Refined Asymptotics of the Finite-Size Magnetization via a New Conditional Limit Theorem for the Spin

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Abstract

We study the fluctuations of the spin per site around the thermodynamic magnetization in the mean-field Blume-Capel model. Our main theorem generalizes the main result in a previous paper [12] in which the first rigorous confirmation of the statistical mechanical theory of finite-size scaling for a mean-field model is given. In that paper our goal is to determine whether the thermodynamic magnetization is a physically relevant estimator of the finite-size magnetization. This is done by comparing the asymptotic behaviors of these two quantities along parameter sequences converging to either a second-order point or the tricritical point in the mean-field Blume-Capel model. The main result is that the thermodynamic magnetization and the finite-size magnetization are asymptotic when the parameter α governing the speed at which the sequence approaches criticality is below a certain threshold α_0. Our main theorem in the present paper on the fluctuations of the spin per site around the thermodynamic magnetization is based on a new conditional limit theorem for the spin, which is closely related to a new conditional central limit theorem for the spin.

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1 Introduction

The purpose of this paper is to analyze the asymptotic behavior of the fluctuations of the spin per site around the thermodynamic magnetization along parameter sequences having physical relevance in the mean-field Blume-Capel model. This research culminates a series of papers that study the phase-transition structure of the model via analytic techniques and probabilistic limit theorems [13, 6, 9, 12]. The mean-field Blume-Capel model is a mean-field version of an important lattice model due to Blume and Capel, to which we will refer as the B–C model [2, 3, 4, 5]. The mean-field B-C model is an important object of study because it is one of the simplest models that exhibits the following complicated phase-transition structure: a curve of second-order points; a curve of first-order points; and a tricritical point, which separates the two curves.

The main theorem in this paper generalizes the main result in [12]. The goal of [12] is to compare the asymptotic behaviors of the thermodynamic magnetization and the finite-size magnetization along parameter sequences converging to either a second-order point or the tricritical point of the mean-field B-C model. Theorem 4.1 in that paper shows that these two quantities are asymptotic when the parameter α governing the speed at which the sequence approaches criticality is below a certain threshold α_0. However, when α exceeds α_0, the thermodynamic magnetization converges to 0 much faster than the finite-size magnetization. These results in [12] are worthwhile because they are the first rigorous confirmations of the statistical mechanical theory of finite-size scaling for a mean-field model [1], [12, §6].

The importance of both the theory of finite-size scaling and the mean-field B-C model motivate us in this paper to refine Theorem 4.1 in [12]. We do this by studying the fluctuations of the spin per site around the thermodynamic magnetization for $0 < \alpha < \alpha_0$, obtaining a more refined asymptotic estimate that yields the conclusion of Theorem 4.1 in [12] as a corollary. This refined asymptotic estimate is stated in (1.4) and is proved in part (a) of Theorem 4.1 below. While Theorem 4.1 in [12] is obtained from a moderate deviation principle, the refinement of that theorem in this paper is obtained from the conditional limit theorem stated in (1.5) and proved in part (b) of Theorem 6.1.

The mean-field B-C model is defined by a canonical ensemble that we denote by $P_{N,\beta,K}$; N is the number of vertices, $\beta > 0$ is the inverse temperature, and $K > 0$ is the interaction strength. $P_{N,\beta,K}$ is defined in (2.1) in terms of the Hamiltonian

$$H_{N,K}(\omega) = \sum_{j=1}^{N} \omega_j^2 - \frac{K}{N}\left(\sum_{j=1}^{N} \omega_j\right)^2.$$

In this formula ω_j is the spin at site $j \in \{1, 2, \ldots, N\}$ and takes values in $\Lambda = \{-1, 0, 1\}$. The configuration space for the model is the set Λ^N containing all sequences $\omega = (\omega_1, \ldots, \omega_N)$ with each $\omega_j \in \Lambda$. Expectation with respect to $P_{N,\beta,K}$ is denoted by $E_{N,\beta,K}$. The finite-size magnetization is defined by $E_{N,\beta,K}\{|S_N/N|\}$, where S_N equals the total spin $\sum_{j=1}^N \omega_j$.

Before discussing the results in this paper, we first summarize the phase-transition structure of the mean-field B-C model as derived in [13]. For $\beta > 0$ and $K > 0$, we denote by $\mathcal{M}_{\beta,K}$ the set of equilibrium values of the magnetization. The set $\mathcal{M}_{\beta,K}$ coincides with the set of global minimum points of the free-energy function $G_{\beta,K}$, which is defined in (2.3)–(2.4). The critical inverse temperature of the mean-field B-C model is $\beta_c = \log 4$. For $0 < \beta \leq \beta_c$ there exists a quantity $K(\beta)$ and for $\beta > \beta_c$ there exists a quantity $K_1(\beta)$ having the following properties. The positive quantity $m(\beta, K)$ appearing in this list is the thermodynamic magnetization.

1. Fix $0 < \beta \leq \beta_c$. Then for $0 < K \leq K(\beta)$, $\mathcal{M}_{\beta,K}$ consists of the unique pure phase 0, and for $K > K(\beta)$, $\mathcal{M}_{\beta,K}$ consists of two nonzero values $\pm m(\beta, K)$.

2. For $0 < \beta \leq \beta_c$, $\mathcal{M}_{\beta,K}$ undergoes a continuous bifurcation at $K = K(\beta)$, changing continuously from $\{0\}$ for $K \leq K(\beta)$ to $\{\pm m(\beta, K)\}$ for $K > K(\beta)$. This continuous bifurcation corresponds to a second-order phase transition.

3. Fix $\beta > \beta_c$. Then for $0 < K < K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of the unique pure phase 0; for $K = K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of 0 and two nonzero values $\pm m(\beta, K_1(\beta))$; and for $K > K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of two nonzero values $\pm m(\beta, K)$.

4. For $\beta > \beta_c$, $\mathcal{M}_{\beta,K}$ undergoes a discontinuous bifurcation at $K = K_1(\beta)$, changing discontinuously from $\{0\}$ for $K < K(\beta)$ to $\{0, \pm m(\beta, K)\}$ for $K = K_1(\beta)$ to $\{\pm m(\beta, K)\}$ for $K > K_1(\beta)$. This discontinuous bifurcation corresponds to a first-order phase transition.

Because of item 2, we refer to the curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$ as the second-order curve and points on this curve as second-order points. Because of item 4, we refer to the curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$ as the first-order curve and points on this curve as first-order points. The point $(\beta_c, K(\beta_c)) = (\log 4, 3/2 \log 4)$, called the tricritical point, separates the second-order curve from the first-order curve. The phase-coexistence region is defined as the set of all points in the positive β-K quadrant for which $\mathcal{M}_{\beta,K}$ consists of more than one value. Therefore the phase-coexistence region consists of all points above the second-order curve, above the tricritical point, on the first-order curve, and above the first-order curve; that is,

$$\{(\beta, K) : 0 < \beta \leq \beta_c, K > K(\beta) \text{ and } \beta > \beta_c, K \geq K_1(\beta)\}.$$

Figure 1 exhibits the sets that describe the phase-transition structure of mean-field B-C model.

