$ are independent and identically distributed so the strong law of large numbers for nonnegative random variables implies that the time of the kth return to y, $R(k) = \min\{n \geq 1 : N_n(y) = k\}$, has

$$\frac{R(k)}{k} \to E_y T_y \leq \infty \tag{1.16}$$

If we do not start at y, then $t_1 < \infty$ and $t_2, t_3, \ldots$ are independent and identically distributed and we again have (1.16). Writing $a_k \sim b_k$ when $a_k/b_k \to 1$ we have $R(k) \sim kE_yT_y$. Taking $k = n/E_yT_y$ we see that there are about n/E_yT_y returns by time n.

Proof. We have already shown (1.16). To turn this into the desired result, we note that from the definition of $R(k)$ it follows that $R(N_n(y)) \leq n < R(N_n(y) + 1)$. Dividing everything by $N_n(y)$ and then multiplying and dividing on the end by $N_n(y) + 1$, we have

$$\frac{R(N_n(y))}{N_n(y)} \leq \frac{n}{N_n(y)} < \frac{R(N_n(y) + 1)}{N_n(y) + 1} \cdot \frac{N_n(y) + 1}{N_n(y)}$$

Letting $n \to \infty$, we have $n/N_n(y)$ trapped between two things that converge to E_yT_y, so

$$\frac{n}{N_n(y)} \to E_y T_y$$

and we have proved the desired result. □

From the last result we immediately get

Theorem 1.21. *If p is an irreducible and has stationary distribution π, then*

$$\pi(y) = 1/E_y T_y$$

Proof. Suppose X_0 has distribution π. From Theorem 1.20 it follows that

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y}$$

Taking expected value and using the fact that $N_n(y) \leq n$, it can be shown that this implies

$$\frac{E_\pi N_n(y)}{n} \to \frac{1}{E_y T_y}$$

but since π is a stationary distribution $E_\pi N_n(y) = n\pi(y)$. □

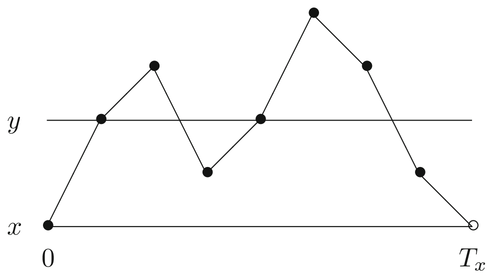


Fig. 1.2 Picture of the cycle trick

A corollary of this result is that for an irreducible chain if there is a stationary distribution it is unique. Our next topic is the existence of stationary measures

Theorem 1.24. *Suppose p is irreducible and recurrent. Let* $x \in S$ *and let* $T_x = \inf\{n \geq 1 : X_n = x\}$.

$$\mu_x(y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n)$$

defines a stationary measure with $0 < \mu_x(y) < \infty$ *for all* y.

Why is this true? This is called the "cycle trick." $\mu_x(y)$ is the expected number of visits to y in $\{0, \ldots, T_x - 1\}$. Multiplying by p moves us forward one unit in time so $\mu_x p(y)$ is the expected number of visits to y in $\{1, \ldots, T_x\}$. Since $X(T_x) = X_0 = x$ it follows that $\mu_x = \mu_x p$ (Fig. 1.2).

Proof. To formalize this intuition, let $\bar{p}_n(x, y) = P_x(X_n = y, T_x > n)$ and interchange sums to get

$$\sum_y \mu_x(y) p(y, z) = \sum_{n=0}^{\infty} \sum_y \bar{p}_n(x, y) p(y, z)$$

Case 1. Consider the generic case first: $z \neq x$.

$$\sum_y \bar{p}_n(x, y) p(y, z) = \sum_y P_x(X_n = y, T_x > n, X_{n+1} = z)$$
$$= P_x(T_x > n + 1, X_{n+1} = z) = \bar{p}_{n+1}(x, z)$$

Here the second equality holds since the chain must be somewhere at time n, and the third is just the definition of $\bar{p}_{n+1}$. Summing from $n = 0$ to ∞, we have

$$\sum_{n=0}^{\infty} \sum_y \bar{p}_n(x, y) p(y, z) = \sum_{n=0}^{\infty} \bar{p}_{n+1}(x, z) = \mu_x(z)$$

since $\bar{p}_0(x, z) = 0$.

Case 2. Now suppose that $z = x$. Reasoning as above we have

$$\sum_y \bar{p}_n(x,y)p(y,x) = \sum_y P_x(X_n = y, T_x > n, X_{n+1} = x) = P_x(T_x = n+1)$$

Summing from $n = 0$ to ∞ we have

$$\sum_{n=0}^{\infty} \sum_y \bar{p}_n(x,y)p(y,x) = \sum_{n=0}^{\infty} P_x(T_x = n+1) = 1 = \mu_x(x)$$

since $P_x(T_x = 0) = 0$.

To check $\mu_x(y) < \infty$, we note that $\mu_x(x) = 1$ and

$$1 = \mu_x(x) = \sum_z \mu_x(z)p^n(z,x) \geq \mu_x(y)p^n(y,x)$$

so if we pick n with $p^n(y,x) > 0$ then we conclude $\mu_x(y) < \infty$.

To prove that $\mu_x(y) > 0$ we note that this is trivial for $y = x$ the point used to define the measure. For $y \neq x$, we borrow an idea from Theorem 1.5. Let $K = \min\{k : p^k(x,y) > 0\}$. Since $p^K(x,y) > 0$ there must be a sequence $y_1, \ldots y_{K-1}$ so that

$$p(x,y_1)p(y_1,y_2)\cdots p(y_{K-1},y) > 0$$

Since K is minimal all the $y_i \neq y$, so $P_x(X_K = y, T_x > K) > 0$ and hence $\mu_x(y) > 0$. □

Our next step is to prove

Theorem 1.22. *Suppose p is irreducible, has stationary distribution π, and $\sum_x |f(x)|\pi(x) < \infty$ then*

$$\frac{1}{n}\sum_{m=1}^{n} f(X_m) \to \sum_x f(x)\pi(x)$$

The key idea here is that by breaking the path at the return times to x we get a sequence of random variables to which we can apply the law of large numbers.

Sketch of Proof. Suppose that the chain starts at x. Let $T_0 = 0$ and $T_k = \min\{n > T_{k-1} : X_n = x\}$ be the time of the kth return to x. By the strong Markov property, the random variables

$$Y_k = \sum_{m=T_{k-1}+1}^{T_k} f(X_m)$$

are independent and identically distributed. By the cycle trick in the proof of Theorem 1.24

$$EY_k = \sum_x \mu_x(y)f(y)$$

Using the law of large numbers for i.i.d. variables

$$\frac{1}{L}\sum_{m=1}^{T_L} f(X_m) = \frac{1}{L}\sum_{k=1}^{L} Y_k \to \sum_x \mu_x(y)f(y)$$

Taking $L = N_n(x) = \max\{k : T_k \le n\}$ and ignoring the contribution from the last incomplete cycle $(N_n(x), n]$

$$\frac{1}{n}\sum_{m=1}^{n} f(X_m) \approx \frac{N_n(x)}{n} \cdot \frac{1}{N_n(x)} \sum_{k=1}^{N_n(x)} Y_k$$

Using Theorem 1.20 and the law of large numbers the above

$$\to \frac{1}{E_x T_x} \sum_y \mu_x(y)f(y) = \sum_y \pi(y)f(y)$$

□

1.8 Proof of the Convergence Theorem*

To prepare for the proof of the convergence theorem, Theorem 1.19, we need the following:

Lemma 1.25. *If there is a stationary distribution, then all states y that have $\pi(y) > 0$ are recurrent.*

Proof. Lemma 1.12 tells us that $E_x N(y) = \sum_{n=1}^{\infty} p^n(x,y)$, so

$$\sum_x \pi(x) E_x N(y) = \sum_x \pi(x) \sum_{n=1}^{\infty} p^n(x,y)$$

Interchanging the order of summation and then using $\pi p^n = \pi$, the above

$$= \sum_{n=1}^{\infty} \sum_x \pi(x) p^n(x,y) = \sum_{n=1}^{\infty} \pi(y) = \infty$$

since $\pi(y) > 0$. Using Lemma 1.11 now gives $E_x N(y) = \rho_{xy}/(1 - \rho_{yy})$, so

$$\infty = \sum_x \pi(x) \frac{\rho_{xy}}{1 - \rho_{yy}} \leq \frac{1}{1 - \rho_{yy}}$$

the second inequality following from the facts that $\rho_{xy} \leq 1$ and π is a probability measure. This shows that $\rho_{yy} = 1$, i.e., y is recurrent. □

The next ingredient is

Lemma 1.26. *If x has period 1, i.e., the greatest common divisor I_x is 1, then there is a number n_0 so that if $n \geq n_0$, then $n \in I_x$. In words, I_x contains all of the integers after some value n_0.*

To prove this we begin by proving

Lemma 1.27. *I_x is closed under addition. That is, if $i, j \in I_x$, then $i + j \in I_x$.*

Proof. If $i, j \in I_x$, then $p^i(x, x) > 0$ and $p^j(x, x) > 0$ so

$$p^{i+j}(x, x) \geq p^i(x, x) p^j(x, x) > 0$$

and hence $i + j \in I_x$. □

Using this we see that in the triangle and square example

$$I_0 = \{3, 4, 6, 7, 8, 9, 10, 11, \ldots\}$$

Note that in this example once we have three consecutive numbers (e.g., 6, 7, 8) in I_0 then $6 + 3, 7 + 3, 8 + 3 \in I_0$ and hence I_0 will contain all the integers $n \geq 6$.

For another unusual example consider the renewal chain (Example 1.37) with $f_5 = f_{12} = 1/2$. $5, 12 \in I_0$ so using Lemma 1.27

$$I_0 = \{5, 10, 12, 15, 17, 20, 22, 24, 25, 27, 29, 30, 32,$$
$$34, 35, 36, 37, 39, 40, 41, 42, 43, \ldots\}$$

To check this note that 5 gives rise to $10 = 5 + 5$ and $17 = 5 + 12$, 10 to 15 and 22, 12 to 17 and 24, etc. Once we have five consecutive numbers in I_0, here 39–43, we have all the rest. The last two examples motivate the following.

Proof. We begin by observing that it is enough to show that I_x will contain two consecutive integers: k and $k + 1$. For then it will contain $2k, 2k + 1, 2k + 2$, and $3k, 3k + 1, 3k + 2, 3k + 3$, or in general $jk, jk + 1, \ldots jk + j$. For $j \geq k - 1$ these blocks overlap and no integers are left out. In the last example $24, 25 \in I_0$ implies $48, 49, 50 \in I_0$ which implies $72, 73, 74, 75 \in I_0$ and $96, 97, 98, 99, 100 \in I_0$, so we know the result holds for $n_0 = 96$. In fact it actually holds for $n_0 = 34$ but it is not important to get a precise bound.

To show that there are two consecutive integers, we cheat and use a fact from number theory: if the greatest common divisor of a set I_x is 1, then there are integers $i_1, \dots i_m \in I_x$ and (positive or negative) integer coefficients c_i so that $c_1 i_1 + \cdots + c_m i_m = 1$. Let $a_i = c_i^+$ and $b_i = (-c_i)^+$. In words the a_i are the positive coefficients and the b_i are -1 times the negative coefficients. Rearranging the last equation gives

$$a_1 i_1 + \cdots + a_m i_m = (b_1 i_1 + \cdots + b_m i_m) + 1$$

and using Lemma 1.27 we have found our two consecutive integers in I_x. □

With Lemmas 1.25 and 1.26 in hand we are ready to tackle the proof of:

Theorem 1.23 (Convergence Theorem). *Suppose p is irreducible, aperiodic, and has stationary distribution π. Then as $n \to \infty$, $p^n(x, y) \to \pi(y)$.*

Proof. Let S be the state space for p. Define a transition probability $\bar{p}$ on $S \times S$ by

$$\bar{p}((x_1, y_1), (x_2, y_2)) = p(x_1, x_2) p(y_1, y_2)$$

In words, each coordinate moves independently.

