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Volume of an N-Simplex by Multiple Integration

For $N = 1, 2, \ldots$, let Δ_N be the N-simplex bounded by the coordinate hyperplanes $x_1 = 0, \ldots, x_N = 0$, and the hyperplane

$$\frac{x_1}{a_1} + \cdots + \frac{x_N}{a_N} = 1 \,. \tag{1}$$

where $a_1, \ldots, a_N$ are positive numbers. Thus, Δ_N intersects the coordinate axes at the points $x_i = a_i$, $i = 1, \ldots, N$. In this note, we prove using multiple integrals that the volume of Δ_N is

$$V(\Delta_N) = \frac{\prod_{i=1}^{N} a_i}{N!} \,. \tag{2}$$

It is felt that the proof may be useful pedagogically to students in advanced calculus or in beginning analysis who are seeing integration in R^N for the first time. It was motivated by a problem in Ref. [1]; p. 560, exer. 1.

We denote by $V_N(a_1, \ldots, a_N)$ the volume of the simplex Δ_N with sides $a_1, \ldots, a_N$. Our proof is by induction on N. Since Δ_1 is a line of length a_1 and Δ_2 a right triangle with sides a_1 and a_2, formula (2) holds for $N = 1, 2$. We now assume that we have proved formula (2) for $N = 1, 2, \ldots, k$ and consider $N = k + 1$. But for $k \geq 2$

$$V_{k+1}(a_1, \ldots, a_{k+1}) =$$
$$\int_0^{a_1} \int_0^{a_2(1-x_1/a_1)} \int_0^{a_3(1-x_1/a_1 - x_2/a_2)} \cdots \int_0^{a_{k+1}(1-x_1/a_1-x_2/a_2-\cdots-x_k/a_k)} dx_{k+1} \cdots dx_3\, dx_2\, dx_1 \,. \tag{3}$$

The upper limit on the integration with respect to each x_i is obtained by solving equation (1) (with $N = k + 1$) for x_i in terms of the other variables and setting $x_j = 0$ for $i < j \leq k + 1$. We make the change of variables $y_1 = x_1/a_1$, $y_2 = x_2/a_2, \ldots, y_{k+1} = x_{k+1}/a_{k+1}$, in the multiple integral in equation (3), obtaining

$$V_{k+1}(a_1, \ldots, a_{k+1}) = a_1 \cdots a_{k+1} \times$$
$$\int_0^{1} \int_0^{(1-y_1)} \int_0^{(1-y_1-y_2)} \cdots \int_0^{(1-y_1-y_2-\cdots-y_k)} dy_{k+1} \cdots dy_3\, dy_2\, dy_1$$
$$= a_1 \cdot a_2 \ldots a_{k+1} V_{k+1}(1, \ldots, 1) \,. \tag{4}$$

The integration over y_{k+1} can be performed, and we find

$$V_{k+1}(1, \ldots, 1) =$$

$$\int_0^1 \int_0^{(1-y_1)} \int_0^{(1-y_1-y_2)} \int_0^{(1-y_1-y_2-\ldots-y_{k-1})} (1 - y_1 - y_2 - \cdots - y_k)\, dy_k \ldots dy_2\, dy_1$$

$$= \int_0^1 \int_0^{(1-y_1)} \int_0^{(1-y_1-y_2)} \int_0^{(1-y_1-y_2-\cdots-y_{k-1})} dy_k \ldots dy_2\, dy_1 - \sum_{i=1}^{k} I_i\,, \tag{5}$$

where

$$I_i = \int_0^1 \int_0^{(1-y_1)} \int_0^{(1-y_1-y_2)} \cdots \int_0^{(1-y_1-y_2-\cdots-y_{k-1})} y_i\, dy_k \ldots dy_2\, dy_1,\ i = 1, \ldots, k\,. \tag{6}$$

The integral in the last line of equation (5) is just $V_k(1, \ldots, 1)$, which equals $1/k!$ by the inductive hypothesis. We shall prove that

$$I_1 = I_2 = \cdots = I_k = \frac{1}{(k+1)!}\,. \tag{7}$$

This will complete the proof, for by equations (4) and (5) we find that

$$V_{k+1}(a_1, \ldots, a_{k+1}) = a_1 \ldots a_{k+1} \left[\frac{1}{k!} - \frac{k}{(k+1)!} \right] = \frac{a_1 \ldots a_{k+1}}{k+1!}\,,$$

as required.

To evaluate I_1, we rewrite the upper limits in (6), obtaining

$$I_1 = \int_0^1 y_1 \left[\int_0^{(1-y_1)} \int_0^{(1-y_1)(1-y_2/(1-y_1))} \ldots \int_0^{(1-y_1)(1-y_2/(1-y_1)-\cdots-y_{k-1}/(1-y_1))} dy_k \cdots dy_2 \right] dy_1\,.$$

But the expression in brackets is just $V_{k-1}(1-y_1, \ldots, 1-y_1)$ [see equation (3)]. Hence,

$$I_1 = \int_0^1 y_1 V_{k-1}(1-y_1, \ldots, 1-y_1)\, dy_1 = \int_0^1 (1-y_1) V_{k-1}(y_1, \ldots, y_1)\, dy_1$$

$$= \frac{1}{(k-1)!} \int_0^1 (1-y_1) y_1^{k-1}\, dy_1 = \frac{1}{(k-1)!} \int_0^1 y^{-1}\, dy_{11}^{\,k} - \int_0^1 y_1^k\, dy_1 = \frac{1}{(k+1)!}\,.$$

The key step here is the use of the induction hypothesis in going from the second to the third equation. To show that $I_i = I_1, i = 2, \ldots, k$, we change the order of integration in I_i so that we integrate with respect to y_i last. This changes the upper limits in

(6), and the resulting expression can be evaluated by exactly the same procedure used for I_1. The proof is complete.

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REFERENCES

[1] GEORGE B. THOMAS, Jr., *Calculus and Analytic Geometry*, Part 2, Addison-Wesley, Reading, Mass. (1972).