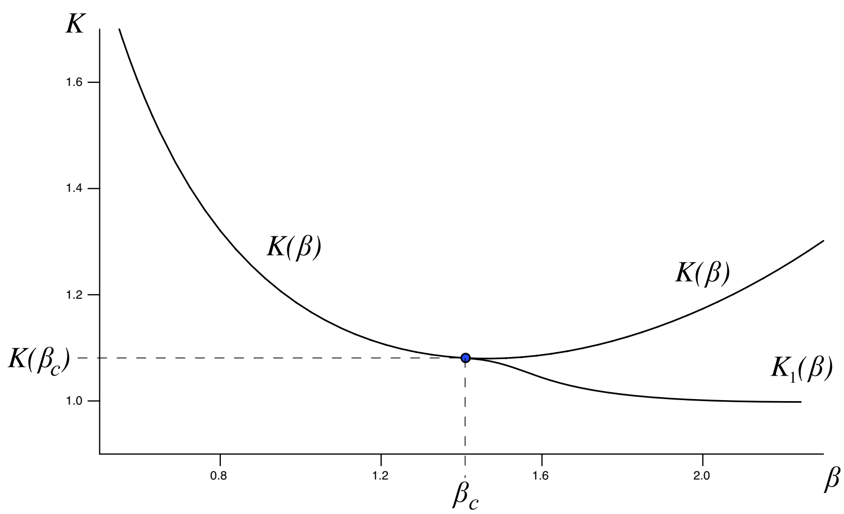

Figure 1: The sets that describe the phase-transition structure of the mean-field B-C model: the second-order curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$, the first-order curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$, and the tricritical point $(\beta_c, K(\beta_c))$. The phase-coexistence region consists of all (β, K) above the second-order curve, above the tricritical point, on the first-order curve, and above the first-order curve.

In order to discuss the contributions of this paper, it is helpful first to explain the main results in [9] and [12]. Those papers focus on positive sequences (β_n, K_n) that lie in the phase-coexistence region for all sufficiently large n, converge to either a second-order point or the tricritical point, and satisfy the hypotheses of Theorem 3.2 in [9]. These sequences are parameterized by $\alpha > 0$ in the sense that the limits

$$b = \lim_{n \to \infty} n^\alpha (\beta_n - \beta) \text{ and } k = \lim_{n \to \infty} n^\alpha (K_n - K(\beta))$$

are assumed to exist and are not both 0. Six specific such sequences are introduced in section 4 of that paper. Theorem 3.2 in [9] states that for any $\alpha > 0$, $m(\beta_n, K_n)$ has the asymptotic behavior

$$m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}, \tag{1.1}$$

where $\theta > 0$ and $\bar{x}$ is the positive global minimum point of a certain polynomial $g(x)$ called the Ginzburg-Landau polynomial. This polynomial is defined in terms of the free-energy function $G_{\beta,K}$ in hypothesis (iii)(a) of Theorem 3.1 below.

One of the surprises in our study of the mean-field B-C model is the appearance of the Ginzburg-Landau polynomial in a number of basic results. These include the asymptotic formula (1.1), the quantity $\bar{z}$ in the asymptotic formula (1.4) and the conditional limit theorem (1.5), the limiting variance in the conditional central limit theorem (1.7), and the rate function in the moderate deviation principle in Theorem 6.2. As we will explain, this conditional central limit theorem is closely related to the main result in this paper, which is the asymptotic formula (1.4).

A straightforward large-deviation calculation summarized in [12, p. 2120] shows that for fixed (β, K) lying in the phase-coexistence region the spin per site S_N/N has the weak-convergence limit

$$P_{N,\beta,K}\{S_N/N \in dx\} \Longrightarrow \left(\frac{1}{2}\delta_{m(\beta,K)} + \frac{1}{2}\delta_{-m(\beta,K)}\right)(dx). \tag{1.2}$$

This implies that

$$\lim_{N\to\infty} E_{N,\beta,K}\{|S_N/N|\} = m(\beta, K).$$

Because the thermodynamic magnetization $m(\beta, K)$ is the limit, as the number of spins goes to ∞, of the finite-size magnetization $E_{N,\beta,K}\{|S_N/N|\}$, the thermodynamic magnetization is a physically relevant estimator of the finite-size magnetization, at least when evaluated at fixed (β, K) in the phase-coexistence region.

The main focus of [12] is to determine whether the thermodynamic magnetization is a physically relevant estimator of the finite-size magnetization in a more general sense, namely, when evaluated along positive sequences that lie in the phase-coexistence region for all sufficiently large n, converge to a second-order point or the tricritical point, and satisfy a set of hypotheses including those of Theorem 3.2 in [9]. The criterion for determining whether $m(\beta_n, K_n)$ is a physically relevant estimator is that as $n \to \infty$, $m(\beta_n, K_n)$ is asymptotic to the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\}$, both of which converge to 0. In this formulation we let $N = n$ in the finite-size magnetization; i.e., we let the number of spins N coincide with the index n parametrizing the sequence (β_n, K_n).

As summarized in Theorems 4.1 and 4.2 in [12], the main finding is that $m(\beta_n, K_n)$ is a physically relevant estimator when the parameter α governing the speed at which (β_n, K_n) approaches criticality is below a certain threshold α_0. The value of α_0 depends on the type of the phase transition — first-order, second-order, or tricritical — that influences the sequence, an issue addressed in section 6 of [9]. For $0 < \alpha < \alpha_0$ this finding is summarized by the limit

$$\lim_{n\to\infty} n^{\theta\alpha}\, |E_{n,\beta_n,K_n}\{|S_n/n|\} - m(\beta_n, K_n)| = 0, \tag{1.3}$$

which in combination with (1.1) implies that

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{x}/n^{\theta\alpha} \sim m(\beta_n, K_n).$$

By contrast, when $\alpha > \alpha_0$, $m(\beta_n, K_n)$ converges to 0 much faster than $E_{n,\beta_n,K_n}\{|S_n/n|\}$. The sequences for which these asymptotic results are valid include the six sequences introduced in [9, §4].

We now turn to the main focus of this paper, which is a refined analysis of the fluctuations of S_n/n around $m(\beta_n, K_n)$ for $0 < \alpha < \alpha_0$. Define $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. As shown in part (a) of Theorem 4.1, for $0 < \alpha < \alpha_0$ and for a class of sequences that includes the first five sequences introduced in [9, §4]

$$E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\} \sim \bar{z}/n^{\kappa}. \tag{1.4}$$

In this formula $\bar{z} = (2/[\pi g^{(2)}(\bar{x})])^{1/2}$, where $g^{(2)}(\bar{x})$ denotes the positive second derivative of the Ginzburg-Landau polynomial g evaluated at its unique positive global minimum point $\bar{x}$. For all $0 < \alpha < \alpha_0$, κ is larger than $\theta\alpha$. Thus the rate $\bar{z}/n^{\kappa}$ at which $E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$ converges to 0 is asymptotically faster than the rate $\bar{x}/n^{\theta\alpha}$ at which $E_{n,\beta_n,K_n}\{|S_n/n|\}$ and $m(\beta_n, K_n)$ converge separately to 0.