Step 1. We will first show that if p is aperiodic and irreducible then $\bar{p}$ is irreducible. Since p is irreducible, there are K, L, so that $p^K(x_1, x_2) > 0$ and $p^L(y_1, y_2) > 0$. Since x_2 and y_2 have period 1, it follows from Lemma 1.26 that if M is large, then $p^{L+M}(x_2, x_2) > 0$ and $p^{K+M}(y_2, y_2) > 0$, so

$$\bar{p}^{K+L+M}((x_1, y_1), (x_2, y_2)) > 0$$

Step 2. Since the two coordinates are independent $\bar{\pi}(a, b) = \pi(a)\pi(b)$ defines a stationary distribution for $\bar{p}$, and Lemma 1.25 implies that all states are recurrent for $\bar{p}$. Let (X_n, Y_n) denote the chain on $S \times S$, and let T be the first time that the two coordinates are equal, i.e., $T = \min\{n \ge 0 : X_n = Y_n\}$. Let $V_{(x,x)} = \min\{n \ge 0 : X_n = Y_n = x\}$ be the time of the first visit to (x, x). Since $\bar{p}$ is irreducible and recurrent, $V_{(x,x)} < \infty$ with probability one. Since $T \le V_{(x,x)}$ for any x we must have

$$P(T < \infty) = 1. \tag{1.17}$$

Step 3. By considering the time and place of the first intersection and then using the Markov property we have

$$\begin{aligned} P(X_n = y, T \le n) &= \sum_{m=1}^{n} \sum_x P(T = m, X_m = x, X_n = y) \\ &= \sum_{m=1}^{n} \sum_x P(T = m, X_m = x) P(X_n = y | X_m = x) \end{aligned}$$

$$= \sum_{m=1}^{n} \sum_{x} P(T = m, Y_m = x) P(Y_n = y | Y_m = x)$$
$$= P(Y_n = y, T \le n)$$

Step 4. To finish up we observe that since the distributions of X_n and Y_n agree on $\{T \le n\}$

$$|P(X_n = y) - P(Y_n = y)| \le P(X_n = y, T > n) + P(Y_n = y, T > n)$$

and summing over y gives

$$\sum_y |P(X_n = y) - P(Y_n = y)| \le 2P(T > n)$$

If we let $X_0 = x$ and let Y_0 have the stationary distribution π, then Y_n has distribution π, and using (1.17) it follows that

$$\sum_y |p^n(x, y) - \pi(y)| \le 2P(T > n) \to 0$$

proving the convergence theorem. □

1.9 Exit Distributions

To motivate developments, we begin with an example.

Example 1.40 (Two Year College). At a local two year college, 60 % of freshmen become sophomores, 25 % remain freshmen, and 15 % drop out. 70 % of sophomores graduate and transfer to a four year college, 20 % remain sophomores and 10 % drop out. What fraction of new students eventually graduate?

We use a Markov chain with state space 1 = freshman, 2 = sophomore, G = graduate, D = dropout. The transition probability is

	1	**2**	**G**	**D**
1	0.25	0.6	0	0.15
2	0	0.2	0.7	0.1
G	0	0	1	0
D	0	0	0	1

Let $h(x)$ be the probability that a student currently in state x eventually graduates. By considering what happens on one step

$$h(1) = 0.25h(1) + 0.6h(2)$$
$$h(2) = 0.2h(2) + 0.7$$

To solve we note that the second equation implies $h(2) = 7/8$ and then the first that

$$h(1) = \frac{0.6}{0.75} \cdot \frac{7}{8} = 0.7$$

Example 1.41 (Tennis). In tennis the winner of a game is the first player to win four points, unless the score is $4 - 3$, in which case the game must continue until one player is ahead by two points and wins the game. Suppose that the server win the point with probability 0.6 and successive points are independent. What is the probability the server will win the game if the score is tied 3–3? if she is ahead by one point? Behind by one point?

We formulate the game as a Markov chain in which the state is the difference of the scores. The state space is $2, 1, 0, -1, -2$ with 2 (win for server) and -2 (win for opponent). The transition probability is

	2	**1**	**0**	**-1**	**-2**
2	1	0	0	0	0
1	.6	0	.4	0	0
0	0	.6	0	.4	0
-1	0	0	.6	0	.4
-2	0	0	0	0	1

If we let $h(x)$ be the probability of the server winning when the score is x, then

$$h(x) = \sum_y p(x, y)h(y)$$

with $h(2) = 1$ and $h(-2) = 0$. This gives us three equations in three unknowns

$$h(1) = .6 + .4h(0)$$
$$h(0) = .6h(1) + .4h(-1)$$
$$h(-1) = .6h(0)$$

Using the first and third equations in the second we have

$$h(0) = .6(.6 + .4h(0)) + .4(.6h(0)) = .36 + .48h(0)$$

so we have $h(0) = 0.36/0.52 = 0.6923$.

The last computation uses special properties of this example. To introduce a general approach, we rearrange the equations to get

$$h(1) - .4h(0) + 0h(-1) = .6$$
$$-.6h(1) + h(0) - .4h(-1) = 0$$
$$0h(1) - .6h(0) + h(-1) = 0$$

which can be written in matrix form as

$$\begin{pmatrix} 1 & -.4 & 0 \\ -.6 & 1 & -.4 \\ 0 & -.6 & 1 \end{pmatrix} \begin{pmatrix} h(1) \\ h(0) \\ h(-1) \end{pmatrix} = \begin{pmatrix} .6 \\ 0 \\ 0 \end{pmatrix}$$

Let $C = \{1, 0, -1\}$ be the nonabsorbing states and let $r(x, y)$ the restriction of p to $x, y \in C$ (i.e., the 3×3 matrix inside the black lines in the transition probability). In this notation then the matrix above is $I - r$, while the right-hand side is $v_i = p(i, a)$. Solving gives

$$\begin{pmatrix} h(1) \\ h(0) \\ h(-1) \end{pmatrix} = (I - r)^{-1} \begin{pmatrix} .6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} .8769 \\ .6923 \\ .4154 \end{pmatrix}$$

General Solution Suppose that the server wins each point with probability w. If the game is tied then after two points, the server will have won with probability w^2, lost with probability $(1 - w)^2$, and returned to a tied game with probability $2w(1 - w)$, so $h(0) = w^2 + 2w(1-w)h(0)$. Since $1-2w(1-w) = w^2 + (1-w)^2$, solving gives

$$h(0) = \frac{w^2}{w^2 + (1 - w)^2}$$

Figure 1.3 graphs this function.

Having worked two examples, it is time to show that we have computed the right answer. In some cases we will want to guess and verify the answer. In those situations it is nice to know that the solution is unique. The next result proves this. Given a set F let $V_F = \min\{n \geq 0 : X_n \in F\}$.

Theorem 1.28. *Consider a Markov chain with state space S. Let A and B be subsets of S, so that $C = S - (A \cup B)$ is finite. Suppose $h(a) = 1$ for $a \in A$, $h(b) = 0$ for $b \in B$, and that for $x \in C$ we have*

$$h(x) = \sum_y p(x, y)h(y) \tag{1.18}$$

If $P_x(V_A \wedge V_B < \infty) > 0$ for all $x \in C$, then $h(x) = P_x(V_a < V_b)$.

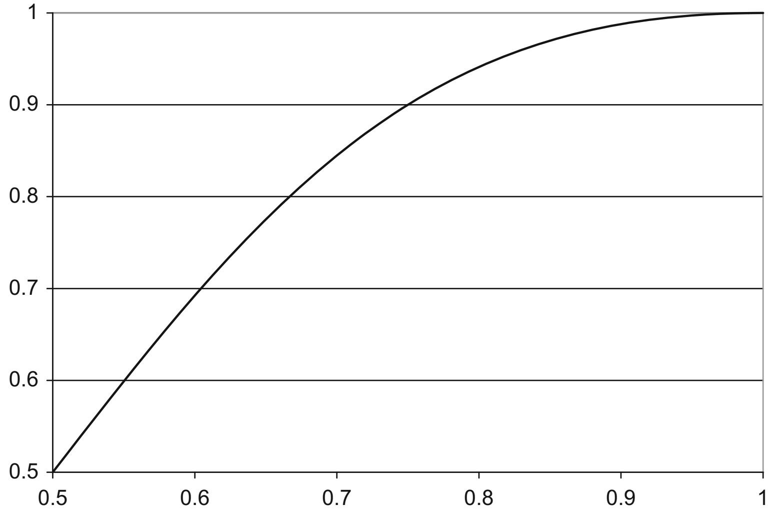


Fig. 1.3 Probability of winning a game in tennis as a function of the probability of winning a point

Proof. Let $T = V_A \wedge V_B$. It follows from Lemma 1.3 that $P_x(T < \infty) = 1$ for all $x \in C$. (1.18) implies that $h(x) = E_x h(X_1)$ when $x \in C$. The Markov property implies

$$h(x) = E_x h(X_{T \wedge n}).$$

We have to stop at time T because the equation is not assumed to be valid for $x \in A \cup B$. Since h is bounded with $h(a) = 1$ for $a \in A$, and $h(b) = 0$ for $b \in B$, it follows from the bounded convergence theorem A.10 that $E_x h(X_{T \wedge n}) \to P_x(V_A < V_B)$ which gives the desired result. □

Example 1.42 (Matching Pennies). Bob, who has 15 pennies, and Charlie, who has 10 pennies, decide to play a game. They each flip a coin. If the two coins match, Bob gets the two pennies (for a profit of 1). If the two coins are different, then Charlie gets the two pennies. They quit when someone has all of the pennies. What is the probability Bob will win the game?

The answer will turn out to be 15/25, Bob's fraction of the total supply of pennies. To explain this, let X_n be the number of pennies Bob has after n plays. X_n is a fair game, i.e., $x = E_x X_1$, or in words the expected number of pennies Bob has is constant in time. Let

$$V_y = \min\{n \geq 0 : X_n = y\}$$

be the time of the first visit to y. Taking a leap of faith the expected number he has at the end of the game should be the same as at the beginning so

$$x = N P_x(V_N < V_0) + 0 P_x(V_0 < V_n)$$

and solving gives

$$P_x(V_N < V_0) = x/N \quad \text{for } 0 \le x \le N \tag{1.19}$$

To prove this note that by considering what happens on the first step

$$h(x) = \frac{1}{2}h(x+1) + \frac{1}{2}h(x-1)$$

Multiplying by 2 and rearranging

$$h(x+1) - h(x) = h(x) - h(x-1)$$

or in words, h has constant slope. Since $h(0) = 0$ and $h(N) = 1$ the slope must be $1/N$ and we must have $h(x) = x/N$.

The reasoning in the last example can be used to study Example 1.9.

Example 1.43 (Wright–Fisher Model with No Mutation). The state space is $S = \{0, 1, \dots N\}$ and the transition probability is

$$p(x,y) = \binom{N}{y} \left(\frac{x}{N}\right)^y \left(\frac{N-x}{N}\right)^{N-y}$$

The right-hand side is the binomial$(N, x/N)$ distribution, i.e., the number of successes in N trials when success has probability x/N, so the mean number of successes is x. From this it follows that if we define $h(x) = x/N$, then

$$h(x) = \sum_y p(x,y)h(y)$$

Taking $a = N$ and $b = 0$, we have $h(a) = 1$ and $h(b) = 0$. Since $P_x(V_a \wedge V_b < \infty) > 0$ for all $0 < x < N$, it follows from Lemma 1.28 that

$$P_x(V_N < V_0) = x/N \tag{1.20}$$

i.e., the probability of fixation to all A's is equal to the fraction of the genes that are A.

Our next topic is non-fair games.

