ON THE GENUS OF GENERALIZED LAGUERRE POLYNOMIALS

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1. INTRODUCTION

The generalized Laguerre polynomials

\[ L_n^{(t)}(x) = \sum_{j=0}^{n} (-x)^j \binom{n}{j} \prod_{k=j+1}^{n} (t + k) \]

belong to one of the three family of orthogonal polynomials, the other two being Jacobi and Legendre. In addition to their important roles in mathematical analysis, these polynomials also feature prominently in algebra and number theory. Schur ([7], [8]) pioneered the study of Galois properties of specializations of these orthogonal polynomials, and Feit [1] used them to solve the inverse Galois problem over \( \mathbb{Q} \) for certain double covers of the alternating group \( A_n \); see ([3], [4]) for other related results. Recently Hajir and Wong [5] proved that for \( n \geq 5 \) and for any number field \( K \), the Galois group of \( L_n^{(\alpha)}(x) \) over \( K \) is \( S_n \) for all but finitely many \( \alpha \in K \). A key ingredient of the proofs is to compute the genus of the function fields in the splitting field of \( L_n^{(t)}(x) \) over the function field \( \mathbb{Q}(t) \). As a by-product of this argument, we showed that for \( n \geq 5 \), \( L_n^{(t)}(x) \) defines an absolutely irreducible plane curve \( \mathcal{L}_n \) of geometric genus \( > 1 \). In light of the importance of \( L_n^{(t)}(x) \) in algebra and other areas of mathematics, in this paper we determine the exact genus of these curves.

**Theorem.** For \( n \geq 1 \), the equation \( L_n^{(t)}(x) = 0 \) defines an absolutely irreducible plane curve of geometric genus \( \left\lfloor \frac{(n-1)/2}{(n-2)/2} \right\rfloor = \left\lfloor (-1 + n/2)^2 \right\rfloor \).

The basic idea is to apply the Riemann-Hurwitz formula to the projection-to-\( t \) map. However, since \( \mathcal{L}_n \) need not be smooth, we first need to resolve its singularity. To do that, we exploit recursions satisfied by \( L_n^{(t)}(x) \) in order to analyze its affine singular locus, and then compute the Newton polygon of \( L_n^{(t)}(x) \) at infinity to determine its desingularization there. Our technique here should be applicable to other families of orthogonal polynomials that satisfy recursions similar to those for \( L_n^{(t)}(x) \).

2. FINITE POINTS OF GENERALIZED LAGUERRE POLYNOMIALS

Fix \( n \geq 2 \). Following Schur [8, p. 54], we homogenize \( L_n^{(t)}(x) \) by setting

\[ F_n(x, \nu, \mu) := (-1)^n n! \mu^n L_n^{(\nu/\mu)}(x/\mu) \]

\[ = x^n - k_1 x^{n-1} + \frac{k_1 k_2}{2} x^{n-2} + \cdots + (-1)^n \frac{k_1 \cdots k_n}{1 \cdot 2 \cdots n} \tag{1} \]

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where \( k_j = j(n + j\mu) \). Denote by \( \mathcal{L}_n \) the projective plane curve \( F_n(x, \nu, \mu) = 0 \). To simplify the notation, we write \( \partial_x F_j \) for \( \partial F_j/\partial x \). Then we have the relations [8, p. 54]

\[
(2) \quad x\partial_x F_n = nF_n + k_n F_{n-1}, \quad (n \geq 1, F_0 := 1);
\]

\[
(3) \quad F_n = (x - \nu - (2n - 1)\mu)F_{n-1} - \mu k_{n-1} F_{n-2}, \quad (n \geq 2).
\]

Set \( \mu = 0 \) and (1) becomes

\[
x^n - nx^{n-1}\nu + \frac{n(n-1)}{2} x^{n-2}\nu^2 \mp \cdots + (-1)^n \nu^n = (x - \nu)^n.
\]

Thus \( \mathcal{L}_n \) has exactly one point along the line at infinity, namely \( [x : \nu : \mu] = [1 : 1 : 0] \).

We now show that \( \mathcal{L}_n \) is absolutely irreducible, and we will determine its singular locus. This is a geometric problem, so for the rest of this paper we will work over \( \mathbb{C} \). Denote by \( \iota_n : \mathcal{L}_n \to \mathbb{P}^1_{\mathbb{C}} \) the morphism sending \( [x : \nu : \mu] \) to \([\nu : \mu]; \) in terms of the affine coordinates \((x, t)\) this is the projection-to-\( t \) map. The next two results are taken from [5]; for completeness we include their proofs.