This asymptotic result generalizes (1.3), which is the conclusion of Theorem 4.1 in [12]. To see this, define $A_n = E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$ and note that

$$|E_{n,\beta_n,K_n}\{|S_n/n|\} - m(\beta_n, K_n)| \le A_n.$$

Equation (1.4) states that $\lim_{n\to\infty} n^{\kappa} A_n = \bar{z}$. Since $\kappa > \theta\alpha$, this implies that

$$0 = \lim_{n\to\infty} n^{\theta\alpha} A_n \ge \lim_{n\to\infty} n^{\theta\alpha} |E_{n,\beta_n,K_n}\{|S_n/n|\} - m(\beta_n, K_n)| = 0.$$

The fact that this second limit equals 0 yields (1.3), which is the conclusion of Theorem 4.1 in [12].

The proof of our main result (1.4) is based on the following new conditional limit stated in part (b) of Theorem 6.1 for $0 < \alpha < \alpha_0$:

$$\lim_{n\to\infty} n^{\kappa} E_{n,\beta_n,K_n}\{|S_n/n - m(\beta_n, K_n)| \,\big|\, S_n/n > \delta m(\beta_n, K_n)\} = \bar{z}. \tag{1.5}$$

The conditioning is on the event $\{S_n/n > \delta m(\beta_n, K_n)\}$, where $\delta \in (0, 1)$ is sufficiently close to 1. This conditioning allows us to study the asymptotic behavior of the system in a neighborhood of the pure states having thermodynamic magnetization $m(\beta_n, K_n)$. According to Lemma 6.3

$$\begin{aligned} &\lim_{n\to\infty} P_{n,\beta_n,K_n}\{S_n/n > \delta m(\beta_n, K_n)\} \\ &\quad = \lim_{n\to\infty} P_{n,\beta_n,K_n}\{S_n/n < -\delta m(\beta_n, K_n)\} = 1/2. \end{aligned} \tag{1.6}$$

These limits are the analog of the weak convergence limit (1.2), showing that as $n \to \infty$ the mass of the P_{n,β_n,K_n}-distribution of S_n/n concentrates at $\pm m(\beta_n, K_n)$. As we show in section 6, the limits (1.6) and (1.5) and a moderate deviation estimate on the probability $P_{n,\beta_n,K_n}\{\delta m(\beta_n, K_n) \geq S_n/n \geq -\delta m(\beta_n, K_n)\}$ yield

$$\lim_{n\to\infty} n^{\kappa} E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\} = \bar{z}.$$

This limit is equivalent to (1.4).

The main result in part (a) of Theorem 4.1 is applied to the first five sequences introduced in [9, §4]. Located in the phase-coexistence region for all sufficiently large n, the first two sequences converge to a second-order point, and the last three sequences converge to the tricritical point. Possible paths followed by these sequences are shown in Figure 2. For each of the five sequences the quantities α_0, θ, and κ appearing in Theorem 4.1 are specified in Table 1.1.

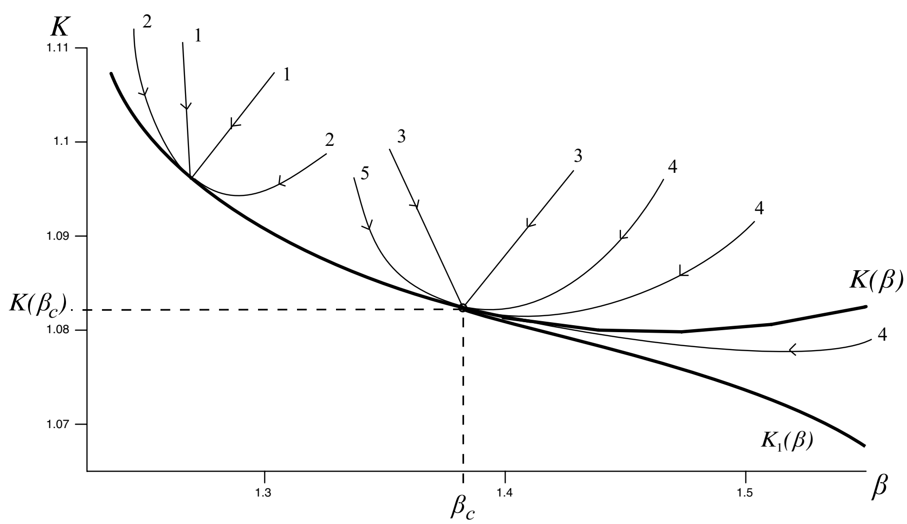

Figure 2: Possible paths for the five sequences converging to a second-order point and to the tricritical point. In section 5 and appendix A these sequences are defined and are shown to satisfy the hypotheses of Theorem 4.1 and Theorem 6.1. The sequences labeled 1–5 in this figure correspond to sequences 1a–5a in Table 1.1 and Table 5.1.

The conditional limit (1.5) is closely related to another result stated in part (a) of Theorem 6.1. This result is a new conditional central limit theorem for $0 < \alpha < \alpha_0$. As in (1.5), the

Seq.	**Defn.**	$\boldsymbol{\alpha_0}$	$\boldsymbol{\theta}$	$\boldsymbol{\kappa}$
1a	(5.5)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}(1-\alpha)$
2a	(5.6)	$\frac{1}{2p}$	$\frac{p}{2}$	$\frac{1}{2}(1-p\alpha)$
3a	(5.7)	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{2}(1-\alpha)$
4a	(5.8)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}(1-2\alpha)$
5a	(5.10)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}(1-2\alpha)$

Table 1.1: The equations where each of the five sequences is defined and the values of α_0, θ, and κ for each sequence.

conditioning is on the event $\{S_n/n > \delta m(\beta_n, K_n)\}$, where $\delta \in (0,1)$ is sufficiently close to 1. Under a set of hypotheses satisfied by the first five sequences introduced in [9, §4], part (a) of Theorem 6.1 states that when conditioned on $\{S_n/n > \delta m(\beta_n, K_n)\}$, the P_{n,β_n,K_n}-distributions of $n^\kappa(S_n/n - m(\beta_n, K_n))$ converge weakly to a normal random variable $N(0, 1/g^{(2)}(\bar{x}))$ with mean 0 and variance $1/g^{(2)}(\bar{x})$; in symbols,

$$P_{n,\beta_n,K_n}\{n^\kappa(S_n/n - m(\beta_n, K_n)) \in dx \bigm| S_n/n > \delta m(\beta_n, K_n)\} \\ \Longrightarrow N(0, 1/g^{(2)}(\bar{x})). \tag{1.7}$$

Since $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ is less than 1/2 [Thm. 6.1(c)], the scaling in this result is non-classical. An equivalent formulation is that for any bounded, continuous function f

$$\begin{aligned} &\lim_{n\to\infty} E_{n,\beta_n,K_n}\{f(n^\kappa(S_n/n - m(\beta_n, K_n))) \bigm| S_n/n > \delta m(\beta_n, K_n)\} \\ &\quad= \lim_{n\to\infty} E_{n,\beta_n,K_n}\{f(S_n/n^{1-\kappa} - n^\kappa m(\beta_n, K_n)) \bigm| S_n/n > \delta m(\beta_n, K_n)\} \\ &\quad= E\{f(N(0, 1/g^{(2)}(\bar{x})))\} \\ &\quad= \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} f(x)\exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx. \end{aligned} \tag{1.8}$$

Through the term $g^{(2)}(\bar{x})$ this conditional central limit theorem and the asymptotic formula (1.4) exhibit a sensitive dependence on the choice of the sequence (β_n, K_n), which lies in the phase coexistence region for all sufficiently large n and converges to a second-order point or the tricritical point. This contrasts sharply with the central limit theorem that is valid for an arbitrary sequence (β_n, K_n) that converges to a point (β, K) in the single-phase region defined by $\{(\beta, K) : 0 < \beta \leq \beta_c, 0 < K < K(\beta)\}$. In this situation it is proved in Theorem 5.5 in [6] that

$$P_{n,\beta_n,K_n}\{S_n/n^{1/2} \in dx\} \Longrightarrow N(0, \sigma^2(\beta, K)),$$

where the limiting variance $\sigma^2(\beta, K)$ depends only on (β, K) and not on the sequence (β_n, K_n).