Example 1.44 (Gambler's Ruin). Consider a gambling game in which on any turn you win \$1 with probability $p \neq 1/2$ or lose \$1 with probability $1 - p$. Suppose further that you will quit playing if your fortune reaches \$$N$. Of course, if your fortune reaches \$0, then the casino makes you stop. Let

$$h(x) = P_x(V_N < V_0)$$

be the happy event that our gambler reaches the goal of \$$N$ before going bankrupt when starting with \$$x$. Thanks to our definition of V_x as the minimum of $n \geq 0$ with $X_n = x$ we have $h(0) = 0$, and $h(N) = 1$. To calculate $h(x)$ for $0 < x < N$, we set $q = 1 - p$ to simplify the formulas, and consider what happens on the first step to arrive at

$$h(x) = ph(x+1) + qh(x-1) \tag{1.21}$$

To solve this we rearrange to get $p(h(x+1) - h(x)) = q(h(x) - h(x-1))$ and conclude

$$h(x+1) - h(x) = \frac{q}{p} \cdot (h(x) - h(x-1)) \tag{1.22}$$

If we set $c = h(1) - h(0)$, then (1.22) implies that for $x \geq 1$

$$h(x) - h(x-1) = c\left(\frac{q}{p}\right)^{x-1}$$

Summing from $x = 1$ to N, we have

$$1 = h(N) - h(0) = \sum_{x=1}^{N} h(x) - h(x-1) = c \sum_{x=1}^{N} \left(\frac{q}{p}\right)^{x-1}$$

Now for $\theta \neq 1$ the partial sum of the geometric series is

$$\sum_{j=0}^{N-1} \theta^j = \frac{1-\theta^N}{1-\theta} \tag{1.23}$$

To check this note that

$$(1-\theta)(1+\theta+\cdots\theta^{N-1}) = (1+\theta+\cdots\theta^{N-1}) \\ -(\theta+\theta^2+\cdots\theta^N) = 1-\theta^N$$

Using (1.23) we see that $c = (1-\theta)/(1-\theta^N)$ with $\theta = q/p$. Summing and using the fact that $h(0) = 0$, we have

$$h(x) = h(x) - h(0) = c\sum_{i=0}^{x-1} \theta^i = c \cdot \frac{1-\theta^x}{1-\theta} = \frac{1-\theta^x}{1-\theta^N}$$

Recalling the definition of $h(x)$ and rearranging the fraction we have

$$P_x(V_N < V_0) = \frac{\theta^x - 1}{\theta^N - 1} \qquad \text{where } \theta = \tfrac{1-p}{p} \tag{1.24}$$

To see what (1.24) says in a concrete example, we consider:

Example 1.45 (Roulette). If we bet \$1 on red on a roulette wheel with 18 red, 18 black, and 2 green (0 and 00) holes, we win \$1 with probability $18/38 = 0.4737$ and lose \$1 with probability 20/38. Suppose we bring \$50 to the casino with the hope of reaching \$100 before going bankrupt. What is the probability we will succeed?

Here $\theta = q/p = 20/18$, so (1.24) implies

$$P_{50}(V_{100} < V_0) = \frac{\left(\frac{20}{18}\right)^{50} - 1}{\left(\frac{20}{18}\right)^{100} - 1}$$

Using $(20/18)^{50} = 194$, we have

$$P_{50}(V_{100} < V_0) = \frac{194 - 1}{(194)^2 - 1} = \frac{1}{194 + 1} = 0.005128$$

Now let's turn things around and look at the game from the viewpoint of the casino, i.e., $p = 20/38$. Suppose that the casino starts with the rather modest capital of $x = 100$. (1.24) implies that the probability they will reach N before going bankrupt is

$$\frac{(9/10)^{100} - 1}{(9/10)^N - 1}$$

If we let $N \to \infty$, $(9/10)^N \to 0$ so the answer converges to

$$1 - (9/10)^{100} = 1 - 2.656 \times 10^{-5}$$

If we increase the capital to \$200, then the failure probability is squared, since to become bankrupt we must first lose \$100 and then lose our second \$100. In this case the failure probability is incredibly small: $(2.656 \times 10^{-5})^2 = 7.055 \times 10^{-10}$.

From the last analysis we see that if $p > 1/2$, $q/p < 1$ and letting $N \to \infty$ in (1.24) gives

$$P_x(V_0 = \infty) = 1 - \left(\frac{q}{p}\right)^x \quad \text{and} \quad P_x(V_0 < \infty) = \left(\frac{q}{p}\right)^x. \tag{1.25}$$

To see that the form of the last answer makes sense, note that to get from x to 0 we must go $x \to x - 1 \to x_2 \ldots \to 1 \to 0$, so

$$P_x(V_0 < \infty) = P_1(V_0 < \infty)^x.$$

Example 1.46. If we rearrange the matrix for the seven state chain in Example 1.14 we get

	2	**3**	**1**	**5**	**4**	**6**	**7**
2	.2	.3	.1	0	.4	0	0
3	0	.5	0	.2	.3	0	0
1	0	0	.7	.3	0	0	0
5	0	0	.6	.4	0	0	0
4	0	0	0	0	.5	.5	0
6	0	0	0	0	0	.2	.8
7	0	0	0	0	1	0	0

We will now use what we have learned about exit distributions to find $\lim_{n\to\infty} p^n(i,j)$.

The first step is to note that by our formula for two state chains the stationary distribution on the closed irreducible set $A = \{1,5\}$ is $2/3, 1/3$. With a little more work one concludes that the stationary distribution on $B = \{4,6,7\}$ is $8/17, 5/17, 4/17$ the third row of

$$\begin{pmatrix} .5 & -.5 & 1 \\ 0 & .8 & 1 \\ -1 & 0 & 1 \end{pmatrix}^{-1}$$

If we collapse the closed irreducible sets into a single state, then we have:

	2	**3**	**A**	**B**
2	.2	.3	.1	.4
3	0	.5	.2	.3

If we let r be the part of the transition probability for states 2 and 3, then

$$(I-r)^{-1}\begin{pmatrix} .1 \\ .2 \end{pmatrix} = \begin{pmatrix} 11/40 \\ 3/5 \end{pmatrix}$$

Combining these observations we see that the limit is

	2	**3**	**1**	**5**	**4**	**6**	**7**
2	0	0	11/60	11/120	29/85	29/136	29/170
3	0	0	4/15	2/15	24/85	15/85	12/85
1	0	0	2/3	1/3	0	0	0
5	0	0	2/3	1/3	0	0	0
4	0	0	0	0	8/17	5/17	4/17
6	0	0	0	0	8/17	5/17	4/17
7	0	0	0	0	8/17	5/17	4/17

1.10 Exit Times

To motivate developments we begin with an example.

Example 1.47 (Two Year College). In Example 1.40 we introduced a Markov chain with state space 1 = freshman, 2 = sophomore, G = graduate, D = dropout, and transition probability

	1	**2**	**G**	**D**
1	0.25	0.6	0	0.15
2	0	0.2	0.7	0.1
G	0	0	1	0
D	0	0	0	1

On the average how many years does a student take to graduate or drop out?

Let $g(x)$ be the expected time for a student starting in state x. $g(G) = g(D) = 0$. By considering what happens on one step

$$g(1) = 1 + 0.25g(1) + 0.6g(2)$$
$$g(2) = 1 + 0.2g(2)$$

where the 1+ is due to the fact that after the jump has been made one year has elapsed. To solve for g, we note that the second equation implies $g(2) = 1/0.8 = 1.25$ and then the first that

$$g(1) = \frac{1 + 0.6(1.25)}{0.75} = \frac{1.75}{0.75} = 2.3333$$

Example 1.48 (Tennis). In Example 1.41 we formulated the last portion of the game as a Markov chain in which the state is the difference of the scores. The state space was $S = \{2, 1, 0, -1, -2\}$ with 2 (win for server) and -2 (win for opponent). The transition probability was

	2	**1**	**0**	**-1**	**-2**
2	1	0	0	0	0
1	.6	0	.4	0	0
0	0	.6	0	.4	0
-1	0	0	.6	0	.4
-2	0	0	0	0	1

Let $g(x)$ be the expected time to complete the game when the current state is x. By considering what happens on one step

$$g(x) = 1 + \sum_y p(x, y)g(y)$$

Since $g(2) = g(-2) = 0$, if we let $r(x, y)$ be the restriction of the transition probability to $1, 0, -1$ we have

$$g(x) - \sum_y r(x, y) g(y) = 1$$

Writing $\mathbf{1}$ for a 3×1 matrix (i.e., column vector) with all 1's we can write this as

$$(I - r)g = \mathbf{1}$$

so $g = (I - r)^{-1}\mathbf{1}$.

There is another way to see this. If $N(y)$ is the number of visits to y at times $n \geq 0$, then from (1.12)

$$E_x N(y) = \sum_{n=0}^{\infty} r^n(x, y)$$

To see that this is $(I - r)^{-1}(x, y)$ note that $(I - r)(I + r + r^2 + r^3 + \cdots)$

$$= (I + r + r^2 + r^3 + \cdots) - (r + r^2 + r^3 + r^4 \cdots) = I$$

If T is the duration of the game, then $T = \sum_y N(y)$ so

$$E_x T = (I - r)^{-1}\mathbf{1} \tag{1.26}$$

To solve the problem now we note that

$$I - r = \begin{pmatrix} 1 & -.4 & 0 \\ -.6 & 1 & -.4 \\ 0 & -.6 & 1 \end{pmatrix} \qquad (I - r)^{-1} = \begin{pmatrix} 19/13 & 10/13 & 4/13 \\ 15/13 & 25/13 & 10/13 \\ 9/13 & 15/13 & 19/13 \end{pmatrix}$$

so $E_0 T = (15 + 25 + 10)/13 = 50/13 = 3.846$ points. Here the three terms in the sum are the expected number of visits to -1, 0, and 1.

Having worked two examples, it is time to show that we have computed the right answer. In some cases we will want to guess and verify the answer. In those situations it is nice to know that the solution is unique. The next result proves this.

Theorem 1.29. *Let $V_A = \inf\{n \geq 0 : X_n \in A\}$. Suppose $C = S - A$ is finite, and that $P_x(V_A < \infty) > 0$ for any $x \in C$. If $g(a) = 0$ for all $a \in A$, and for $x \in C$ we have*

$$g(x) = 1 + \sum_y p(x, y) g(y) \tag{1.27}$$

Then $g(x) = E_x(V_A)$.

Proof. It follows from Lemma 1.3 that $E_x V_A < \infty$ for all $x \in C$. (1.27) implies that $g(x) = 1 + E_x g(X_1)$ when $x \notin A$. The Markov property implies

$$g(x) = E_x(V_A \wedge n) + E_x g(X_{V_A \wedge n}).$$

We have to stop at time T because the equation is not valid for $x \in A$. It follows from the definition of the expected value that $E_x(V_A \wedge n) \uparrow E_x V_A$. Since g is bounded and $g(a) = 0$ for $a \in A$, we have $E_x g(X_{T \wedge n}) \to 0$. □

Example 1.49 (Waiting Time for TT). Let T_{TT} be the (random) number of times we need to flip a coin before we have gotten Tails on two consecutive tosses. To compute the expected value of T_{TT} we will introduce a Markov chain with states 0, 1, 2 = the number of Tails we have in a row.

Since getting a Tails increases the number of Tails we have in a row by 1, but getting a Heads sets the number of Tails we have in a row to 0, the transition matrix is

	0	**1**	**2**
0	1/2	1/2	0
1	1/2	0	1/2
2	0	0	1

Since we are not interested in what happens after we reach 2 we have made 2 an absorbing state. If we let $V_2 = \min\{n \geq 0 : X_n = 2\}$ and $g(x) = E_x V_2$, then one step reasoning gives

$$g(0) = 1 + .5g(0) + .5g(1)$$
$$g(1) = 1 + .5g(0)$$

Plugging the second equation into the first gives $g(0) = 1.5 + .75g(0)$, so $.25g(0) = 1.5$ or $g(0) = 6$. To do this with the previous approach we note

$$I - r = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{pmatrix} \qquad (I - r)^{-1} = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$$

so $E_0 V_2 = 6$.

Example 1.50 (Waiting Time for HT). Let T_{HT} be the (random) number of times we need to flip a coin before we have gotten a Heads followed by a Tails. Consider X_n is Markov chain with transition probability:

	HH	**HT**	**TH**	**TT**
HH	1/2	1/2	0	0
HT	0	0	1/2	1/2
TH	1/2	1/2	0	0
TT	0	0	1/2	1/2

If we eliminate the row and the column for *HT*, then

$$I - r = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -1/2 & 1/2 \end{pmatrix} \quad (I - r)^{-1}\mathbf{1} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

To compute the expected waiting time for our original problem, we note that after the first two tosses we have each of the four possibilities with probability 1/4 so

$$ET_{HT} = 2 + \frac{1}{4}(0 + 2 + 2 + 4) = 4$$

Why is $\boldsymbol{ET_{TT} = 6}$ while $\boldsymbol{ET_{HT} = 4}$? To explain we begin by noting that $E_y T_y = 1/\pi(y)$ and the stationary distribution assigns probability 1/4 to each state. One can verify this and check that convergence to equilibrium is rapid by noting that all the entries of p^2 are equal to 1/4. Our identity implies that

$$E_{HT} T_{HT} = \frac{1}{\pi(HT)} = 4$$

To get from this to what we wanted to calculate, note that if we start with a H at time -1 and a T at time 0, then we have nothing that will help us in the future, so the expected waiting time for a *HT* when we start with nothing is the same.