**Lemma 1.** Suppose for some integer \( j \in [0, n] \) and some point \( z = [x(z) : \nu(z) : \mu(z)] \in \mathbb{P}^1_{\mathbb{C}} \) with \( x(z)\mu(z) \neq 0 \), we have

\[
(4) \quad F_{n-j}|_z = \partial_x F_{n-j}|_z = 0 \quad \text{and} \quad k_{n-j} \neq 0.
\]

Then \( F_{n-j-1}|_z = 0 \) and \( k_{n-j-1} \neq 0 \). Moreover, if \( j \leq n-2 \), then \( \partial_x F_{n-j-1}|_z = 0 \).

**Proof.** Since \( \mu(z) \neq 0 \), without loss of generality we can set \( \mu(z) = 1 \).

Suppose \( n \geq j + 1 \); then substitute into (2) the first two relations in (4), we get

\[
0 = k_{n-j} F_{n-j-1}|_z, \quad \text{whence}
\]

\[
(5) \quad F_{n-j-1}|_z = 0.
\]

Next, suppose \( k_{n-j-1} = 0 \). When we use the expansion (1) to evaluate (5), we see that \( x(z) = 0 \), a contradiction. Finally, suppose \( n \geq j + 2 \). Substituting (5) along with the first relation in (4) into (3), we get

\[
0 = -\mu(z) k_{n-j-1} F_{n-j-2}|_z.
\]

Substitute this and (5) back into (2) and we get \( x\partial_x F_{n-j-1}|_z = 0 \). As \( x(z) \neq 0 \), that means \( \partial_x F_{n-j-1}|_z = 0 \). This completes the proof of the Lemma.

**Lemma 2.** For \( n \geq 3 \) the curve \( \mathcal{L}_n \) has no finite singular point.

**Proof.** Using the relations (2) and (3), Schur [8, p. 54] showed that \( F_n \), viewed as a polynomial in \( x \), has discriminant

\[
(6) \quad \mu^{-\frac{n(n-1)}{2}} n! k_3 k_5^2 \cdots k_{n-1}^n.
\]

We are interested in the finite points on \( \mathcal{L}_n \), so for the rest of the proof we can set \( \mu = 1 \). Clearly it suffices to consider only those points on \( \mathcal{L}_n \) lying above the branched locus of \( \iota_n \).

Suppose \((x_0, \nu_0)\) is a finite singular point. By (6) we have \( \nu_0 \in \{-2, \ldots, -n\} \), and

\[
(7) \quad F_n|_z = \partial_x F_n|_z = \partial_\nu F_n|_z = 0.
\]
We claim that \( x_0 \neq 0 \). Suppose otherwise; set \( \partial_\nu F_n = 0 \) and then substitute \( x = 0 \) (recall that \( \mu = 1 \)), we get
\[
0 = (-1)^n \frac{\partial}{\partial \nu} \prod_{k=2}^{n} (\nu + k) = (-1)^n \sum_{m=2}^{n} \prod_{k=2, k \neq m}^{n} (\nu + k).
\]
Set \( \nu = \nu_0 \) and this becomes
\[
\prod_{k=2, k \neq \nu_0}^{n} (\nu_0 + k) = 0,
\]
a contradiction. Thus \( x_0 \neq 0 \). Also, if \( k_n = 0 \) then from (1) we get \( x_0 = 0 \), a contradiction. Thus \( k_n \neq 0 \). That means the hypothesis of Lemma 1 are satisfied for \( j = 0 \). Apply the previous Lemma \( (n-1) \) times and we see that \( F_0|_{(x_0, \nu_0)} = 0 \). This is a contradiction since \( F_0 = 1 \) by definition. Thus \( L_n \) has no finite singular point. \( \Box \)

**Lemma 3.** \( L_n \) is absolutely irreducible, so it makes sense to speak of its geometric genus.

**Proof.** Suppose otherwise; write \( C_1, \ldots, C_r \) for its reduced, \( \mathbb{C} \)-irreducible components. No two \( C_i \) are the same; otherwise \( L_n \) would have infinitely many singular points, contradicting Lemma 2 and the fact that \( L_n \) has only one point along the line at infinity. Every \( C_i \) is a projective plane curve, so any two \( C_i \) intersect non-trivially over \( \mathbb{C} \), by Bezout’s theorem, whence by Lemma 2, any two \( C_i \) intersect precisely at \([1 : 1 : 0]\). But then every \( C_i \) has a \( \mathbb{Q} \)-rational point, and hence every \( C_i \) is defined over \( \mathbb{Q} \). Thus \( L_n \) is \( \mathbb{Q} \)-reducible. This contradicts Schur’s result [7] that \( L_n(0)(x) \) is \( \mathbb{Q} \)-irreducible. \( \Box \)