Formally, the conditional limit (1.5) follows from the conditional central limit theorem (1.8) if one replaces the bounded, continuous function f by the absolute value function. Then (1.8) would imply

$$\begin{aligned}
&\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\{|S_n/n - m(\beta_n, K_n)| \mid S_n/n > \delta m(\beta_n, K_n)\} \\
&\quad = \lim_{n\to\infty} E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m(\beta_n, K_n)| \mid S_n/n > \delta m(\beta_n, K_n)\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2} g^{(2)}(\bar{x}) x^2] dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2} g^{(2)}(\bar{x}) x^2] dx = \bar{z}.
\end{aligned}$$

In order to justify this formal derivation, one needs a uniform integrability estimate. In fact, we can derive the conditional limit (1.5) from a related weak convergence result via a more circuitous route. The related weak convergence result, proved in Lemma 7.7, involves two extra summands defined in terms of a sequence of scaled normal random variables W_n. We prove the conditional limit (1.5) by two steps: the uniform integrability-type result in Proposition 8.2 allows us to replace the bounded, continuous function f in Lemma 7.7 by the absolute value function; the calculations in Lemmas 8.3 and 8.4 show that in the limit $n \to \infty$ the extra summands involving the normal random variables W_n do not affect the limit. As we show at the end of section 7 in [8], we also use the weak convergence result in Lemma 7.7 to prove the conditional central limit theorem (1.7) by an analogous but more straightforward argument. Again, a key step is to show that in the limit $n \to \infty$ the extra summands involving the normal random variables W_n do not affect the limit.

The conditional limit (1.5) is stated in part (b) of Theorem 6.1, the proof of which is subtle and complicated. In this proof Lemma 7.5 is key. There we obtain two basic estimates that allow us to apply the Dominated Convergence Theorem to prove the weak convergence result in Lemma 7.7, from which part (b) of Theorem 6.1 will be deduced. The value of κ can be motivated from the calculation underlying the proof of part (a) of Lemma 7.5.

The contents of this paper are as follows. In section 2 we define the mean-field B-C model and summarize its phase-transition structure in Theorems 2.1 and 2.2. For a class of sequences (β_n, K_n) lying in the phase-coexistence region for all sufficiently large n and converging either to a second-order point or to the tricritical point, Theorem 3.1 in section 3 describes the asymptotic behavior of $m(\beta_n, K_n) \to 0$ as stated in (1.1). Theorem 3.2 in section 3 states one of the main results of [12], which is that as $n \to 0$, $m(\beta_n, K_n)$ is asymptotic to $E_{n,\beta_n,K_n}\{|S_n/n|\}$ for $0 < \alpha < \alpha_0$, proving that for this range of α the thermodynamic magnetization $m(\beta_n, K_n)$ is a physically relevant estimator of the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\}$.

The main result in this paper is given in section 4. According to part (a) of Theorem 4.1, for $0 < \alpha < \alpha_0$

$$E_{n,\beta_n,K_n}\{|S_n/n - m(\beta_n, K_n)|\} \sim \bar{z}/n^\kappa,$$

where $\bar{z} = (2/[\pi g^{(2)}(\bar{x})])^{1/2}$ and $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. Part (a) of Theorem 4.1 is applied in section 5 to five specific sequences (β_n, K_n). The first two sequences converge to a second-order point, and the last three sequences converge to the tricritical point. In section 6 part (a) of Theorem 6.1 states the conditional central limit theorem (1.7), and part (b) of that theorem states the conditional limit (1.5). In section 7 we derive a number of lemmas that are applied in section 8 to part (b) of Theorem 6.1. In section 8 we prove part (b) of Theorem 6.1 using these lemmas together with Lemmas 8.1, 8.3, and 8.4 and the weaker form of the standard uniform integrability estimate in Proposition 8.2. In appendix A we prove that sequences 1a–5a satisfy the limits in hypothesis (iii′) of Theorem 4.1. In appendix B we prove the moderate deviation principle in part (a) of Theorem 6.2. This result is used in the proof of one of our main results in part (a) of Theorem 4.1.

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2 Phase-Transition Structure of the Mean-Field B-C Model

For $N \in \mathbb{N}$ the mean-field Blume-Capel model is defined on the complete graph on N vertices $1, 2, \ldots, N$. The spin at site $j \in \{1, 2, \ldots, N\}$ is denoted by ω_j, a quantity taking values in $\Lambda = \{-1, 0, 1\}$. The Hamiltonian for this model is defined by

$$H_{N,K}(\omega) = \sum_{j=1}^{N} \omega_j^2 - \frac{K}{N}\left(\sum_{j=1}^{N} \omega_j\right)^2,$$

where $K > 0$ is a positive parameter representing the interaction strength and $\omega = (\omega_1, \ldots, \omega_N) \in \Lambda^N$. We will refer to this model as the mean-field B-C model.

Let P_N be the product measure on Λ^N with identical one-dimensional marginals $\rho = \frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$. Then P_N assigns the probability 3^{-N} to each $\omega \in \Lambda^N$. For inverse temperature $\beta > 0$ and for $K > 0$, the canonical ensemble for the mean-field B-C model is the sequence of probability measures that assign to each subset B of Λ^N the probability

$$\begin{aligned} P_{N,\beta,K}(B) &= \frac{1}{Z_N(\beta, K)} \cdot \int_B \exp[-\beta H_{N,K}] dP_N \\ &= \frac{1}{Z_N(\beta, K)} \cdot \sum_{\omega \in B} \exp[-\beta H_{N,K}(\omega)] \cdot 3^{-N}, \end{aligned} \tag{2.1}$$

where

$$Z_N(\beta, K) = \int_{\Lambda^N} \exp[-\beta H_{N,K}] dP_N = \sum_{\omega \in \Lambda^N} \exp[-\beta H_{N,K(\omega)}] \cdot 3^{-N}.$$

It is useful to rewrite this measure in a different form. Define $S_N(\omega) = \sum_{j=1}^N \omega_j$ and let $P_{N,\beta}$ be the product measure on Λ_N with identical one-dimensional marginals

$$\rho_\beta(d\omega_j) = \frac{1}{Z(\beta)} \cdot \exp(-\beta \omega_j^2) \rho(d\omega_j),$$

where $Z(\beta) = \int_\Lambda \exp(-\beta\omega_j^2)\rho(d\omega_j) = (1 + 2e^{-\beta})/3$. Define

$$P_{N,\beta}(d\omega) = \prod_{j=1}^N \rho_\beta(d\omega_j) = \frac{1}{[Z(\beta)]^N} \prod_{j=1}^N \exp(-\beta\omega_j^2)\rho(d\omega_j)$$

and

$$\tilde{Z}_N(\beta, K) = \int_{\Lambda^N} \exp[N\beta K (S_N/N)^2] dP_{N,\beta} = \frac{Z_N(\beta, K)}{[Z(\beta)]^N}.$$

