1. Let $r_{T_1, T_2}$ denote the forward rate from (year) $T_1$ to $T_2$. Today is time 0 and today’s zero rates are as follows: the 3-month rate is 3%, the 5-month rate is 3.4%, the 8-month rate is 3.9%, the 12-month rate is 3.4%, the 14-month rate is 3.5%. Compute (to one hundredth of one-percent precision) the following rates:

$$ r_{\frac{5}{12}, \frac{14}{12}} $$ $$ r_{0, \frac{1}{12}} $$ $$ r_{\frac{5}{12}, \frac{9}{12}} $$

If not enough information is available for any of these, indicate what additional information you would need to know to compute the rate. Show your work.

**Solution:** $r_{\frac{5}{12}, \frac{14}{12}}$ satisfies

$$ e^{(r_{0, \frac{1}{12}})\times\frac{14}{12}} = e^{(r_{0, \frac{5}{12}})\times\frac{5}{12}} e^{(r_{\frac{5}{12}, \frac{14}{12}})\times\frac{14-5}{12}} $$

which implies $r_{\frac{5}{12}, \frac{14}{12}} = 3.55\%$.

$r_{0, \frac{1}{12}} = 3.4\%$ is the given 12-month rate.

$r_{\frac{5}{12}, \frac{9}{12}}$ requires knowledge of the 9-month rate.

2. Eighteen months ago, a bank had entered a swap, (original) lifetime of 5 years, agreeing to pay coupons on a fixed bond based on a fixed 5% per annum (annual compounding) and receive coupons on a floating bond based on the one-year LIBOR (annual compounding). Coupons to be exchanged every year. Both bonds have principal (face value) of 1 million dollars. Six months ago, the one-year LIBOR was 5.3% with continuous compounding. Today, all rates are 4% with continuous compounding. Compute the value of the swap to the bank.

**Solution:** The swap expires in 3.5 years, with the next coupons exchanged in 6 months. The value of the fixed coupon bond is

$$ B_{fix} = 50000e^{-0.04\times0.5} + 50000e^{-0.04\times1.5} + 50000e^{-0.04\times2.5} + 1050000e^{-0.04\times3.5} = 1054160.$$
To compute the value of the float coupon bond we must first figure out the one-year LIBOR rate six month ago, in terms of annual compounding.

\[(1 + r_{an}) = e^{r_{cts}} = e^{0.053} \rightarrow r_{an} = 0.0544.\]

The value of the float coupon bond is thus

\[B_{fl} = (1000000 + 54400)e^{-0.04 \times 0.5} = 1033550.\]

The value of the swap is

\[V_{swap} = B_{fl} - B_{fix} = -20610.\]

3. The up-front cost of storing potatoes for 6 months is $500 per 10,000 pounds. The spot price of potatoes is $1.10 per pound. The forward price of delivering 10,000 pounds in 6 months is $20,000. Today, you can borrow money (for any length of time) at 5%, and invest money (for any length of time) at 4%. Describe an arbitrage opportunity involving 30,000 pounds of potatoes. What is (in 6 months) the the risk-free profit associated to this opportunity?

**Solution:**

It seems like the forward price is too high relative to the spot price. But let us confirm this with an arbitrage opportunity.

At time 0, we take a short position promising to deliver 30000 pounds of potatoes in 6 months for $60000. We must buy the potatoes and store them, so we borrow

\[3 \times 500 + 30000 \times 1.1 = 34500\]

at 5% for 6 months. With this cash, we buy the potatoes for $33000 and store them paying the up-front cost of $1500.

In 6 months, we deliver the potatoes for $60000 and repay our loan of \(34500e^{0.05 \times 0.5} = 35373\) for a risk-free profit of $24627.

Note that there is not a well-defined answer for the present value of this profit, since it is unclear if we should discount at 4% or 5% (or something else). The grading reflected this.

4. Consider Bond A which is an 8%-coupon bearing bond maturing in 15 months, with principal $100. Coupons are paid annually. Consider Bond B which is a 0%-coupon bond with principal $100 maturing in one year. Consider Bond C whose current price is $95 and duration is 9 months. Suppose the zero rates are the following: 3 month is 5.0%, 6 month is 5.1%, 9 month is 5.2%, 12 month is 5.3%, 15 month is 5.4%.
(a) What are the prices and durations of Bonds A, B and C?

**Solution:**

Let $P_X$ and $D_X$ denote the price and duration of bond/portfolio $X$.

\[
\begin{align*}
P_A &= 8e^{-0.05 \times 0.25} + 108e^{-0.54 \times 1.25} = 108.85 \\
D_A &= \frac{0.25 \times 8e^{-0.05 \times 0.25} + 1.25 \times 108e^{-0.54 \times 1.25}}{108.85} = 1.17 \text{ years.}
\end{align*}
\]

$P_B = 100e^{-0.053} = 94.83$. $D_B = 1$ since there are no coupons. $P_C = 95$, $D_C = 0.75$ are given.

(b) Your portfolio, which we call $E$, is made up of the following: a long position in 5 Bond A’s; a short position in 2 Bond B’s; and a long position in 4 Bond C’s. Compute the value (price) and duration of Portfolio $E$.

**Solution:**

\[
\begin{align*}
P_E &= 5P_A - 2P_B + 4P_C = 734.51 \\
D_E &= \frac{5P_AD_A - 2P_BD_B + 4P_CD_C}{P_E} = 1.002 \text{ years.}
\end{align*}
\]

(c) Suppose the Federal reserve decreases all rates by 25 bips. Estimate the new value of your portfolio.

**Solution:**

Let $\Delta r = -0.0025$ be the rate change.

\[
P_E^{\text{new}} \approx P_E - D_E P_E \Delta r = 736.42
\]

5. The spot price of Google stock is $300, today March 12, 2009. Google will pay each shareholder a $10/share dividend in four months and a $10/share dividend in eight months. The 1-month, 2-month, ..., 6-month zero rates are all 3%. The 7-month, 8-month, ..., 12-month zero rates are all 5%.

(a) Compute the forward price of Google stock with delivery date March 12, 2010.

**Solution:**

\[
\begin{align*}
F_{\text{March}12} &= S_{\text{March}12} e^{(\text{rate} \times \text{time until delivery})} \\
&= (300 - 10e^{-0.03 \times 4/12} - 10e^{-0.05 \times 8/12}) e^{0.05 \times 1} = 294.81.
\end{align*}
\]
(b) Suppose 6 months later, on September 12, 2009, the forward price of Google stock with delivery date March 12, 2010 is now $295. Assume all rates are 4%. What does that imply the spot price of Google should be, assuming no-arbitrage?

Solution:

\[ F_{Sept12} = S_{Sept12}^{eff} e^{(rate \times time \ until \ delivery)} \]

so

\[ 295 = (S_{Sept12} - 10e^{-0.04 \times 2/12})e^{0.04 \times 0.5/12} \]

so \( S_{Sept12} = 299.09 \).

(c) Suppose, on March 12, 2009, you had taken a short position on a Google forward with delivery March 12, 2010. What would the value of your position be on September 12, 2009?

Solution: The value of the position is what you will get for the share on 3/12/10, less what anyone else will get for the Google share who enters a forward contract today (9/12/09), discounted by 6 months from 3/12/10 back to 9/12/09. This equals

\[ (F_{March12} - F_{Sept12})e^{-rate \times time \ until \ delivery} = (294.81 - 295)e^{-0.04 \times 0.5} = 0.186. \]