When we consider *TT*, our identity again gives

$$E_{TT} T_{TT} = \frac{1}{\pi(TT)} = 4$$

However, this time if we start with a T at time -1 and a T at time 0, then a T at time 1 will give us a *TT* and a return to *TT* at time 1; while if we get a H at time 1, then we have wasted 1 turn and we have nothing that can help us later, so

$$4 = E_{TT} T_{TT} = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1 + ET_{TT})$$

Solving gives $ET_{TT} = 6$, so it takes longer to observe *TT*. The reason for this, which can be seen in the last equation, is that once we have one *TT*, we will get another one with probability 1/2, while occurrences of *HT* cannot overlap.

In the Exercises 1.49 and 1.50 we will consider waiting times for three coin patterns. The most interesting of these is $ET_{HTH} = ET_{THT}$.

Example 1.51. On the average how many times do we need to roll one die in order to see a run of six rolls with all different numbers? To formulate the problem we need some notation. Let X_n, $n \geq 1$ be independent and equal to 1, 2, 3, 4, 5, 6, with probability 1/6 each. Let $K_n = \max\{k : X_n, X_{n-1}, \ldots X_{n-k+1}\}$ are all different. To explain the last definition note that if the last five rolls are

$$\frac{6}{n-4} \quad \frac{2}{n-3} \quad \frac{6}{n-2} \quad \frac{5}{n-1} \quad \frac{1}{n}$$

then $K_n = 4$. K_n is a Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$. To begin to compute the transition probability note that if in the example drawn above $X_{n+1} \in \{3, 4\}$ then $K_{n+1} = 5$ while if $X_{n+1} = 1, 5, 6, 2$ then $K_{n+1} = 1, 2, 3, 4$. Extending the reasoning we see that the transition probability is

	1	**2**	**3**	**4**	**5**	**6**
1	1/6	5/6	0	0	0	0
2	1/6	1/6	4/6	0	0	0
3	1/6	1/6	1/6	3/6	0	0
4	1/6	1/6	1/6	1/6	2/6	0
5	1/6	1/6	1/6	1/6	1/6	1/6
6	1/6	1/6	1/6	1/6	1/6	1/6

If we let r be the portion of the transition probability for states {1, 2, 3, 4, 5}, then

$$(I-r)^{-1} = \begin{pmatrix} 14.7 & 23.5 & 23 & 15 & 6 \\ 13.5 & 23.5 & 23 & 15 & 6 \\ 13.2 & 22 & 23 & 15 & 6 \\ 12.6 & 21 & 21 & 15 & 6 \\ 10.8 & 18 & 18 & 12 & 6 \end{pmatrix} \tag{1.28}$$

To see why the last column consists of 6's recall that multiplying $(I-r)^{-1}$ by the column vector $(0, 0, 0, 0, 1/6)$ gives the probability we exist at 6 which is 1. In next to last column $E_x N(5) = 15$ for $x = 1, 2, 3, 4$, since in each case we will wander around for a while before we hit 5.

Using (1.28) the expected exit times are

$$(I-r)^{-1}\mathbf{1} = \begin{pmatrix} 82.2 \\ 81 \\ 79.2 \\ 75.6 \\ 64.8 \end{pmatrix}$$

To check the order of magnitude of the answer we note that the probability six rolls of a die will be different is

$$\frac{6!}{6^6} = .015432 = \frac{1}{64.8}$$

To connect this with the last entry note that

$$E_5 T_6 = E_6 T_6 = \frac{1}{\pi(6)}$$

Example 1.52 (Duration of Fair Games). Consider the gambler's ruin chain in which $p(i, i+1) = p(i, i-1) = 1/2$. Let $\tau = \min\{n : X_n \notin (0, N)\}$. We claim that

$$E_x \tau = x(N - x) \tag{1.29}$$

To see what formula (1.29) says, consider matching pennies. There $N = 25$ and $x = 15$, so the game will take $15 \cdot 10 = 150$ flips on the average. If there are twice as many coins, $N = 50$ and $x = 30$, then the game takes $30 \cdot 20 = 600$ flips on the average, or four times as long.

There are two ways to prove this.

Verify the Guess Let $g(x) = x(N - x)$. Clearly, $g(0) = g(N) = 0$. If $0 < x < N$, then by considering what happens on the first step we have

$$g(x) = 1 + \frac{1}{2}g(x+1) + \frac{1}{2}g(x-1)$$

If $g(x) = x(N - x)$, then the right-hand side is

$$\begin{aligned}
&= 1 + \frac{1}{2}(x+1)(N-x-1) + \frac{1}{2}(x-1)(N-x+1) \\
&= 1 + \frac{1}{2}[x(N-x) - x + N - x - 1] + \frac{1}{2}[x(N-x) + x - (N-x+1)] \\
&= 1 + x(N-x) - 1 = x(N-x)
\end{aligned}$$

Derive the Answer (1.27) implies that

$$g(x) = 1 + (1/2)g(x+1) + (1/2)g(x-1)$$

Rearranging gives

$$g(x+1) - g(x) = -2 + g(x) - g(x-1)$$

Setting $g(1) - g(0) = c$ we have $g(2) - g(1) = c - 2$, $g(3) - g(2) = c - 4$ and in general that

$$g(k) - g(k-1) = c - 2(k-1)$$

Using $g(0) = 0$ and summing we have

$$0 = g(N) = \sum_{k=1}^{N} c - 2(k-1) = cN - 2 \cdot \frac{N(N-1)}{2}$$

since, as one can easily check by induction, $\sum_{j=1}^{m} j = m(m+1)/2$. Solving gives $c = (N-1)$. Summing again, we see that

$$g(x) = \sum_{k=1}^{x} (N-1) - 2(k-1) = x(N-1) - x(x+1) = x(N-x)$$

Example 1.53 (Duration of Nonfair Games). Consider the gambler's ruin chain in which $p(i, i+1)p$ and $p(i, i-1) = q$, where $p \neq q$. Let $\tau = \min\{n : X_n \notin (0, N)\}$. We claim that

$$E_x\tau = \frac{x}{q-p} - \frac{N}{q-p} \cdot \frac{1 - (q/p)^x}{1 - (q/p)^N} \tag{1.30}$$

This time the derivation is somewhat tedious, so we will just verify the guess. We want to show that $g(x) = 1 + pg(x+1) + qg(x-1)$. Plugging the formula into the right-hand side:

$$\begin{aligned} &= 1 + p\frac{x+1}{q-p} + q\frac{x-1}{q-p} - \frac{N}{q-p}\left[p \cdot \frac{1-(q/p)^{x+1}}{1-(q/p)^N} + q\frac{1-(q/p)^{x-1}}{1-(q/p)^N}\right] \\ &= 1 + \frac{x}{q-p} + \frac{p-q}{q-p} - \frac{N}{q-p}\left[\frac{p+q-(q/p)^x(q+p)}{1-(q/p)^N}\right] \end{aligned}$$

which $= g(x)$ since $p + q = 1$.

To see what this says note that if $p < q$ then $q/p > 1$ so

$$\frac{N}{1-(q/p)^N} \to 0 \quad \text{and} \quad g(x) = \frac{x}{q-p} \tag{1.31}$$

To see this is reasonable note that our expected value on one play is $p - q$, so we lose an average of $q - p$ per play, and it should take an average of $x/(q-p)$ to lose x dollars.

When $p > q$, $(q/p)^N \to 0$, so doing some algebra

$$g(x) \approx \frac{N-x}{p-q}[1-(q/p)^x] + \frac{x}{p-q}(q/p)^x$$

Using (1.25) we see that the probability of not hitting 0 is $1-(q/p)^x$. In this case, since our expected winnings per play is $p-q$, it should take about $(N-x)/(p-q)$ plays to get to N. The second term represents the contribution to the expected value from paths that end at 0, but it is hard to explain why the term has exactly this form.

1.11 Infinite State Spaces*

In this section we consider chains with an infinite state space. The major new complication is that recurrence is not enough to guarantee the existence of a stationary distribution.

Example 1.54 (Reflecting Random Walk). Imagine a particle that moves on $\{0, 1, 2, \ldots\}$ according to the following rules. It takes a step to the right with probability p. It attempts to take a step to the left with probability $1-p$, but if it is at 0 and tries to jump to the left, it stays at 0, since there is no -1 to jump to. In symbols,

$$\begin{aligned} p(i, i+1) &= p \quad \text{when } i \geq 0 \\ p(i, i-1) &= 1-p \quad \text{when } i \geq 1 \\ p(0,0) &= 1-p \end{aligned}$$

This is a birth and death chain, so we can solve for the stationary distribution using the detailed balance equations:

$$p\pi(i) = (1-p)\pi(i+1) \quad \text{when } i \geq 0$$

Rewriting this as $\pi(i+1) = \pi(i) \cdot p/(1-p)$ and setting $\pi(0) = c$, we have

$$\pi(i) = c\left(\frac{p}{1-p}\right)^i \tag{1.32}$$

There are now three cases to consider:

$p < 1/2$: $p/(1-p) < 1$. $\pi(i)$ decreases exponentially fast, so $\sum_i \pi(i) < \infty$, and we can pick c to make π a stationary distribution. To find the value of c to make π a probability distribution we recall

$$\sum_{i=0}^{\infty} \theta^i = 1/(1-\theta) \quad \text{when } \theta < 1.$$

Taking $\theta = p/(1-p)$ and hence $1-\theta = (1-2p)/(1-p)$, we see that the sum of the $\pi(i)$ defined in $(*)$ is $c(1-p)/(1-2p)$, so

$$\pi(i) = \frac{1-2p}{1-p} \cdot \left(\frac{p}{1-p}\right)^i = (1-\theta)\theta^i \tag{1.33}$$

To confirm that we have succeeded in making the $\pi(i)$ add up to 1, note that if we are flipping a coin with a probability θ of Heads, then the probability of getting i Heads before we get our first Tails is given by $\pi(i)$.

The reflecting random walk is clearly irreducible. To check that it is aperiodic note that $p(0,0) > 0$ implies 0 has period 1, and then Lemma 1.17 implies that all states have period 1. Using the convergence theorem, Theorem 1.19, now we see that

I. *When* $p < 1/2$, $P(X_n = j) \to \pi(j)$, *the stationary distribution in (1.33).*

Using Theorem 1.21 now,

$$E_0 T_0 = \frac{1}{\pi(0)} = \frac{1}{1-\theta} = \frac{1-p}{1-2p} \tag{1.34}$$

It should not be surprising that the system stabilizes when $p < 1/2$. In this case movements to the left have a higher probability than to the right, so there is a drift back toward 0. On the other hand, if steps to the right are more frequent than those to the left, then the chain will drift to the right and wander off to ∞.

II. *When* $p > 1/2$ *all states are transient.*

(1.25) implies that if $x > 0$, $P_x(T_0 < \infty) = ((1-p)/p)^x$.

To figure out what happens in the borderline case $p = 1/2$, we use results from Sects. 1.8 and 1.9. Recall we have defined $V_y = \min\{n \geq 0 : X_n = y\}$ and (1.19) tells us that if $x > 0$

$$P_x(V_N < V_0) = x/N$$

If we keep x fixed and let $N \to \infty$, then $P_x(V_N < V_0) \to 0$ and hence

$$P_x(V_0 < \infty) = 1$$

In words, for any starting point x, the random walk will return to 0 with probability 1. To compute the mean return time, we note that if $\tau_N = \min\{n : X_n \notin (0,N)\}$, then we have $\tau_N \leq V_0$ and by (1.29) we have $E_1 \tau_N = N - 1$. Letting $N \to \infty$ and combining the last two facts shows $E_1 V_0 = \infty$. Reintroducing our old hitting time $T_0 = \min\{n > 0 : X_n = 0\}$ and noting that on our first step we go to 0 or to 1 with probability 1/2 shows that

$$E_0 T_0 = (1/2) \cdot 1 + (1/2) E_1 V_0 = \infty$$

Summarizing the last two paragraphs, we have

III. *When* $p = 1/2$, $P_0(T_0 < \infty) = 1$ *but* $E_0 T_0 = \infty$.

Thus when $p = 1/2$, 0 is recurrent in the sense we will certainly return, but it is not recurrent in the following sense:

x is said to be **positive recurrent** if $E_x T_x < \infty$.

If a state is recurrent but not positive recurrent, i.e., $P_x(T_x < \infty) = 1$ but $E_x T_x = \infty$, then we say that x is **null recurrent**.

In our new terminology, our results for reflecting random walk say

If $p < 1/2$, 0 is positive recurrent
If $p = 1/2$, 0 is null recurrent
If $p > 1/2$, 0 is transient

In reflecting random walk, null recurrence thus represents the borderline between recurrence and transience. This is what we think in general when we hear the term. To see the reason we might be interested in positive recurrence recall that by Theorem 1.21

$$\pi(x) = \frac{1}{E_x T_x}$$

If $E_x T_x = \infty$, then this gives $\pi(x) = 0$. This observation motivates

Theorem 1.30. *For an irreducible chain the following are equivalent:*

(i) Some state is positive recurrent.
(ii) There is a stationary distribution π.
(iii) All states are positive recurrent.