Then we have

$$P_{N,\beta,K}(d\omega) = \frac{1}{\tilde{Z}_N(\beta, K)} \exp[N\beta K(S_N(\omega)/N)^2] P_{N,\beta}(d\omega). \tag{2.2}$$

For $t \in \mathbb{R}$ and $x \in \mathbb{R}$ we also define the cumulate generating function

$$c_\beta(t) = \log \int_\Lambda \exp(t\omega_1)\rho_\beta(d\omega_1) = \log\left[\frac{1 + e^{-\beta}(e^t + e^{-t})}{1 + 2e^{-\beta}}\right] \tag{2.3}$$

and the free-energy function

$$G_{\beta,K}(x) = \beta K x^2 - c_\beta(2\beta K x). \tag{2.4}$$

We denote by $\mathcal{M}_{\beta,K}$ the set of equilibrium macrostates of the mean-field B-C model. As shown in Proposition 3.4 in [13], $\mathcal{M}_{\beta,K}$ can be characterized as the set of global minimum points of $G_{\beta,K}$:

$$\mathcal{M}_{\beta,K} = \{x \in [-1, 1] : x \text{ is the global minimum points of } G_{\beta,K}(x)\}.$$

In [13] $\mathcal{M}_{\beta,K}$ is denoted by $\tilde{\mathcal{E}}_{\beta,K}$.

The critical inverse temperature for the mean-field B-C model is $\beta_c = \log 4$. For $0 < \beta \leq \beta_c$, the next theorem states that $\mathcal{M}_{\beta,K}$ exhibits a continuous bifurcation as K increases through a value $K(\beta)$. This bifurcation corresponds to a second-order phase transition, and the curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$ is called the second-order curve. The point $(\beta_c, K(\beta_c))$ is called the tricritical point. Theorem 2.1 is proved in Theorem 3.6 in [13], where $K(\beta)$ is denoted by $K_c^{(2)}(\beta)$.

Theorem 2.1. *For* $0 < \beta \le \beta_c$*, we define*

$$K(\beta) = 1/[2\beta c''_\beta(0)] = (e^\beta + 2)/(4\beta).$$

For these values of β*,* $\mathcal{M}_{\beta,K}$ *has the following structure.*

(a) *For* $0 < K \le K(\beta)$*,* $\mathcal{M}_{\beta,K} = \{0\}$*.*

(b) *For* $K > K(\beta)$*, there exists* $m(\beta,K) > 0$ *such that* $\mathcal{M}_{\beta,K} = \{\pm m(\beta,K)\}$*.*

(c) $m(\beta,K)$ *is a positive, increasing, continuous function for* $K > K_c(\beta)$*, and as* $K \to (K(\beta))^+$*,* $m(\beta,K) \to 0$*. Therefore,* $\mathcal{M}_{\beta,K}$ *exhibits a continuous bifurcation at* $K(\beta)$*.*

For $\beta > \beta_c$, the next theorem states that $\mathcal{M}_{\beta,K}$ exhibits a discontinuous bifurcation as K increases through a value $K_1(\beta)$. This bifurcation corresponds to a first-order phase transition, and the curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$ is called the first-order curve. Theorem 2.2 is proved in Theorem 3.8 in [13], where $K_1(\beta)$ is denoted by $K_c^{(1)}(\beta)$.

Theorem 2.2. *For* $\beta > \beta_c$*,* $\mathcal{M}_{\beta,K}$ *has the following structure in terms of the quantity* $K_1(\beta)$*, denoted by* $K_c^{(1)}(\beta)$ *in* [13] *and defined implicitly for* $\beta > \beta_c$ *on page* 2231 *of* [13].

(a) *For* $0 < K < K_1(\beta)$*,* $\mathcal{M}_{\beta,K} = \{0\}$*.*

(b) *For* $K = K_1(\beta)$ *there exists* $m(\beta, K_1(\beta)) > 0$ *such that* $\mathcal{M}_{\beta,K_1(\beta)} = \{0, \pm m(\beta, K_1(\beta))\}$*.*

(c) *For* $K > K_1(\beta)$ *there exists* $m(\beta,K) > 0$ *such that* $\mathcal{M}_{\beta,K} = \{\pm m(\beta,K)\}$*.*

(d) $m(\beta,K)$ *is a positive, increasing, continuous function for* $K \ge K_1(\beta)$*, and as* $K \to K_1(\beta)^+$*,* $m(\beta,K) \to m(\beta, K_1(\beta)) > 0$*. Therefore,* $\mathcal{M}_{\beta,K}$ *exhibits a discontinuous bifurcation at* $K_1(\beta)$*.*

The positive quantity $m(\beta,K)$ in Theorems 2.1 and 2.2 is called the thermodynamic magnetization. In the next section we describe the asymptotic behavior of the finite-size magnetization for suitable sequences (β_n, K_n) and relate this to the asymptotic behavior of the thermodynamic magnetization $m(\beta_n, K_n)$.

3 Asymptotic Behavior of $E_{n,\beta_n,K_n}\{|S_n/n|\}$

For $\beta > 0$ and $K > 0$ the finite-size magnetization is defined as

$$E_{N,\beta,K}\{|S_N/N|\} = \int_{\Omega_N} |S_N/N| dP_{N,\beta,K},$$

where $P_{N,\beta,K}$ denotes the measure defined in (2.1)–(2.2). In this section we describe the asymptotic behavior of $E_{n,\beta_n,K_n}\{|S_n/n|\}$ for suitable sequences (β_n, K_n) lying in the phase-coexistence region. In this formulation we let $N = n$ in the finite-size magnetization; i.e., we let the number of spins N coincide with the index n parametrizing the sequence (β_n, K_n).

The phase-coexistence region is defined as the set of all points in the positive β-K quadrant for which $\mathcal{M}_{\beta,K}$ cosists of more than one value. According to Theorems 2.1 and 2.2, the phase-coexistence region consists of all points above the second-order curve, above the tricritical point, on the first-order curve, and above the first-order curve; that is,

$$\{(\beta, K) : 0 < \beta \leq \beta_c, K > K(\beta) \text{ and } \beta > \beta_c, K \geq K_1(\beta)\}.$$

For a class of sequences (β_n, K_n) lying in the phase-coexistence region for all sufficiently large n and converging either to a second-order point or to the tricritical point, Theorem 3.1 describes the asymptotic behavior of the thermodynamic magnetization $m(\beta_n, K_n) \to 0$. The asymptotic behavior is related to the unique positive, global minimum point of the Ginzburg-Landau polynomial, which is defined in hypothesis (iii) of the theorem.

Theorem 3.1 is a special case of the main theorem in [9], Theorem 3.2. In that paper we describe six different sequences that satisfy the hypotheses of Theorem 3.1. The first five of these sequences are revisited in section 5 of this paper, where we show that they satisfy the hypotheses of our main theorem, Theorem 4.1. These five sequences, labeled 1a–5a, are summarized in Table 5.1. The main conclusion of Theorem 3.1 about the rate at which $m(\beta_n, K_n) \to 0$ will be used in the proofs of a number of results in this paper.