### 3. Singularity at infinity

We now analyze the singularity locus of \( L_n \). By Lemma 2 it suffices to focus on the unique point at infinity, \([1 : 1 : 0]\). First, we move this point to the origin on an affine coordinate patch by dehomogenizing \( F_n \) via
\[
G_n(w, \mu) = F_n(1, w + 1, \mu).
\]

**Lemma 4.** For \( n \geq 2 \), the Newton polygon of \( G_n(w, \mu) \) is as follow (note that we only plot the minimal number of vertices needed to determine the polygon):

- **Case: \( n \) even**

  \[
  \begin{align*}
  & (0, n) \\
  & (\frac{n}{2}, 0) \\
  & (0, n)
  \end{align*}
  \]

  \( \mu \)

- **Case: \( n = 2m - 1 \)**

  \[
  \begin{align*}
  & (0, n) \\
  & (m - 1, 1) \\
  & (m, 0)
  \end{align*}
  \]

  \( \mu \)
Thus the lowest power of $\mu$ is non-zero. This coefficient is not contained in the terms
remaining in the recursion (3), which now says that every monomial $\mu^a w^b$ in $G_{n-1}$ satisfies $\alpha + 2\beta \geq n \geq \alpha + \beta$.

(b) $NT(G_n)$ contains the two (resp. three) vertices depicted in the diagram above for $n$ even (resp. $n$ odd).

Using (1) we find that
\begin{align*}
G_2(w, \mu) &= -\mu + w^2 + 3\mu w + 2\mu^2, \\
G_3(w, \mu) &= \mu(3w + 7\mu) + (-11\mu^2 w - 6\mu^3 - w^3 - 6\mu w^2).
\end{align*}

This verifies both claims for $2 \leq n \leq 4$. To handle the general case we make use of the recursion (3), which now says that
\begin{equation}
G_n = -wG_{n-1} - (2n - 1)\mu G_{n-1} - (n - 1)^2 \mu^2 G_{n-2} - (n - 1)w\mu G_{n-2} - (n - 1)\mu G_{n-2}.
\end{equation}

We begin with Claim (a). The condition $n \geq \alpha + \beta$ trivially holds, since any monomial $\mu^a w^b$ in $G_n$ must have degree $\leq n$. As for the other condition in (8), note that by induction, every monomial $\mu^a w^b$ in $G_{n-1}$ satisfies $a + 2\beta \geq n - 1$, so every monomial $w^a \mu^b$ in $w G_{n-1}$ satisfies $(\alpha - 1) + 2\beta \geq n - 1$, whence $\alpha + \beta \geq n$. The same argument shows that this condition also holds for the other four terms in (10). This completes the proof of Claim (a).

Degree considerations alone show that none of the last four terms in (10) has a $w^n$ term. By induction $(0, n - 1)$ is a vertex of $NT(G_{n-1})$, so $G_{n-1}$ has a $w^{n-1}$ term. Thus $wG_{n-1}$, and hence $G_n$, has a $w^n$ term. So $(0, n)$ is a vertex for $NT(G_n)$. Combine this with the first inequality in Claim (a) and we see that the carrier of $NT(G_n)$ must lie to the right of the line through $(0, n)$ of slope $-2$. It remains to show that the rest of the vertices depicted in the Lemma do appear in $NT(G_n)$.

We claim that
\begin{enumerate}
  \item[(c)] $\left[\frac{n+1}{2}\right]$ = the smallest exponent $e$ so that $\mu^e$ appears in $G_n$,
  \item[(d)] $\epsilon_n :=$ the coefficient of this term has sign $(-1)^{\left[\frac{n+1}{2}\right]}$, and
  \item[(e)] $|\epsilon_n| > n|\epsilon_{n-1}|$ for $n \geq 3$.
\end{enumerate}

By (9) these hold for $n = 2, 3$. Now, suppose $n \geq 4$. The first and the fourth terms in (10) do not contain any power of $\mu$. By induction on Claim (c), the lowest power of $\mu$ in the remaining terms are
\begin{align*}
\begin{array}{c|c|c|c}
\text{term} & \frac{\mu G_{n-1}}{n \text{ even}} & \frac{\mu^2 G_{n-2}}{n \text{ odd}} & \frac{\mu G_{n-2}}{n \text{ odd}} \\
\hline
\frac{\mu^{n/2}}{n \text{ even}} & \frac{\mu^{1+n/2}}{n \text{ odd}} & \frac{\mu^{n/2}}{n \text{ odd}} \\
\hline
\frac{\mu^{(n+1)/2}}{n \text{ odd}} & \frac{\mu^{(n+3)/2}}{n \text{ odd}} & \frac{\mu^{(n+1)/2}}{n \text{ odd}}.
\end{array}
\end{align*}

