

Quantum mechanical soft springs and reverse correlation inequalities

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Various properties of one-dimensional Schrödinger operators with "soft spring" potentials are derived as a consequence of the fact that the GHS and other correlation inequalities are reversed for certain general Ising modules.

We consider the Hamiltonian, $H_V = -d^2/dx^2 + V(x)$, of a one-dimensional quantum mechanical particle under the influence of a "spring" force, $-dV/dx$. In classical mechanics, a distinction is sometimes drawn between the qualitatively different motions due to "hard springs," where d^2V/dx^2 is increasing in $|x|$, and "soft springs," where d^2V/dx^2 is decreasing in $|x|$ (see Ref. 1, Chap. II). The introduction of statistical mechanical techniques into constructive quantum field theory in recent years (see Ref. 2) has led to some interesting "spin-off" results concerning quantum mechanical hard springs, and the main purpose of this paper is to give the corresponding results for soft springs.

Our soft spring potentials will be real-valued functions in the class

$$V_s = \{V \mid V(x) = \text{const} + \int_0^x G(y) dy \text{ with } G(y) = -G(-y) \nabla y, \\ G \text{ concave on } [0, \infty), \text{ and } a_V \equiv \lim_{y \rightarrow \infty} G(y) > 0\}. \quad (1)$$

We further define

$$\exp(-V_s) = \{f \mid f = \exp(-V) \text{ for some } V \in V_s\}. \quad (2)$$

For $-a_V < a < a_V$, $H_V - ax$ [considered as an operator on $L^2(\mathbb{R}, dx)$] has nondegenerate eigenvalues which we list in increasing order as $E_0(a) < E_1(a) < \dots$ and a normalized ground state Ω^a [$(H_V - ax)\Omega^a = E_0(a)\Omega^a$] which we choose to be positive.

Theorem 1: Suppose $V \in V_s$. Then

$$M(a) \equiv (\Omega^a, x \Omega^a) \text{ is convex on } [0, a_V), \quad (3)$$

$$E_1(a) - E_0(a) \text{ is nonincreasing on } [0, a_V), \quad (4)$$

$$E_1(0) - E_0(0) \geq E_2(0) - E_1(0), \quad (5)$$

$$U \in \exp(-V_s) \Rightarrow \exp(-tH_V)U \in \exp(-V_s), \text{ for } t \geq 0, \quad (6)$$

$$\Omega^0 \in \exp(-V_s). \quad (7)$$

Remark 2: In the case of hard spring potentials, the analogues of (3) and (4) were first derived in Ref. 3 and the analog of (5) in Ref. 4 for V a quartic polynomial. These three results were then extended to a larger class of V 's in Refs. 5 and 6, and finally to all hard spring potentials in Refs. 7 and 8. The analogues of (6) and (7) for hard springs are given in Ref. 8. We do not include a proof of Theorem 1 since (3)–(7) follows

from the "reverse" correlation inequalities [in particular (14) and (15)] given below for certain general Ising models in exactly the same way as the hard spring results follow from the usual GHS and Lebowitz inequalities (see Ref. 9, Chap. IX, and Ref. 8 for details).

Remark 3: Property (6) for H_V can be expressed in terms of the diffusion process determined by H_V , exactly as was done for the analogous hard spring result in Ref. 8. Property (5) and its hard spring analog suggest some general relation between convexity properties of V and those of the spectrum of H_V . In particular, we suggest the existence of a natural class of V 's for which $E_{j+1}(0) - E_j(0)$ is nonincreasing (resp., nondecreasing) in j .

A general Ising model (with pair interactions) is a collection of "spin" random variables, $\{X_i; i = 1, \dots, N\}$, with joint probability distribution,

$$Z^{-1} \exp\left(\sum_{i=1}^N h_i x_i + \sum_{i,j=1}^N J_{ij} x_i x_j\right) \prod_{i=1}^N \rho_i(dx_i), \quad (8)$$

where each ρ_i is a measure in \mathcal{E} , the set of even Borel measures ρ on \mathbb{R} such that $\int \exp(kx^2)\rho(dx) < \infty$ for some $k > 0$, where Z is chosen so that (8) is a probability measure, and where the J_{ij} 's are real and so small that Z is finite for all real h_i 's. We shall always assume that $J_{ij}, h_i \geq 0$ for all i, j .