Theorem 3.1. *Let (β_n, K_n) be a positive sequence that converges either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. We assume that (β_n, K_n) satisfies the following four hypotheses.*

(i) *(β_n, K_n) lies in the phase-coexistence region for all sufficiently large n.*

(ii) *The sequence (β_n, K_n) is parametrized by $\alpha > 0$. This parameter regulates the speed of approach of (β_n, K_n) to the second-order point or the tricritical point in the following sense:*

$$b = \lim_{n\to\infty} n^\alpha(\beta_n - \beta) \text{ and } k = \lim_{n\to\infty} n^\alpha(K_n - K(\beta))$$

both exist, and b and k are not both 0; if $b \neq 0$, then b equals 1 or -1.

(iii) *There exists an even polynomial g of degree 4 or 6 satisfying $g(x) \to \infty$ as $|x| \to \infty$ together with the following two properties; g is called the Ginzburg-Landau polynomial.*

(a) *There exist $\alpha_0 > 0$ and $\theta > 0$ such that for all $\alpha > 0$*

$$\lim_{n\to\infty} n^{\alpha/\alpha_0} G_{\beta_n,K_n}(x/n^{\theta\alpha}) = g(x)$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *There exists* $\bar{x} > 0$ *such that the set of global minimum points of* g *equals* $\{\pm\bar{x}\}$.

(iv) *Consider* $\alpha_0 > 0$ *and* $\theta > 0$ *in hypothesis* (iii)(a). *There exists a polynomial* H *satisfying* $H(x) \to \infty$ *as* $|x| \to \infty$ *together with the following property*: *for all* $\alpha > 0$ *there exists* $R > 0$ *such that for all* $n \in \mathbb{N}$ *sufficiently large and for all* $x \in \mathbb{R}$ *satisfying* $|x/n^{\theta\alpha}| < R$, $n^{\alpha/\alpha_0} G_{\beta_n,K_n}(x/n^{\theta\alpha}) \geq H(x)$.

Under hypotheses (i)–(iv), *for any* $\alpha > 0$

$$m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}; \text{ i.e., } \lim_{n\to\infty} n^{\theta\alpha} m(\beta_n, K_n) = \bar{x}.$$

If $b \neq 0$, *then this becomes* $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^\theta$.

Theorem 3.2 restates Theorem 4.1 in [12]. The hypotheses are those of Theorem 3.1 for all $0 < \alpha < \alpha_0$ together with the inequality $0 < \theta\alpha_0 < 1/2$. These hypotheses are satisfied by sequences 1a–5a in Table 5.1 as well as by a sixth sequence described in Theorem 4.6 in [9]. Part (a) of the next theorem gives the rate at which $E_{n,\beta_n,K_n}\{|S_n/n|\} \to 0$ for $0 < \alpha < \alpha_0$, and part (b) states that for the same values of α, $E_{n,\beta_n,K_n}\{|S_n/n|\} \sim m(\beta_n, K_n)$. Thus Theorem 3.2 shows that the asymptotic behavior of $E_{n,\beta_n,K_n}\{|S_n/n|\}$ coincides with that of $m(\beta_n, K_n)$ for $0 < \alpha < \alpha_0$. Theorem 4.2 in [12] shows that for $\alpha > \alpha_0$, $m(\beta_n, K_n)$ converges to 0 asymptotically faster than $E_{n,\beta_n,K_n}\{|S_n/n|\}$.

Theorem 3.2. *Let* (β_n, K_n) *be a positive sequence parametrized by* $\alpha > 0$ *and converging either to a second-order point* $(\beta, K(\beta))$, $0 < \beta < \beta_c$, *or to the tricritical point* $(\beta_c, K(\beta_c))$. *We assume that* (β_n, K_n) *satisfies the hypotheses of Theorem* 3.1 *for all* $0 < \alpha < \alpha_0$. *We also assume the inequality* $0 < \theta\alpha_0 < 1/2$. *The following conclusions hold.*

(a) *For all* $0 < \alpha < \alpha_0$

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{x}/n^{\theta\alpha}; \text{ i.e., } \lim_{n\to\infty} n^{\theta\alpha} E_{n,\beta_n,K_n}\{|S_n/n|\} = \bar{x}.$$

(b) *For all* $0 < \alpha < \alpha_0$, $E_{n,\beta_n,K_n}\{|S_n/n|\} \sim m(\beta_n, K_n)$.

In Theorem 4.1 in the next section we state our main result on the rate at which $E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$ converges to 0 for $0 < \alpha < \alpha_0$. We then explain how Theorem 4.1 generalizes Theorem 3.2.

4 Asymptotic Behavior of $E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$

We denote by E_{n,β_n,K_n} expectation with respect to the measure P_{n,β_n,K_n}. Theorem 4.1 is our main result. In this theorem we investigate the asymptotic behavior of the expectation

$E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$ under the hypotheses of Theorem 3.1 and an additional hypothesis (iii′). Part (a) of Theorem 4.1 states that the expected value of the fluctuations of $|S_n/n|$ around $m(\beta_n, K_n)$ is asymptotic to $\bar{z}/n^\kappa$, where $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ and $\bar{z} > 0$ is given explicitly. Compared with the conclusion of Theorem 3.2, part (a) of Theorem 4.1 is a more refined statement. As we showed in the introduction, it yields the conclusion of Theorem 3.2 as a corollary. The rate $\bar{z}/n^\kappa$ at which $E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$ converges to 0 is much faster than the rate $\bar{x}/n^{\theta\alpha}$ at which $E_{n,\beta_n,K_n}\{|S_n/n|\}$ and $m(\beta_n, K_n)$ converge to 0 separately. We comment on the hypotheses of Theorem 4.1 at the end of this section.

Part (a) of Theorem 4.1 is proved in section 6. Part (b) of Theorem 4.1 asserts that the hypotheses of this theorem are satisfied by sequences 1a–5a in Table 5.1. This is discussed in section 5. For each of these sequences the Ginzburg-Landau polynomial has degree 4 or 6.

Theorem 4.1. *Let (β_n, K_n) be a positive sequence converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. We assume that (β_n, K_n) satisfies the hypotheses of Theorem* 3.1 *for all* $0 < \alpha < \alpha_0$. *We also assume the following additional hypothesis on the Ginzburg-Landau polynomial g.*

(iii′) *Assume that g has degree 4. Then $\theta\alpha_0$ lies in the interval $[1/4, 1/2)$. In addition, for all $0 < \alpha < \alpha_0$ and for $j = 2, 3, 4$*

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}(m(\beta_n, K_n)) = g^{(j)}(\bar{x}) > 0.$$

Assume that g has degree 6. Then $\theta\alpha_0$ lies in the interval $[1/6, 1/2)$. In addition, for all $0 < \alpha < \alpha_0$ and for $j = 2, 3, 4, 5, 6$

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}(m(\beta_n, K_n)) = g^{(j)}(\bar{x}) > 0.$$

For $\alpha \in (0, \alpha_0)$ we also define $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. Then for all $0 < \alpha < \alpha_0$ the following conclusions hold.