Proof. The stationary measure constructed in Theorem 1.24 has total mass

$$\begin{aligned}\sum_y \mu(y) &= \sum_{n=0}^{\infty} \sum_y P_x(X_n = y, T_x > n) \\ &= \sum_{n=0}^{\infty} P_x(T_x > n) = E_x T_x\end{aligned}$$

so (i) implies (ii). Noting that irreducibility implies $\pi(y) > 0$ for all y and then using $\pi(y) = 1/E_y T_y$ shows that (ii) implies (iii). It is trivial that (iii) implies (i). □

Our next example may at first seem to be quite different. In a branching process 0 is an absorbing state, so by Theorem 1.5 all the other states are transient. However, as the story unfolds we will see that branching processes have the same trichotomy as random walks do.

Example 1.55 (Branching Processes). Consider a population in which each individual in the nth generation gives birth to an independent and identically distributed number of children. The number of individuals at time n, X_n is a Markov chain with transition probability given in Example 1.8. As announced there, we are interested in the question:

Q. What is the probability the species avoids extinction?

Here "extinction" means becoming absorbed state at 0. As we will now explain, whether this is possible or not can be determined by looking at the average number of offspring of one individual:

$$\mu = \sum_{k=0}^{\infty} kp_k$$

If there are m individuals at time $n-1$, then the mean number at time n is $m\mu$. More formally the conditional expectation given X_{n-1}

$$E(X_n|X_{n-1}) = \mu X_{n-1}$$

Taking expected values of both sides gives $EX_n = \mu EX_{n-1}$. Iterating gives

$$EX_n = \mu^n EX_0 \tag{1.35}$$

If $\mu < 1$, then $EX_n \to 0$ exponentially fast. Using the inequality

$$EX_n \geq P(X_n \geq 1)$$

it follows that $P(X_n \geq 1) \to 0$ and we have

I. *If $\mu < 1$, then extinction occurs with probability 1.*

To treat the cases $\mu \geq 1$ we will use a one-step calculation. Let ρ be the probability that this process dies out (i.e., reaches the absorbing state 0) starting from $X_0 = 1$. If there are k children in the first generation, then in order for extinction to occur, the family line of each child must die out, an event of probability ρ^k, so we can reason that

$$\rho = \sum_{k=0}^{\infty} p_k \rho^k \tag{1.36}$$

If we let $\phi(\theta) = \sum_{k=0}^{\infty} p_k \theta^k$ be the generating function of the distribution p_k, then the last equation can be written simply as $\rho = \phi(\rho)$ (Fig. 1.4).

The equation in (1.36) has a trivial root at $\rho = 1$ since $\phi(\rho) = \sum_{k=0}^{\infty} p_k \rho^k = 1$. The next result identifies the root that we want:

Lemma 1.31. *The extinction probability ρ is the smallest solution of the equation $\phi(x) = x$ with $0 \leq x \leq 1$.*

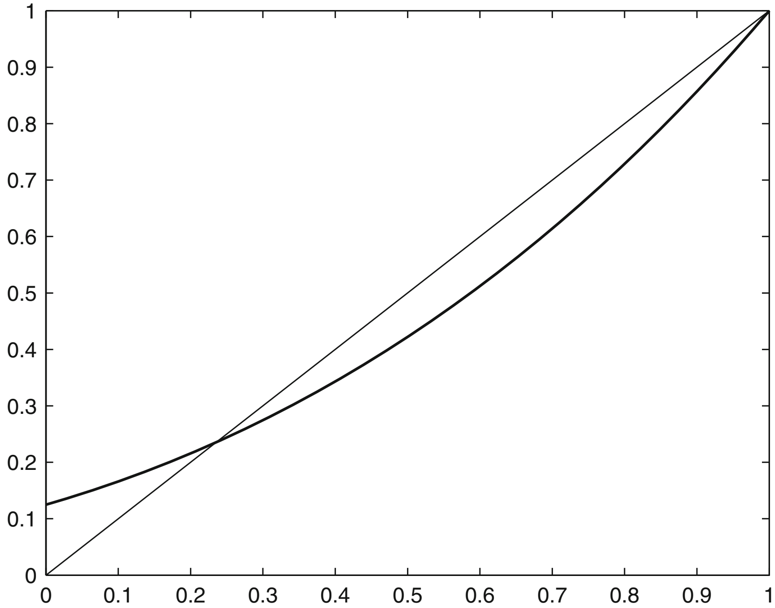


Fig. 1.4 Generating function for Binomial(3,1/2)

Proof. Extending the reasoning for (1.36) we see that in order for the process to hit 0 by time n, all of the processes started by first-generation individuals must hit 0 by time $n-1$, so

$$P(X_n = 0) = \sum_{k=0}^{\infty} p_k P(X_{n-1} = 0)^k$$

From this we see that if $\rho_n = P(X_n = 0)$ for $n \geq 0$, then $\rho_n = \phi(\rho_{n-1})$ for $n \geq 1$.

Since 0 is an absorbing state, $\rho_0 \leq \rho_1 \leq \rho_2 \leq \ldots$ and the sequence converges to a limit ρ_∞. Letting $n \to \infty$ in $\rho_n = \phi(\rho_{n-1})$ implies that $\rho_\infty = \phi(\rho_\infty)$, i.e., ρ_∞ is a solution of $\phi(x) = x$. To complete the proof now let ρ be the smallest solution. Clearly $\rho_0 = 0 \leq \rho$. Using lthe fact that ϕ is increasing, it follows that $\rho_1 = \phi(\rho_0) \leq \phi(\rho) = \rho$. Repeating the argument we have $\rho_2 \leq \rho$, $\rho_3 \leq \rho$ and so on. Taking limits we have $\rho_\infty \leq \rho$. However, ρ is the smallest solution, so we must have $\rho_\infty = \rho$. □

To see what this says, let us consider a concrete example.

Example 1.56 (Binary Branching). Suppose $p_2 = a$, $p_0 = 1 - a$, and the other $p_k = 0$. In this case $\phi(\theta) = a\theta^2 + 1 - a$, so $\phi(x) = x$ means

$$0 = ax^2 - x + 1 - a = (x-1)(ax - (1-a))$$

The roots are 1 and $(1-a)/a$. If $a \leq 1/2$, then the smallest root is 1, while if $a > 1/2$ the smallest root is $(1-a)/a$.

Noting that $a \leq 1/2$ corresponds to mean $\mu \leq 1$ in binary branching motivates the following guess:

II. *If $\mu > 1$, then there is positive probability of avoiding extinction.*

Proof. In view of Lemma 1.31, we only have to show there is a root < 1. We begin by discarding a trivial case. If $p_0 = 0$, then $\phi(0) = 0$, 0 is the smallest root, and there is no probability of dying out. If $p_0 > 0$, then $\phi(0) = p_0 > 0$. Differentiating the definition of ϕ, we have

$$\phi'(x) = \sum_{k=1}^{\infty} p_k \cdot kx^{k-1} \quad \text{so} \quad \phi'(1) = \sum_{k=1}^{\infty} kp_k = \mu$$

If $\mu > 1$, then the slope of ϕ at $x = 1$ is larger than 1, so if ϵ is small, then $\phi(1-\epsilon) < 1-\epsilon$. Combining this with $\phi(0) > 0$ we see there must be a solution of $\phi(x) = x$ between 0 and $1-\epsilon$. See the figure in the proof of (7.6). □

Turning to the borderline case:

III. *If $\mu = 1$ and we exclude the trivial case $p_1 = 1$, then extinction occurs with probability 1.*

Proof. By Lemma 1.31 we only have to show that there is no root < 1. To do this we note that if $p_1 < 1$, then for $y < 1$

$$\phi'(x) = \sum_{k=1}^{\infty} p_k \cdot kx^{k-1} < \sum_{k=1}^{\infty} p_k k = 1$$

so if $x < 1$ then $\phi(x) = \phi(1) - \int_x^1 \phi'(y)\,dy > 1 - (1-x) = x$. Thus $\phi(x) > x$ for all $x < 1$. □

Note that in binary branching with $a = 1/2$, $\phi(x) = (1+x^2)/2$, so if we try to solve $\phi(x) = x$ we get

$$0 = 1 - 2x + x^2 = (1-x)^2$$

i.e., a double root at $x = 1$. In general when $\mu = 1$, the graph of ϕ is tangent to the diagonal (x, x) at $x = 1$. This slows down the convergence of ρ_n to 1 so that it no longer occurs exponentially fast.

In more advanced treatments, it is shown that if the offspring distribution has mean 1 and variance $\sigma^2 > 0$, then

$$P_1(X_n > 0) \sim \frac{2}{n\sigma^2}$$

This is not easy even for the case of binary branching, so we refer to reader to Sect. 1.9 of Athreya and Ney (1972) for a proof. We mention the result here because it allows us to see that the expected time for the process to die out $\sum_n P_1(T_0 > n) = \infty$. If we modify the branching process, so that $p(0,1) = 1$ then in the modified process

If $\mu < 1$, 0 is positive recurrent
If $\mu = 1$, 0 is null recurrent
If $\mu > 1$, 0 is transient

1.12 Chapter Summary

A Markov chain with transition probability p is defined by the property that given the present state the rest of the past is irrelevant for predicting the future:

$$P(X_{n+1} = y | X_n = x, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = p(x, y)$$

The m step transition probability

$$p^m(x, y) = P(X_{n+m} = y | X_n = x)$$

is the mth power of the matrix p.

Recurrence and Transience

The first thing we need to determine about a Markov chain is which states are recurrent and which are transient. To do this we let $T_y = \min\{n \geq 1 : X_n = y\}$ and let

$$\rho_{xy} = P_x(T_y < \infty)$$

When $x \neq y$ this is the probability X_n ever visits y starting at x. When $x = y$ this is the probability X_n returns to y when it starts at y. We restrict to times $n \geq 1$ in the definition of T_y so that we can say: y is recurrent if $\rho_{yy} = 1$ and transient if $\rho_{yy} < 1$.

Transient states in a finite state space can all be identified using

Theorem 1.5. *If $\rho_{xy} > 0$, but $\rho_{yx} < 1$, then x is transient.*

Once the transient states are removed we can use

Theorem 1.7. *If C is a finite closed and irreducible set, then all states in C are recurrent.*

Here A is closed if $x \in A$ and $y \notin A$ implies $p(x, y) = 0$, and B is irreducible if $x, y \in B$ implies $\rho_{xy} > 0$.

The keys to the proof of Theorem 1.7 are: (i) If x is recurrent and $\rho_{xy} > 0$, then y is recurrent, and (ii) In a finite closed set there has to be at least one recurrent state. To prove these results, it was useful to know that if $N(y)$ is the number of visits to y at times $n \geq 1$ then

$$\sum_{n=1}^{\infty} p^n(x, y) = E_x N(y) = \frac{\rho_{xy}}{1 - \rho_{yy}}$$

so y is recurrent if and only if $E_y N(y) = \infty$.

Theorems 1.5 and 1.7 allow us to decompose the state space and simplify the study of Markov chains.

Theorem 1.8. *If the state space S is finite, then S can be written as a disjoint union $T \cup R_1 \cup \cdots \cup R_k$, where T is a set of transient states and the R_i, $1 \leq i \leq k$, are closed irreducible sets of recurrent states.*

Stationary Distributions

A stationary measure is a nonnegative solution of $\mu p = \mu$. A stationary distribution is a nonnegative solution of $\pi p = \pi$ normalized so that the entries sum to 1. The first question is: do these things exist?

Theorem 1.24. *Suppose p is irreducible and recurrent. Let $x \in S$ and let $T_x = \inf\{n \geq 1 : X_n = x\}$.*

$$\mu_x(y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n)$$

defines a stationary measure with $0 < \mu_x(y) < \infty$ for all y.

If the state space S is finite and irreducible, there is a unique stationary distribution. More generally if $E_x T_x < \infty$, i.e., x is positive recurrent, then $\mu_x(y)/E_x T_x$ is a stationary distribution. Since $\mu_x(x) = 1$ we see that

$$\pi(x) = \frac{1}{E_x T_x}$$

If there are k states, then the stationary distribution π can be computed by the following procedure. Form a matrix A by taking the first $k - 1$ columns of $p - I$ and adding a final column of 1's. The equations $\pi p = \pi$ and $\pi_1 + \cdots \pi_k = 1$ are equivalent to

$$\pi A = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix}$$

so we have

$$\pi = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix} A^{-1}$$

or π is the bottom row of A^{-1}.