In order to discuss our correlation inequalities, we consider four independent copies, $\{X_i^\alpha\}$ ($\alpha = 1, \dots, 4$), of the $\{X_i\}$ and define $T_i = (X_i^1 + X_i^2)/\sqrt{2}$, $Q_i = (X_i^1 - X_i^2)/\sqrt{2}$, $W_i^\alpha = \sum_{\beta=1}^4 A_{\alpha\beta} X_i^\beta$, and $Y_i^\alpha = \sum_{\beta=1}^4 B_{\alpha\beta} X_i^\beta$, where A and B are the following 4×4 matrices:

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \\ B = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}. \quad (9)$$

Given a finite measure ρ on \mathbb{R} and an invertible 4×4 matrix T , we define $\rho(d\mathbf{x})$ as $\prod_{\alpha=1}^4 \rho(dx^\alpha)$ and $\rho_T(d\mathbf{x})$ as $\rho(d[T^{-1}\mathbf{x}])$, where $\mathbf{x} = (x^1, \dots, x^4) \in \mathbb{R}^4$. We define \mathcal{G}_+ (resp., \mathcal{G}_-) as the set of measures ρ in \mathcal{E} such that for

$$\begin{aligned} \mu &= \rho_A \text{ (resp. , } \rho_B) \\ \int_{\mathbb{R}^4} (x^1)^{l^1} \cdots (x^4)^{l^4} \mu(d\mathbf{x}) &\geq 0, \\ \forall l^1, \dots, l^4 &= 0, 1, 2, \dots \end{aligned} \quad (10)$$

We also define \mathcal{G}_s (resp. , \mathcal{G}_h) as the set of finite even measures ρ such that $\rho_A \geq \rho_B$ (resp. , $\rho_B \geq \rho_A$) on $\mathbb{R}_+^4 = \{\mathbf{x} : \mathbf{x}^\alpha > 0, \forall \alpha\}$. (Note that in Ref. 8, \mathcal{G}_h is denoted by \mathcal{G} .) We denote a multi-index (m_1, \dots, m_n) by m , $\prod_{i=1}^n X_i^{m_i}$ by X^m , an expectation $E(H)$ by $\langle H \rangle$, and $\langle X_{i_1} \cdots X_{i_k} \rangle$ by $\langle i_1 \cdots i_k \rangle$.

Theorem 4: A measure ρ in \mathcal{E} belongs to \mathcal{G}_+ if it belongs to \mathcal{G}_s . If each ρ_i in (8) belongs to \mathcal{G}_+ , then for any multi-indices m^1, \dots, m^4, m, n , and any i_1, \dots, i_4 ,

$$\left\langle \prod_{\alpha=1}^4 (W^\alpha)^{m^\alpha} \right\rangle \geq 0, \quad (11)$$

$$\langle Q^m Q^n \rangle - \langle Q^m \rangle \langle Q^n \rangle \geq 0, \quad (12)$$

$$\langle T^m Q^n \rangle - \langle T^m \rangle \langle Q^n \rangle \geq 0, \quad (13)$$

$$\begin{aligned} \langle i_1 i_2 i_3 \rangle - \langle i_1 \rangle \langle i_2 i_3 \rangle - \langle i_2 \rangle \langle i_1 i_3 \rangle \\ - \langle i_3 \rangle \langle i_1 i_2 \rangle + 2 \langle i_1 \rangle \langle i_2 \rangle \langle i_3 \rangle \geq 0, \end{aligned} \quad (14)$$

$$\langle i_1 i_2 i_3 i_4 \rangle - \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle - \langle i_1 i_3 \rangle \langle i_2 i_4 \rangle - \langle i_1 i_4 \rangle \langle i_2 i_3 \rangle \geq 0, \quad (15)$$

when

$$h_j = 0, \forall j.$$

Remark 5: These results are the “reverse” of the usual correlation inequalities which were originally proved when each $\rho_i(dx) = \delta(x-1) + \delta(x+1)$ in Refs. 10, 11, and 12 and then extended to measures in \mathcal{G}_- in Refs. 5–7. In the case of \mathcal{G}_- , the direction of the inequalities (13)–(15) is changed to give the usual GHS and Lebowitz inequalities, W^α in (11) is replaced by Y^α to give the usual Ellis–Monroe inequality, and (12) remains the same.

Proof: The proof is essentially identical to that of the usual inequalities as given in Refs. 6 and 7. We only note that in deriving (12)–(13) from (11), use must be made of the fact that the sign of any two of the bottom three rows of A may be changed without altering the validity of (11).

The next theorem completely characterizes measures in \mathcal{G}_s and is analogous to the characterization of \mathcal{G}_h given in Ref. 8.

Theorem 6: For a finite, even, not identically zero, Borel measure ρ on \mathbb{R} , the following three statements are equivalent:

(i) Either $\rho(dx) = C\delta(x)$ for some $C > 0$, or else $\rho(dx) = f(x)dx$ for some $f \in \exp(-V_s)$,

(ii) $\rho \in \mathcal{G}_s$,

(iii) For any $b > 0$

$$\left(\frac{d^3}{dh^3} \right) \ln \int_{-\infty}^{\infty} \exp(hx - bx^2) \rho(dx) \geq 0 \text{ for } h \geq 0. \quad (16)$$

Proof: The proof of Theorem 2.4 of Ref. 8 directly yields that (i) \Rightarrow (ii) and (ii) \Rightarrow (iii), and it reduces the proof of (iii) \Rightarrow (i) to showing that if $\rho_b - \rho$ weakly with $\rho_b(dx) = f_b(x)dx$ and $f_b \in \exp(-V_s)$, then ρ must be as in (i). This latter fact is easily derived by using the proofs of Lemmas 4.6 and 4.10 of Ref. 8.

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