(a) *We have the asymptotic behavior*

$$E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\} \sim \bar{z}/n^\kappa,$$

where $\bar{z} = \left(2/(\pi g^{(2)}(\bar{x}))\right)^{1/2}$; i.e., $\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\} = \bar{z}$.

(b) *The hypotheses of this theorem are satisfied by sequences* 1a–5a *in Table* 5.1.

The hypotheses of Theorem 4.1 are those of Theorem 3.1 together with the additional hypothesis (iii′) for all $0 < \alpha < \alpha_0$. The latter hypothesis takes two related forms depending on whether g has degree 4 or degree 6. In this hypothesis, the assumption on $\theta\alpha_0$ yields the inequality $0 < \theta\alpha_0 < 1/2$, which is required by the moderate deviation principle stated in Theorem 6.2. Hypothesis (iii′) also assumes both the asymptotic behavior of certain derivatives of $n^{\alpha/\alpha_0} G_{\beta_n,K_n}$ evaluated at $m(\beta_n, K_n)$ and the positivity of the corresponding derivatives of g evaluated at the positive global minimum point $\bar{x}$. These assumptions are needed in the proof of Lemma 7.5, a key result needed to prove part (b) of Theorem 6.1, which in turn yields part (a) of Theorem 4.1. The proof of that lemma also requires the fact assumed in hypothesis (iii′) that $\theta\alpha_0$ lies in the interval [1/4, 1/2) or [1/6, 1/2) depending on whether g has degree 4 or degree 6.

In the next section we outline how to verify the hypotheses of Theorem 4.1 for sequences 1a–5a in Table 5.1.

5 Verification of Hypotheses of Theorem 4.1 for Sequences 1a–5a

Table 5.1 summarizes five sequences (β_n, K_n) introduced in section 4 of [9]. Depending on the inequalities on the coefficients, sequences 1, 2, 3, and 5 each have two cases labeled a and b, and sequence 4 has three cases labeled a, b, and c. All five sequences 1a–5a lie in the phase-coexistence region for all sufficiently large n as required by hypothesis (i) of Theorem 3.1.

The hypotheses of Theorem 4.1 consist of the hypotheses of Theorem 3.1 for all $0 < \alpha < \alpha_0$ and hypothesis (iii′). Hypothesis (iii′) takes two forms depending on the degree of the Ginzburg-Landau polynomial g. When g has degree 4, $\theta\alpha_0$ is assumed to lie in the interval $[1/4, 1/2)$ and for all $\alpha \in (0, \alpha_0)$ and for $j = 2, 3, 4$

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}(m(\beta_n, K_n)) = g^{(j)}(\bar{x}) > 0. \tag{5.1}$$

When g has degree 6, $\theta\alpha_0$ is assumed to lie in the interval $[1/6, 1/2)$ and for all $\alpha \in (0, \alpha_0)$ and for $j = 2, 3, 4, 5, 6$

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}(m(\beta_n, K_n)) = g^{(j)}(\bar{x}) > 0. \tag{5.2}$$

In this section we verify for sequences 1a–5a that when g has degree 4, we have $\theta\alpha_0 \in [1/4, 1/2)$ and $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4$ and that when g has degree 6, we have $\theta\alpha_0 \in [1/6, 1/2)$ and $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4, 5, 6$. The verification of the limits in (5.1) and (5.2) is carried out in appendix A.

Sequence 6 introduced in Theorem 4.6 in [9] does not satisfy hypothesis (iii′) in Theorem 4.1. In this case g has degree 4, but $\theta\alpha_0$ does not lie in the interval $[1/4, 1/2)$.

The first two sequences converge to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, and the last three sequences converge to the tricritical point $(\beta_c, K(\beta_c))$. For each sequence 1a–5a, the

Seq.	Defn.	Case	Ineq.	Region	$\mathcal{M}_g$	Thm. in [9]
1	(5.5)	a	$K'(\beta)b - k < 0$	Ph-CR	$\{\pm\bar{x}\}$	Thm.4.1
		b	$K'(\beta)b - k > 0$	1-PhR	$\{0\}$	
2	(5.6)	a	$(K^{(p)}(\beta) - \ell)b^p < 0$	Ph-CR	$\{\pm\bar{x}\}$	Thm.4.2
		b	$(K^{(p)}(\beta) - \ell)b^p < 0$	1-PhR	$\{0\}$	
3	(5.7)	a	$K'(\beta_c)b - k < 0$	Ph-CR	$\{\pm\bar{x}\}$	Thm.4.3
		b	$K'(\beta_c)b - k > 0$	1-PhR	$\{0\}$	
4	(5.8)	a	$\ell > \ell_c, \tilde{\ell} \in \mathbb{R}$	Ph-CR	$\{\pm\bar{x}\}$	Thm.4.4
		b	$\ell = \ell_c, \tilde{\ell} > K_1'''(\beta_c)$	Ph-CR	$\{0, \pm\bar{x}\}$	
		c	$\ell < \ell_c, \tilde{\ell} \in \mathbb{R}$	1-PhR	$\{0\}$	
5	(5.10)	a	$\ell > K''(\beta_c)$	Ph-CR	$\{\pm\bar{x}\}$	Thm.4.5
		b	$\ell < K''(\beta_c)$	1-PhR	$\{0\}$	

Table 5.1: The equation where each of the 5 sequences is defined and the inequalities on the coefficients guaranteeing that each sequence lies in the phase-coexistence region (Ph-CR) or in the single-phase region (1-PhR). The next-to-last column states the structure of the set $\mathcal{M}_g$ of global minimum points of the Ginzburg-Landau polynomial g for each sequence in terms of a positive number $\bar{x}$ that can be explicitly calculated. The theorems in [9] where this information is verified are also given.

hypotheses of Theorem 3.1 are verified in Theorems 4.1–4.5 in [9]. We follow the same method used in that paper to verify hypothesis (iii′) in Theorem 4.1 for sequences 1a–5a. Hypothesis (iii′) of Theorem 4.1 takes two forms depending on whether the degree of the Ginzburg-Landau polynomial g is 4 or 6. We must verify that $\theta\alpha_0$ lies in a certain interval and that

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}(m(\beta_n, K_n)) = g^{(j)}(\bar{x}) > 0 \tag{5.3}$$

for $j = 2, 3, 4$ when g has degree 4 and for $j = 2, 3, 4, 5, 6$ when g has degree 6. The function $G_{\beta,K}$ is defined in (2.3)–(2.4).

It is straightforward to show that the limit in (5.3) holds for a given j provided the following limit holds uniformly for x in compact subsets of $\mathbb{R}$:

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}(x/n^{\theta\alpha}) = g^{(j)}(x). \tag{5.4}$$

The proof that the uniform convergence in (5.4) implies the limit in (5.3) uses the fact that $n^{\theta\alpha} m(\beta_n, k_n) \to \bar{x}$ [Thm. 3.1]. The uniform convergence in (5.4) can be obtained formally by taking the j-th derivative of the uniform convergence limits in hypothesis (iii)(a) of Theorem 3.1:

$$\lim_{n\to\infty} n^{\alpha/\alpha_0} G_{\beta_n,K_n}(x/n^{\theta\alpha}) = g(x).$$

The verification of the uniform convergence limits in (5.4), and thus the verification of the limits (5.1) and (5.2) in hypothesis (iii′), depend on asymptotic properties of the Taylor expansions of $G^{(j)}_{\beta_n,K_n}(x/n^{\theta\alpha})$. This analysis closely parallels the proof of Theorem 3.1, which is based on a similar analysis of the Taylor expansions of $G_{\beta_n,K_n}(x/n^{\theta\alpha})$ carried out in [9]. The straightforward but tedious calculations can be found in appendix A.