In two situations, the stationary distribution is easy to compute. (i) If the chain is doubly stochastic, i.e., $\sum_x p(x,y) = 1$, and has k states, then the stationary distribution is uniform $\pi(x) = 1/k$. (ii) π is a stationary distribution if the detailed balance condition holds

$$\pi(x)p(x,y) = \pi(y)p(y,x)$$

Birth and death chains, defined by the condition that $p(x,y) = 0$ if $|x-y| > 1$ always have stationary distributions with this property. If the state space is $\ell, \ell+1, \dots r$ then π can be found by setting $\pi(\ell) = c$, solving for $\pi(x)$ for $\ell < x \le r$, and then choosing c to make the probabilities sum to 1.

Convergence Theorems

Transient states y have $p^n(x,y) \to 0$, so to investigate the convergence of $p^n(x,y)$ it is enough, by the decomposition theorem, to suppose the chain is irreducible and all states are recurrent. The period of a state is the greatest common divisor of $I_x = \{n \ge 1 : p^n(x,x) > 0\}$. If the period is 1, x is said to be aperiodic. A simple sufficient condition to be aperiodic is that $p(x,x) > 0$. To compute the period it is useful to note that if $\rho_{xy} > 0$ and $\rho_{yx} > 0$ then x and y have the same period. In particular all of the states in an irreducible set have the same period.

The three main results about the asymptotic behavior of Markov chains are

Theorem 1.19. *Suppose p is irreducible, aperiodic, and has a stationary distribution π. Then as $n \to \infty$, $p^n(x,y) \to \pi(y)$.*

Theorem 1.20. *Suppose p is irreducible and recurrent. If $N_n(y)$ be the number of visits to y up to time n, then*

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y}$$

Theorem 1.22. *Suppose p is irreducible, has stationary distribution π, and $\sum_x |f(x)|\pi(x) < \infty$ then*

$$\frac{1}{n}\sum_{m=1}^{n} f(X_m) \to \sum_x f(x)\pi(x)$$

Exit Distributions

If F is a subset of S let $V_F = \min\{n \geq 0 : X_n \in F\}$.

Theorem 1.28. *Consider a Markov chain with finite state space S. Let $A, B \subset S$, so that $C = S - \{a, b\}$ is finite, and $P_x(V_A \wedge V_B < \infty) > 0$ for all $x \in C$. If $h(a) = 1$ for $a \in A$, $h(b) = 0$ for $b \in B$, and*

$$h(x) = \sum_y p(x, y)h(y) \quad \textit{for } x \in C$$

then $h(x) = P_x(V_A < V_B)$.

Let $r(x, y)$ be the part of the matrix $p(x, y)$ with $x, y \in C$ and for $x \in C$ let $v(x) = \sum_{y \in A} p(x, y)$ which we think of as a column vector. Since $h(a) = 1$ for $a \in A$ and $h(b) = 0$ for $b \in B$, the equation for h can be written for $x \in C$ as

$$h(x) = v(x) + \sum_y r(x, y)h(y)$$

and the solution is

$$h = (I - r)^{-1}v.$$

Exit Times

Theorem 1.29. *Consider a Markov chain with finite state space S. Let $A \subset S$ so that $C = S - A$ is finite and $P_x(V_A < \infty) > 0$ for $x \in C$. If $g(a) = 0$ for all $a \in A$, and*

$$g(x) = 1 + \sum_y p(x, y)g(y) \quad \textit{for } x \in C$$

then $g(x) = E_x(V_A)$.

Since $g(x) = 0$ for $x \in A$ the equation for g can be written for $x \in C$ as

$$g(x) = 1 + \sum_y r(x, y)g(y)$$

so if we let $\mathbf{1}$ be a column vector consisting of all 1's then the last equation says $(I - r)g = \mathbf{1}$ and the solution is

$$g = (I - r)^{-1}\mathbf{1}.$$

$(I - r)^{-1}(x, y)$ is the expected number of visits to y starting from x. When we multiply by $\mathbf{1}$ we sum over all $y \in C$, so the result is the expected exit time.

1.13 Exercises

Understanding the Definitions

1.1. A fair coin is tossed repeatedly with results $Y_0, Y_1, Y_2, \ldots$ that are 0 or 1 with probability 1/2 each. For $n \geq 1$ let $X_n = Y_n + Y_{n-1}$ be the number of 1's in the $(n-1)$th and nth tosses. Is X_n a Markov chain?

1.2. Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.

1.3. We repeated roll two four sided dice with numbers 1, 2, 3, and 4 on them. Let Y_k be the sum on the kth roll, $S_n = Y_1 + \cdots + Y_n$ be the total of the first n rolls, and $X_n = S_n \pmod 6$. Find the transition probability for X_n.

1.4. The 1990 census showed that 36 % of the households in the District of Columbia were homeowners while the remainder were renters. During the next decade 6 % of the homeowners became renters and 12 % of the renters became homeowners. What percentage were homeowners in 2000? in 2010?

1.5. Consider a gambler's ruin chain with $N = 4$. That is, if $1 \leq i \leq 3, p(i, i+1) = 0.4$, and $p(i, i-1) = 0.6$, but the endpoints are absorbing states: $p(0, 0) = 1$ and $p(4, 4) = 1$ Compute $p^3(1, 4)$ and $p^3(1, 0)$.

1.6. A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel, then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4. (a) Find the transition matrix for the chain. (b) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.

1.7. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$. (a) Compute its transition probability. (b) Compute the two-step transition probability. (c) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday.

1.8. Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains. Give reasons for your answers.

(*a*)	**1**	**2**	**3**	**4**	**5**
1	.4	.3	.3	0	0
2	0	.5	0	.5	0
3	.5	0	.5	0	0
4	0	.5	0	.5	0
5	0	.3	0	.3	.4

(*b*)	**1**	**2**	**3**	**4**	**5**	**6**
1	.1	0	0	.4	.5	0
2	.1	.2	.2	0	.5	0
3	0	.1	.3	0	0	.6
4	.1	0	0	.9	0	0
5	0	0	0	.4	0	.6
6	0	0	0	0	.5	.5

(*c*)	**1**	**2**	**3**	**4**	**5**
1	0	0	0	0	1
2	0	.2	0	.8	0
3	.1	.2	.4	.3	0
4	0	.4	0	.6	0
5	.2	0	0	0	.8

(*d*)	**1**	**2**	**3**	**4**	**5**	**6**
1	.8	0	0	.2	0	0
2	0	.5	0	0	.5	0
3	0	0	.3	.4	.1	.2
4	.1	0	0	.9	0	0
5	0	.2	0	0	.8	0
6	0	.3	0	.3	0	.4

(*e*)	**1**	**2**	**3**	**4**	**5**
1	1	0	0	0	0
2	0	2/3	0	1/3	0
3	1/8	1/4	5/8	0	0
4	0	1/6	0	5/6	0
5	1/3	0	1/3	0	1/3

1.9. Find the stationary distributions for the Markov chains with transition matrices:

(*a*)	**1**	**2**	**3**
1	.5	.4	.1
2	.2	.5	.3
3	.1	.3	.6

(*b*)	**1**	**2**	**3**
1	.5	.4	.1
2	.3	.4	.3
3	.2	.2	.6

(*c*)	**1**	**2**	**3**
1	.6	.4	0
2	.2	.4	.2
3	0	.2	.8

1.10. Find the stationary distributions for the Markov chains on $\{1,2,3,4\}$ with transition matrices:

$$(a)\begin{pmatrix} .7 & 0 & .3 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .4 & 0 & .6 \end{pmatrix} \qquad (b)\begin{pmatrix} .7 & .3 & 0 & 0 \\ .2 & .5 & .3 & 0 \\ .0 & .3 & .6 & .1 \\ 0 & 0 & .2 & .8 \end{pmatrix} \qquad (c)\begin{pmatrix} .7 & 0 & .3 & 0 \\ .2 & .5 & .3 & 0 \\ .1 & .2 & .4 & .3 \\ 0 & .3 & 0 & .7 \end{pmatrix}$$

1.11. (a) Find the stationary distribution for the transition probability

	1	**2**	**3**
1	.2	.4	.4
2	.1	.6	.3
3	.2	.6	.2

(b) Does it satisfy the detailed balance condition (1.11)?

1.12. Find the stationary distributions for the chains in exercises (a) 1.2, (b) 1.3, and (c) 1.7.

1.13. Consider the Markov chain with transition matrix:

	1	**2**	**3**	**4**
1	0	0	0.1	0.9
2	0	0	0.6	0.4
3	0.8	0.2	0	0
4	0.4	0.6	0	0

(a) Compute p^2. (b) Find the stationary distributions of p and all of the stationary distributions of p^2. (c) Find the limit of $p^{2n}(x,x)$ as $n \to \infty$.

1.14. Do the following Markov chains converge to equilibrium?

(a)	**1**	**2**	**3**	**4**
1	0	0	1	0
2	0	0	.5	.5
3	.3	.7	0	0
4	1	0	0	0

(b)	**1**	**2**	**3**	**4**
1	0	1	.0	0
2	0	0	0	1
3	1	0	0	0
4	1/3	0	2/3	0

(c)	**1**	**2**	**3**	**4**	**5**	**6**
1	0	.5	.5	0	0	0
2	0	0	0	1	0	0
3	0	0	0	.4	0	.6
4	1	0	0	0	0	0
5	0	1	0	0	0	0
6	.2	0	0	0	.8	0

(d)	**1**	**2**	**3**	**4**	**5**	**6**
1	0	0	1	0	0	0
2	1	0	0	0	0	0
3	0	.5	0	0	.5	0
4	0	.5	0	0	.5	0.
5	0	0	0	0	0	1
6	0	0	0	1	0	0

1.15. Find $\lim_{n\to\infty} p^n(i,j)$ for the chains in parts (c), (d), and (e) of Problem 1.8.

Two State Markov Chains

1.16. Market research suggests that in a five year period 8 % of people with cable television will get rid of it, and 26 % of those without it will sign up for it. Compare the predictions of the Markov chain model with the following data on the fraction of people with cable TV: 56.4 % in 1990, 63.4 % in 1995, and 68.0 % in 2000. What is the long run fraction of people with cable TV?

1.17. A sociology professor postulates that in each decade 8 % of women in the work force leave it and 20 % of the women not in it begin to work. Compare the predictions of his model with the following data on the percentage of women working: 43.3 % in 1970, 51.5 % in 1980, 57.5 % in 1990, and 59.8 % in 2000. In the long run what fraction of women will be working?

1.18. A rapid transit system has just started operating. In the first month of operation, it was found that 25 % of commuters are using the system while 75 % are travelling by automobile. Suppose that each month 10 % of transit users go back to using their cars, while 30 % of automobile users switch to the transit system. (a) Compute the three step transition probability p^3. (b) What will be the fractions using rapid transit in the fourth month? (c) In the long run?

1.19. A regional health study indicates that from one year to the next, 75 % percent of smokers will continue to smoke while 25 % will quit. 8 % of those who stopped smoking will resume smoking while 92 % will not. If 70 % of the population were smokers in 1995, what fraction will be smokers in 1998? in 2005? in the long run?

1.20. Three of every four trucks on the road are followed by a car, while only one of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

1.21. In a test paper the questions are arranged so that 3/4's of the time a True answer is followed by a True, while 2/3's of the time a False answer is followed by a False. You are confronted with a 100 question test paper. Approximately what fraction of the answers will be True.

1.22. In unprofitable times corporations sometimes suspend dividend payments. Suppose that after a dividend has been paid the next one will be paid with probability 0.9, while after a dividend is suspended the next one will be suspended with probability 0.6. In the long run what is the fraction of dividends that will be paid?

1.23. Census results reveal that in the USA 80 % of the daughters of working women work and that 30 % of the daughters of nonworking women work. (a) Write the transition probability for this model. (b) In the long run what fraction of women will be working?

1.24. When a basketball player makes a shot then he tries a harder shot the next time and hits (H) with probability 0.4, misses (M) with probability 0.6. When he misses he is more conservative the next time and hits (H) with probability 0.7, misses (M) with probability 0.3. (a) Write the transition probability for the two state Markov chain with state space $\{H, M\}$. (b) Find the long-run fraction of time he hits a shot.

1.25. Folk wisdom holds that in Ithaca in the summer it rains 1/3 of the time, but a rainy day is followed by a second one with probability 1/2. Suppose that Ithaca weather is a Markov chain. What is its transition probability?