Thus the lowest power of $\mu$ in $G_n$ is $\mu^{\left[\frac{n+1}{2}\right]}$, provided that the coefficient of this term in $G_n$ is non-zero. This coefficient is
\begin{align*}
&= -(2n - 1)\epsilon_{n-1} - (n - 1)\epsilon_{n-2} & \text{by (10)} \\
&= (-1)^{\left[\frac{n+1}{2}\right]}[(2n - 1)|\epsilon_{n-1}| - (n - 1)|\epsilon_{n-2}|] & \text{induction on Claim (d)} \\
&\neq 0 & \text{induction on Claim (e)}.
\end{align*}
Thus the $\mu^{[\frac{n+1}{2}]}$ coefficient in $G_n$ is non-zero and has sign $(-1)^{[\frac{n+1}{2}]}$. In particular, $([\frac{n+1}{2}], 0)$ is a vertex of $NT(G_n)$. Furthermore,

\[(11) \quad |\epsilon_n| = (2n - 1)|\epsilon_{n-1}| - (n - 1)|\epsilon_{n-2}| > n|\epsilon_{n-1}|,
\]
by induction on Claim (e). This completes the proof of Claims (c)-(e). Note that Claim (c) together with the discussion immediately preceding it implies that $NT(G_n)$ for even $n$ is exactly as is depicted in the Lemma.

Finally, we claim that

(f) $[\frac{n}{2}] = \text{the smallest exponent } d \text{ so that } w\mu^d \text{ appears in } G_n,$

(g) $\delta_n := \text{the coefficient of this term has sign } (-1)^{[\frac{n-1}{2}]}$, and

(h) $|\delta_n| > |\epsilon_n|$ for even $n \geq 2$.

For $n = 2$ and $3$ these hold by (9). Next, suppose $n \geq 4$. By induction on Claims (c) and (f), the monomial in each of the five terms in (10) of the form $w\mu^b$ with $b$ minimal are

<table>
<thead>
<tr>
<th>term</th>
<th>$wG_{n-1}$</th>
<th>$\mu G_{n-1}$</th>
<th>$\mu^2 G_{n-2}$</th>
<th>$w\mu G_{n-2}$</th>
<th>$\mu G_{n-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>monomial</td>
<td>$w \cdot \mu^{[\frac{n-2}{2}]}$</td>
<td>$\mu \cdot w\mu^{[\frac{n-4}{2}]}$</td>
<td>$\mu^2 \cdot w\mu^{[\frac{n-6}{2}]}$</td>
<td>$w \cdot \mu^{[\frac{n-4}{2}]}$</td>
<td>$\mu \cdot w\mu^{[\frac{n-6}{2}]}$</td>
</tr>
<tr>
<td>$(n = 2m - 1)$</td>
<td>$w\mu^{m-1}$</td>
<td>$w\mu^m$</td>
<td>$w\mu^m$</td>
<td>$w\mu^{m-1}$</td>
<td>$w\mu^m$</td>
</tr>
<tr>
<td>$(n = 2m)$</td>
<td>$w\mu^{m/2}$</td>
<td>$w\mu^{m/2}$</td>
<td>$w\mu^{m/2}$</td>
<td>$w\mu^{m/2}$</td>
<td>$w\mu^{m/2}$</td>
</tr>
</tbody>
</table>

Thus by (10), the coefficient of $w\mu^{[\frac{n}{2}]}$ in $G_n$ is

\[
\begin{aligned}
&\{ - (\epsilon_{n-1} + (n - 1)\delta_{n-2}) & n = 2m - 1 \\
&\{ - (\epsilon_{n-1} + (2n - 1)\delta_{n-1} + (n - 1)\epsilon_{n-2} + (n - 1)\delta_{n-2}) & n \text{ even}
\end{aligned}
\]

By induction on Claims (d) and (g), this becomes

\[
\begin{aligned}
&\{ - (\frac{1}{2})^{[\frac{n}{2}]}|\epsilon_{n-1}| + (\frac{1}{2})^{[\frac{n-2}{2}]}(n - 1)|\delta_{n-2}|) & n = 2m - 1 \\
&\{ - (\frac{1}{2})^{[\frac{n}{2}]}|\epsilon_{n-1}| + (\frac{1}{2})^{[\frac{n-2}{2}]}(2n - 1)|\delta_{n-1}| +