Chains with Three or More States

1.26. (a) Suppose brands A and B have consumer loyalties of .7 and .8, meaning that a customer who buys A one week will with probability .7 buy it again the next week, or try the other brand with .3. What is the limiting market share for each of these products? (b) Suppose now there is a third brand with loyalty .9, and that a consumer who changes brands picks one of the other two at random. What is the new limiting market share for these three products?

1.27. A midwestern university has three types of health plans: a health maintenance organization (*HMO*), a preferred provider organization (*PPO*), and a traditional fee for service plan (*FFS*). Experience dictates that people change plans according to the following transition matrix

	HMO	**PPO**	**FFS**
HMO	.85	.1	.05
PPO	.2	.7	.1
FFS	.1	.3	.6

In 2000, the percentages for the three plans were *HMO*:30 %, *PPO*:25 %, and *FFS*:45 %. (a) What will be the percentages for the three plans in 2001? (b) What is the long run fraction choosing each of the three plans?

1.28. Bob eats lunch at the campus food court every week day. He either eats Chinese food, Quesadila, or Salad. His transition matrix is

	C	**Q**	**S**
C	.15	.6	.25
Q	.4	.1	.5
S	.1	.3	.6

He had Chinese food on Monday. (a) What are the probabilities for his three meal choices on Friday (four days later). (b) What are the long run frequencies for his three choices?

1.29. The liberal town of Ithaca has a "free bikes for the people program." You can pick up bikes at the library (L), the coffee shop (C), or the cooperative grocery store (G). The director of the program has determined that bikes move around according to the following Markov chain

	L	**C**	**G**
L	.5	.2	.3
C	.4	.5	.1
G	.25	.25	.5

On Sunday there are an equal number of bikes at each place. (a) What fraction of the bikes are at the three locations on Tuesday? (b) on the next Sunday? (c) In the long run what fraction are at the three locations?

1.30. A plant species has red, pink, or white flowers according to the genotypes RR, RW, and WW, respectively. If each of these genotypes is crossed with a pink (*RW*) plant, then the offspring fractions are

	RR	**RW**	**WW**
RR	.5	.5	0
RW	.25	.5	.25
WW	0	.5	.5

What is the long run fraction of plants of the three types?

1.31. The weather in a certain town is classified as rainy, cloudy, or sunny and changes according to the following transition probability is

	R	**C**	**S**
R	1/2	1/4	1/4
C	1/4	1/2	1/4
S	1/2	1/2	0

In the long run what proportion of days in this town are rainy? cloudy? sunny?

1.32. A sociologist studying living patterns in a certain region determines that the pattern of movement between urban (U), suburban (S), and rural areas (R) is given by the following transition matrix.

	U	**S**	**R**
U	.86	.08	.06
S	.05	.88	.07
R	.03	.05	.92

In the long run what fraction of the population will live in the three areas?

1.33. In a large metropolitan area, commuters either drive alone (A), carpool (C), or take public transportation (T). A study showed that transportation changes according to the following matrix:

	A	C	T
A	.8	.15	.05
C	.05	.9	.05
S	.05	.1	.85

In the long run what fraction of commuters will use the three types of transportation?

1.34. (a) Three telephone companies A, B, and C compete for customers. Each year customers switch between companies according to the following transition probability

	A	B	C
A	.75	.05	.20
B	.15	.65	.20
C	.05	.1	.85

What is the limiting market share for each of these companies?

1.35. In a particular county voters declare themselves as members of the Republican, Democrat, or Green party. No voters change directly from the Republican to Green party or vice versa. Other transitions occur according to the following matrix:

	R	D	G
R	.85	.15	0
D	.05	.85	.10
G	0	.05	.95

In the long run what fraction of voters will belong to the three parties?

1.36. An auto insurance company classifies its customers into three categories: poor, satisfactory, and excellent. No one moves from poor to excellent or from excellent to poor in one year.

	P	S	E
P	.6	.4	0
S	.1	.6	.3
E	0	.2	.8

What is the limiting fraction of drivers in each of these categories?

1.37. A professor has two light bulbs in his garage. When both are burned out, they are replaced, and the next day starts with two working light bulbs. Suppose that when both are working, one of the two will go out with probability .02 (each has probability .01 and we ignore the possibility of losing two on the same day).

However, when only one is there, it will burn out with probability .05. (i) What is the long-run fraction of time that there is exactly one bulb working? (ii) What is the expected time between light bulb replacements?

1.38. An individual has three umbrellas, some at her office, and some at home. If she is leaving home in the morning (or leaving work at night) and it is raining, she will take an umbrella, if one is there. Otherwise, she gets wet. Assume that independent of the past, it rains on each trip with probability 0.2. To formulate a Markov chain, let X_n be the number of umbrellas at her current location. (a) Find the transition probability for this Markov chain. (b) Calculate the limiting fraction of time she gets wet.

1.39. Let X_n be the number of days since David last shaved, calculated at 7:30 a.m. when he is trying to decide if he wants to shave today. Suppose that X_n is a Markov chain with transition matrix

	1	**2**	**3**	**4**
1	1/2	1/2	0	0
2	2/3	0	1/3	0
3	3/4	0	0	1/4
4	1	0	0	0

In words, if he last shaved k days ago, he will not shave with probability 1/(k+1). However, when he has not shaved for 4 days his mother orders him to shave, and he does so with probability 1. (a) What is the long-run fraction of time David shaves? (b) Does the stationary distribution for this chain satisfy the detailed balance condition?

1.40. *Reflecting random walk on the line.* Consider the points $1, 2, 3, 4$ to be marked on a straight line. Let X_n be a Markov chain that moves to the right with probability $2/3$ and to the left with probability $1/3$, but subject this time to the rule that if X_n tries to go to the left from 1 or to the right from 4 it stays put. Find (a) the transition probability for the chain, and (b) the limiting amount of time the chain spends at each site.

1.41. At the end of a month, a large retail store classifies each of its customer's accounts according to current (0), 30–60 days overdue (1), 60–90 days overdue (2), more than 90 days (3). Their experience indicates that the accounts move from state to state according to a Markov chain with transition probability matrix:

	0	**1**	**2**	**3**
0	.9	.1	0	0
1	.8	0	.2	0
2	.5	0	0	.5
3	.1	0	0	.9

In the long run what fraction of the accounts are in each category?

1.42. At the beginning of each day, a piece of equipment is inspected to determine its working condition, which is classified as state 1 = new, 2, 3, or 4 = broken. Suppose that a broken machine requires three days to fix. To incorporate this into the Markov chain we add states 5 and 6 and write the following transition matrix:

	1	**2**	**3**	**4**	**5**	**6**
1	.95	.05	0	0	0	0
2	0	.9	.1	0	0	0
3	0	0	.875	.125	0	0
4	0	0	0	0	1	0
5	0	0	0	0	0	1
6	1	0	0	0	0	0

(a) Find the fraction of time that the machine is working. (b) Suppose now that we have the option of performing preventative maintenance when the machine is in state 3, and that this maintenance takes one day and returns the machine to state 1. This changes the transition probability to

	1	**2**	**3**
1	.95	.05	0
2	0	.9	.1
3	1	0	0

Find the fraction of time the machine is working under this new policy.

1.43. *Landscape dynamics.* To make a crude model of a forest we might introduce states 0 = grass, 1 = bushes, 2 = small trees, 3 = large trees, and write down a transition matrix like the following:

	0	**1**	**2**	**3**
0	$1/2$	$1/2$	0	0
1	$1/24$	$7/8$	$1/12$	0
2	$1/36$	0	$8/9$	$1/12$
3	$1/8$	0	0	$7/8$

The idea behind this matrix is that if left undisturbed a grassy area will see bushes grow, then small trees, which of course grow into large trees. However, disturbances such as tree falls or fires can reset the system to state 0. Find the limiting fraction of land in each of the states.

Exit Distributions and Times

1.44. The Markov chain associated with a manufacturing process may be described as follows: A part to be manufactured will begin the process by entering step 1. After step 1, 20 % of the parts must be reworked, i.e., returned to step 1, 10 % of the parts are thrown away, and 70 % proceed to step 2. After step 2, 5 % of the parts must be returned to the step 1, 10 % to step 2, 5 % are scrapped, and 80 % emerge to be sold for a profit. (a) Formulate a four-state Markov chain with states 1, 2, 3, and 4 where 3 = a part that was scrapped and 4 = a part that was sold for a profit. (b) Compute the probability a part is scrapped in the production process.

1.45. A bank classifies loans as paid in full (F), in good standing (G), in arrears (A), or as a bad debt (B). Loans move between the categories according to the following transition probability:

	F	**G**	**A**	**B**
F	1	0	0	0
G	.1	.8	.1	0
A	.1	.4	.4	.1
B	0	0	0	1

What fraction of loans in good standing are eventually paid in full? What is the answer for those in arrears?

1.46. A warehouse has a capacity to hold four items. If the warehouse is neither full nor empty, the number of items in the warehouse changes whenever a new item is produced or an item is sold. Suppose that (no matter when we look) the probability that the next event is "a new item is produced" is 2/3 and that the new event is a "sale" is 1/3. If there is currently one item in the warehouse, what is the probability that the warehouse will become full before it becomes empty.

1.47. The Duke football team can Pass, Run, throw an Interception, or Fumble. Suppose the sequence of outcomes is Markov chain with the following transition matrix.

	P	**R**	**I**	**F**
P	0.7	0.2	0.1	0
R	0.35	0.6	0	0.05
I	0	0	1	0
F	0	0	0	1

The first play is a pass. (a) What is the expected number of plays until a fumble or interception? (b) What is the probability the sequence of plays ends in an interception.

1.48. Six children (Dick, Helen, Joni, Mark, Sam, and Tony) play catch. If Dick has the ball, he is equally likely to throw it to Helen, Mark, Sam, and Tony. If Helen has the ball, she is equally likely to throw it to Dick, Joni, Sam, and Tony. If Sam has the ball, he is equally likely to throw it to Dick, Helen, Mark, and Tony. If either Joni or Tony gets the ball, they keep throwing it to each other. If Mark gets the ball, he runs away with it. (a) Find the transition probability and classify the states of the chain. (b) Suppose Dick has the ball at the beginning of the game. What is the probability Mark will end up with it?

1.49. To find the waiting time for *HHH* we let the state of our Markov chains be the number of consecutive heads we have at the moment. The transition probability is

	0	**1**	**2**	**3**
0	0.5	0.5	0	0
1	0.5	0	0.5	0
2	0.5	0	0	0.5
3	0	0	0	1

Find $E_0 T_3$.

(b) (10 points) Consider now the chain where the state gives the outcomes of the last three tosses. It has state space $\{HHH, HHT, \dots TTT\}$. Use the fact that $E_{HHH} T_{HHH} = 8$ to find the answer to part (a).

1.50. To find the waiting time for *HTH* we let the state of our Markov chains be the part of the pattern we have so far. The transition probability is

	0	**H**	**HT**	**HTH**
0	0.5	0.5	0	0
H	0	0.5	0.5	0
HT	0.5	0	0	0.5
HTH	0	0	0	1

(a) Find $E_0 T_{HTH}$. (b) use the reasoning for part (b) of the previous exercise to conclude $E_0 T_{HHT} = 8$ and $E_0 T_{HTT} = 8/$

1.51. *Sucker bet.* Consider the following gambling game. Player 1 picks a three coin pattern (for example, *HTH*) and player 2 picks another (say *THH*). A coin is flipped repeatedly and outcomes are recorded until one of the two patterns appears. Somewhat surprisingly player 2 has a considerable advantage in this game. No matter what player 1 picks, player 2 can win with probability $\geq 2/3$. Suppose without loss of generality that player 1 picks a pattern that begins with H:

case	Player 1	Player 2	Prob. 2 wins
1	HHH	THH	7/8
2	HHT	THH	3/4
3	HTH	HHT	2/3
4	HTT	HHT	2/3

Verify the results in the table. You can do this by solving six equations in six unknowns but this is not the easiest way.

1.52. At the New York State Fair in Syracuse, Larry encounters a carnival game where for one dollar he may buy a single coupon allowing him to play a guessing game. On each play, Larry has an even chance of winning or losing a coupon. When he runs out of coupons he loses the game. However, if he can collect three coupons, he wins a surprise. (a) What is the probability Larry will win the surprise? (b) What is the expected number of plays he needs to win or lose the game.