&\quad (\frac{1}{2})^{[\frac{n-1}{2}]}(n - 1)|\epsilon_{n-2}| + (\frac{1}{2})^{[\frac{n-1}{2}]}(n - 1)|\delta_{n-2}|) & n \text{ even}
\end{aligned}
\]

By Claims (c) and (g), this coefficient is non-zero and has sign $(-1)^{[\frac{n-2}{2}]}$; invoke Claim (h) and we see that the same holds for even $n$. Thus $([\frac{n}{2}], 1)$ belongs to the carrier of in $NT(G_n)$. Invoke Claim (h) again and we see that for even $n$,

\[
|\delta_n| > (2n - 1)|\delta_{n-1}|
\]

\[
= (2n - 1)(|\epsilon_{n-1}| + (n - 1)|\delta_{n-2}|).
\]

Recall (11) and we see that $|\delta_n| > |\epsilon_n|$ for even $n$. This completes the proof of Claims (f)-(h), and hence the Lemma. 

Denote by $\sigma : \overline{\mathcal{L}_n} \to \mathcal{L}_n$ the canonical desingularization of $\mathcal{L}_n$. This is birational map, so $\overline{\mathcal{L}_n} = \sigma \mathcal{L}_n$ also has degree $n$. \qed
Lemma 5. For \( n \geq 2 \), the inverse image under \( \tau_\infty \) of the point at infinity \([1 : 0] \in \mathbb{P}_K^1 \) has \( \left[ \frac{n+1}{2} \right] \) distinct points. If \( n \) is even, then each such point has ramification index \( 2 \) with respect to \( \tau_\infty \). If \( n = 2m - 1 \), then \( m - 1 \) of these points have ramification index \( 2 \), and the remaining point has ramification index \( 1 \).

Proof. We give the argument for \( n = 2m - 1 \); the even case is similar.

By Lemma 4, the Newton polygon of \( G_n(w, \mu) \) consists of a line segment of slope \(-2\) and with \( w \)-displacement \( m - 1 \), plus a line segment of slope \(-1\) and with \( w \)-displacement \( 1 \). The slope \(-2\) line segment corresponds to Puiseux series of the form
\[
(12) \quad cw^{1/2} + (\text{higher order terms}),
\]
where \( c \) is a non-zero root of the polynomial whose terms are those in \( G_n(w, \mu) \) of the form \( \mu^a w^b \) with \( a + 2b = n \); and the slope \(-1\) segment corresponds to Puiseux series of the form
\[
(13) \quad c' w + (\text{higher order terms}),
\]
where \( c' \) is a non-zero root of the polynomial whose terms are those in \( G_n(w, \mu) \) of the form \( \mu^\alpha w^\beta \) with \( \alpha + \beta = n \). Note that there are two choices for \( w^{1/2} \), so all together this gives \( \leq 2m - 1 \) Puiseux series around \((w, \mu) = (0,0)\). On the other hand, exactly \( n \) distinct branches of \( \mathcal{L}_n \) pass through \((0,0)\), so there are exactly \( n = 2m - 1 \) Puiseux series: \( m - 1 \) pairs of the form (12), and a single series of the form (13). Recall [6, Remark 7.30] and the Lemma follows. \( \square \)

Proof of the Theorem. The Theorem is trivially true for \( n \leq 2 \), so from now on we assume that \( n \geq 3 \). Dehomogenize \( F_n \) by setting \( \mu = 1 \); then by (6), \( \tau_\nu \) is ramified above \( \nu = -2, \ldots, -n \). With \( \nu_0 \) chosen as such, \( F_n(x, \nu_0, 1) \) becomes
\[
(14) \quad x^{\left| \nu_0 \right|} \cdot (\text{polynomial of degree } n - |\nu_0| \text{ in } x \text{ with a non-zero constant term}).
\]
By (6), this degree \( n - |\nu_0| \) factor has distinct roots. Thus the fiber of \( \tau_\nu \) above \( \nu = \nu_0 \) consists of \( n - |\nu_0| \) distinct points with ramification index \( 1 \), plus one point with ramification index \( |\nu_0| \). Recall Lemma 5 about the ramification at infinity into the calculation at the end of section 2, we see that geometric genus of \( \mathcal{L}_n \) is equal to
\[
1 - n + \frac{1}{2} \left( \sum_{k=2}^{n} (k - 1) + \left[ \frac{n}{2} \right] (2 - 1) \right) = 1 - n + \frac{n(n - 1)}{4} + \frac{1}{2} \left[ \frac{n}{2} \right],
\]
and the Theorem follows. \( \square \)

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[1] W. Feit, \( \hat{A}_5 \) and \( \hat{A}_7 \) are Galois groups over number fields. J. Algebra 104 (1986) 231-260.


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