1.53. The Megasoft company gives each of its employees the title of programmer (P) or project manager (M). In any given year 70 % of programmers remain in that position 20 % are promoted to project manager and 10 % are fired (state X). 95 % of project managers remain in that position while 5 % are fired. How long on the average does a programmer work before they are fired?

1.54. At a nationwide travel agency, newly hired employees are classified as beginners (B). Every six months the performance of each agent is reviewed. Past records indicate that transitions through the ranks to intermediate (I) and qualified (Q) are according to the following Markov chain, where F indicates workers that were fired:

	B	**I**	**Q**	**F**
B	.45	.4	0	.15
I	0	.6	.3	.1
Q	0	0	1	0
F	0	0	0	1

(a) What fraction eventually become qualified? (b) What is the expected time until a beginner is fired or becomes qualified?

1.55. At a manufacturing plant, employees are classified as trainee (R), technician (T), or supervisor (S). Writing Q for an employee who quits we model their progress through the ranks as a Markov chain with transition probability

	R	**T**	**S**	**Q**
R	.2	.6	0	.2
T	0	.55	.15	.3
S	0	0	1	0
Q	0	0	0	1

(a) What fraction of recruits eventually make supervisor? (b) What is the expected time until a trainee audits or becomes supervisor?

1.56. Customers shift between variable rate loans (V), thirty year fixed-rate loans (30), fifteen year fixed-rate loans (15), or enter the states paid in full (P), or foreclosed according to the following transition matrix:

	V	**30**	**15**	**P**	**F**
V	.55	.35	0	.05	.05
30	.15	.54	.25	.05	.01
15	.20	0	.75	.04	.01
P	0	0	0	1	0
F	0	0	0	0	1

(a) For each of the three loan types find (a) the expected time until paid or foreclosed. (b) the probability the loan is paid.

1.57. 3. Two barbers and two chairs. Consider the following chain

	0	**1**	**2**	**3**	**4**
0	0	1	0	0	0
1	0.6	0	0.4	0	0
2	0	0.75	0	0.25	0
3	0	0	0.75	0	0.25
4	0	0	0	0.75	0.25

(a) Find the stationary distribution. (b) Compute $P_x(V_0 < V_4)$ for $x = 1, 2, 3$. (c) Let $\tau = \min\{V_0, V_4\}$. Find $E_x\tau$ for $x = 1, 2, 3$.

More Theoretical Exercises

1.58. Consider a general chain with state space $S = \{1, 2\}$ and write the transition probability as

	1	**2**
1	$1-a$	a
2	b	$1-b$

Use the Markov property to show that

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)\left\{P(X_n = 1) - \frac{b}{a+b}\right\}$$

and then conclude

$$P(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n \left\{ P(X_0 = 1) - \frac{b}{a+b} \right\}$$

This shows that if $0 < a + b < 2$, then $P(X_n = 1)$ converges exponentially fast to its limiting value $b/(a+b)$.

1.59. *Bernoulli–Laplace model of diffusion.* Consider two urns each of which contains m balls; b of these $2m$ balls are black, and the remaining $2m - b$ are white. We say that the system is in state i if the first urn contains i black balls and $m - i$ white balls while the second contains $b - i$ black balls and $m - b + i$ white balls. Each trial consists of choosing a ball at random from each urn and exchanging the two. Let X_n be the state of the system after n exchanges have been made. X_n is a Markov chain. (a) Compute its transition probability. (b) Verify that the stationary distribution is given by

$$\pi(i) = \binom{b}{i}\binom{2m-b}{m-i} \Big/ \binom{2m}{m}$$

(c) Can you give a simple intuitive explanation why the formula in (b) gives the right answer?

1.60. *Library chain.* On each request the ith of n possible books is the one chosen with probability p_i. To make it quicker to find the book the next time, the librarian moves the book to the left end of the shelf. Define the state at any time to be the sequence of books we see as we examine the shelf from left to right. Since all the books are distinct this list is a permutation of the set $\{1, 2, \ldots n\}$, i.e., each number is listed exactly once. Show that

$$\pi(i_1, \ldots, i_n) = p_{i_1} \cdot \frac{p_{i_2}}{1 - p_{i_1}} \cdot \frac{p_{i_3}}{1 - p_{i_1} - p_{i_2}} \cdots \frac{p_{i_n}}{1 - p_{i_1} - \cdots p_{i_{n-1}}}$$

is a stationary distribution.

1.61. *Random walk on a clock.* Consider the numbers $1, 2, \ldots 12$ written around a ring as they usually are on a clock. Consider a Markov chain that at any point jumps with equal probability to the two adjacent numbers. (a) What is the expected number of steps that X_n will take to return to its starting position? (b) What is the probability X_n will visit all the other states before returning to its starting position?

1.62. *King's random walk.* This and the next example continue Example 1.30. A king can move one squares horizontally, vertically, or diagonally. Let X_n be the sequence of squares that results if we pick one of king's legal moves at random. Find (a) the stationary distribution and (b) the expected number of moves to return to corner (1,1) when we start there.

1.63. *Queen's random walk.* A queen can move any number of squares horizontally, vertically, or diagonally. Let X_n be the sequence of squares that results if we pick one of queen's legal moves at random. Find (a) the stationary distribution and (b) the expected number of moves to return to corner (1,1) when we start there.

1.64. *Wright–Fisher model.* Consider the chain described in Example 1.7.

$$p(x, y) = \binom{N}{y} (\rho_x)^y (1 - \rho_x)^{N-y}$$

where $\rho_x = (1 - u)x/N + v(N - x)/N$. (a) Show that if $u, v > 0$, then $\lim_{n\to\infty} p^n(x, y) = \pi(y)$, where π is the unique stationary distribution. There is no known formula for $\pi(y)$, but you can (b) compute the mean $\nu = \sum_y y\pi(y) = \lim_{n\to\infty} E_x X_n$.

1.65. *Ehrenfest chain.* Consider the Ehrenfest chain, Example 1.2, with transition probability $p(i, i + 1) = (N - i)/N$, and $p(i, i - 1) = i/N$ for $0 \leq i \leq N$. Let $\mu_n = E_x X_n$. (a) Show that $\mu_{n+1} = 1 + (1 - 2/N)\mu_n$. (b) Use this and induction to conclude that

$$\mu_n = \frac{N}{2} + \left(1 - \frac{2}{N}\right)^n (x - N/2)$$

From this we see that the mean μ_n converges exponentially rapidly to the equilibrium value of $N/2$ with the error at time n being $(1 - 2/N)^n(x - N/2)$.

1.66. *Brother–sister mating.* In this genetics scheme two individuals (one male and one female) are retained from each generation and are mated to give the next. If the individuals involved are diploid and we are interested in a trait with two alleles, A and a, then each individual has three possible states AA, Aa, aa or more succinctly 2, 1, 0. If we keep track of the sexes of the two individuals the chain has nine states, but if we ignore the sex there are just six: 22, 21, 20, 11, 10, and 00. (a) Assuming that reproduction corresponds to picking one letter at random from each parent, compute the transition probability. (b) 22 and 00 are absorbing states for the chain. Show that the probability of absorption in 22 is equal to the fraction of A's in the state. (c) Let $T = \min\{n \geq 0 : X_n = 22 \text{ or } 00\}$ be the absorption time. Find $E_x T$ for all states x.

1.67. Roll a fair die repeatedly and let $Y_1, Y_2, \ldots$ be the resulting numbers. Let $X_n = |\{Y_1, Y_2, \ldots, Y_n\}|$ be the number of values we have seen in the first n rolls for $n \geq 1$ and set $X_0 = 0$. X_n is a Markov chain. (a) Find its transition probability. (b) Let $T = \min\{n : X_n = 6\}$ be the number of trials we need to see all 6 numbers at least once. Find ET.

1.68. *Coupon collector's problem.* We are interested now in the time it takes to collect a set of N baseball cards. Let T_k be the number of cards we have to buy before we have k that are distinct. Clearly, $T_1 = 1$. A little more thought reveals that if each time we get a card chosen at random from all N possibilities, then for $k \geq 1$,

$T_{k+1} - T_k$ has a geometric distribution with success probability $(N-k)/N$. Use this to show that the mean time to collect a set of N baseball cards is $\approx N \log N$, while the variance is $\approx N^2 \sum_{k=1}^{\infty} 1/k^2$.

1.69. *Algorithmic efficiency.* The simplex method minimizes linear functions by moving between extreme points of a polyhedral region so that each transition decreases the objective function. Suppose there are n extreme points and they are numbered in increasing order of their values. Consider the Markov chain in which $p(1,1) = 1$ and $p(i,j) = 1/i-1$ for $j < i$. In words, when we leave j we are equally likely to go to any of the extreme points with better value. (a) Use (1.27) to show that for $i > 1$

$$E_i T_1 = 1 + 1/2 + \cdots + 1/(i-1)$$

(b) Let $I_j = 1$ if the chain visits j on the way from n to 1. Show that for $j < n$

$$P(I_j = 1 | I_{j+1}, \ldots I_n) = 1/j$$

to get another proof of the result and conclude that $I_1, \ldots I_{n-1}$ are independent.

Infinite State Space

1.70. *General birth and death chains.* The state space is $\{0, 1, 2, \ldots\}$ and the transition probability has

$$\begin{aligned} p(x, x+1) &= p_x \\ p(x, x-1) &= q_x \qquad \text{for } x > 0 \\ p(x, x) &= r_x \qquad \text{for } x \geq 0 \end{aligned}$$

while the other $p(x,y) = 0$. Let $V_y = \min\{n \geq 0 : X_n = y\}$ be the time of the first visit to y and let $h_N(x) = P_x(V_N < V_0)$. By considering what happens on the first step, we can write

$$h_N(x) = p_x h_N(x+1) + r_x h_N(x) + q_x h_N(x-1)$$

Set $h_N(1) = c_N$ and solve this equation to conclude that 0 is recurrent if and only if $\sum_{y=1}^{\infty} \prod_{x=1}^{y-1} q_x/p_x = \infty$ where by convention $\prod_{x=1}^{0} = 1$.

1.71. To see what the conditions in the last problem say we will now consider some concrete examples. Let $p_x = 1/2$, $q_x = e^{-cx^{-\alpha}}/2$, $r_x = 1/2 - q_x$ for $x \geq 1$ and $p_0 = 1$. For large x, $q_x \approx (1 - cx^{-\alpha})/2$, but the exponential formulation keeps the probabilities nonnegative and makes the problem easier to solve. Show that the chain is recurrent if $\alpha > 1$ or if $\alpha = 1$ and $c \leq 1$ but is transient otherwise.

1.72. Consider the Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition probability

$$p(m, m+1) = \frac{1}{2}\left(1 - \frac{1}{m+2}\right) \qquad \text{for } m \geq 0$$
$$p(m, m-1) = \frac{1}{2}\left(1 + \frac{1}{m+2}\right) \qquad \text{for } m \geq 1$$

and $p(0, 0) = 1 - p(0, 1) = 3/4$. Find the stationary distribution π.

1.73. Consider the Markov chain with state space $\{1, 2, \ldots\}$ and transition probability

$$p(m, m+1) = m/(2m+2) \qquad \text{for } m \geq 1$$
$$p(m, m-1) = 1/2 \qquad \text{for } m \geq 2$$
$$p(m, m) = 1/(2m+2) \qquad \text{for } m \geq 2$$

and $p(1, 1) = 1 - p(1, 2) = 3/4$. Show that there is no stationary distribution.

1.74. Consider the aging chain on $\{0, 1, 2, \ldots\}$ in which for any $n \geq 0$ the individual gets one day older from n to $n+1$ with probability p_n but dies and returns to age 0 with probability $1-p_n$. Find conditions that guarantee that (a) 0 is recurrent, (b) positive recurrent. (c) Find the stationary distribution.

1.75. The opposite of the aging chain is the renewal chain with state space $\{0, 1, 2, \ldots\}$ in which $p(i, i-1) = 1$ when $i > 0$. The only nontrivial part of the transition probability is $p(0, i) = p_i$. Show that this chain is always recurrent but is positive recurrent if and only if $\sum_n n p_n < \infty$.

1.76. Consider a branching process as defined in 1.55, in which each family has exactly three children, but invert Galton and Watson's original motivation and ignore male children. In this model a mother will have an average of 1.5 daughters. Compute the probability that a given woman's descendents will die out.

1.77. Consider a branching process as defined in 1.55, in which each family has a number of children that follows a shifted geometric distribution: $p_k = p(1-p)^k$ for $k \geq 0$, which counts the number of failures before the first success when success has probability p. Compute the probability that starting from one individual the chain will be absorbed at 0.