We now define the five sequences (β_n, K_n) and summarize the verification of the hypotheses of Theorem 4.1 for them.

Sequence 1a

Definition. Given $0 < \beta < \beta_c$, $\alpha > 0$, $b \in \{1, 0, -1\}$, and $k \in \mathbb{R}$, $k \neq 0$, the sequence is defined by

$$\beta_n = \beta + b/n^\alpha \ \text{ and } \ K_n = K(\beta) + k/n^\alpha. \tag{5.5}$$

This sequence converges to the second-order point $(\beta, K(\beta))$ along a ray with slope k/b if $b \neq 0$. We assume that $K'(\beta)b - k < 0$. Under this assumption it is proved in Theorem 4.1 in [9] that sequence 1 satisfies the hypotheses of Theorem 3.1 with $\alpha_0 = 1/2$ and $\theta = 1/2$. When $K'(\beta)b - k < 0$, we refer to sequence 1 as sequence 1a.

Hypothesis (iii′) *in Theorem* 4.1 *for sequence* 1a. Since $\alpha_0 = 1/2$ and $\theta = 1/2$, $\theta\alpha_0$ lies in the interval $[1/4, 1/2)$ as required by hypothesis (iii′). The limits in (5.1) for $j = 2, 3, 4$ are proved in appendix A. We now prove that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4$ using the formulas for g and $\bar{x}$ in Theorem 4.1 in [9]. Let $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/8 \cdot 4!$. Since $0 < \beta < \beta_c = \log 4$, we have $e^\beta < e^{\beta_c} = 4$, which implies $c_4(\beta) > 0$. Since $K'(\beta)b - k < 0$, these formulas yield

$$g^{(2)}(\bar{x}) = 2\beta(K'(\beta)b - k) + 3 \cdot 4c_4(\beta)\bar{x}^2 = 4\beta(k - K'(\beta)b) > 0,$$
$$g^{(3)}(\bar{x}) = 4!c_4(\beta)\bar{x} > 0, \ \text{ and } \ g^{(4)}(\bar{x}) = 4!c_4(\beta) > 0.$$

Thus under the condition $K'(\beta)b - k < 0$ sequence 1a satisfies all the hypotheses of Theorem 4.1.

Sequence 2a

Definition. Given $0 < \beta < \beta_c$, $\alpha > 0$, $b \in \{1, -1\}$, an integer $p \geq 2$, and a real number $\ell \neq K^{(p)}(\beta)$, the sequence is defined by

$$\beta_n = \beta + b/n^\alpha \ \text{ and } \ K_n = K(\beta) + \sum_{j=1}^{p-1} K^{(j)}(\beta)b^j/(j!n^{j\alpha}) + \ell b^p/(p!n^{p\alpha}). \tag{5.6}$$

This sequence converges to the second-order point $(\beta, K(\beta))$ along a curve that coincides with the second-order curve to order $n^{-(p-1)\alpha}$. We assume that $(K^{(p)}(\beta) - \ell)b^p < 0$. Under this assumption it is proved in Theorem 4.2 in [9] that sequence 2 satisfies the hypotheses of Theorem

3.1 with $\alpha_0 = 1/(2p)$ and $\theta = p/2$. When $(K^{(p)}(\beta) - \ell)b^p < 0$, we refer to sequence 2 as sequence 2a.

Hypothesis (iii$'$) *in Theorem* 4.1 *for sequence* 2a. Since $\alpha_0 = 1/(2p)$ and $\theta = p/2$, $\theta\alpha_0$ lies in the interval $[1/4, 1/2)$ as required by hypothesis (iii$'$). The limits in (5.1) for $j = 2, 3, 4$ are proved in appendix A. We now prove that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4$ using the formulas for g and $\bar{x}$ in Theorem 4.2 in [9]. Let $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/8 \cdot 4!$. Since $0 < \beta < \beta_c = \log 4$, we have $e^\beta < e^{\beta_c} = 4$, which implies $c_4(\beta) > 0$. Since $(K^{(p)}(\beta) - \ell)b^p < 0$, these formulas yield

$$g^{(2)}(\bar{x}) = \frac{2}{p!}\beta(K^{(p)}(\beta) - \ell)b^p + 3 \cdot 4c_4(\beta)\bar{x}^2 = \frac{4}{p!}\beta(\ell - K^{(p)}(\beta))b^p > 0,$$

$$g^{(3)}(\bar{x}) = 4!c_4(\beta)\bar{x} > 0, \text{ and } g^{(4)}(\bar{x}) = 4!c_4(\beta) > 0.$$

Thus under the condition $(K^{(p)}(\beta) - \ell)b^p < 0$ sequence 2a satisfies all the hypotheses of Theorem 4.1.

Sequence 3a

Definition. Given $\alpha > 0$, $b \in \{1, 0, -1\}$, and $k \in \mathbb{R}$, $k \neq 0$, the sequence is defined by

$$\beta_n = \beta_c + b/n^\alpha \text{ and } K_n = K(\beta_c) + k/n^\alpha. \tag{5.7}$$

This sequence converges to the tricritical point $(\beta_c, K(\beta_c))$ along a ray with slope k/b if $b \neq 0$. We assume that $K'(\beta_c)b - k < 0$. Under this assumption it is proved in Theorem 4.3 in [9] that sequence 3 satisfies the hypotheses of Theorem 3.1 with $\alpha_0 = 2/3$ and $\theta = 1/4$. When $K'(\beta_c)b - k < 0$, we refer to sequence 3 as sequence 3a.

Hypothesis (iii$'$) *in Theorem* 4.1 *for sequence* 3a. Since $\alpha_0 = 2/3$ and $\theta = 1/4$, $\theta\alpha_0$ lies in the interval $[1/6, 1/2)$ as required by hypothesis (iii$'$). The limits in (5.2) for $j = 2, 3, 4, 5, 6$ are proved in appendix A. We now prove that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4, 5, 6$ using the formulas for g and $\bar{x}$ in Theorem 4.3 in [9]. Let $c_6 = 9/40$. Since $K'(\beta_c)b - k < 0$, these formulas yield

$$g^{(2)}(\bar{x}) = 2\beta_c(K'(\beta_c)b - k) + 5 \cdot 6c_6\bar{x}^4 = 8\beta_c(k - K'(\beta_c)b) > 0,$$

$$g^{(3)}(\bar{x}) = 4 \cdot 5 \cdot 6c_6\bar{x}^3 > 0, \ g^{(4)}(\bar{x}) = 3 \cdot 4 \cdot 5 \cdot 6c_6\bar{x}^2 > 0,$$

$$g^{(5)}(\bar{x}) = 6!c_6\bar{x} > 0, \text{ and } g^{(6)}(\bar{x}) = 6!c_6 > 0.$$

Thus under the condition $K'(\beta_c)b - k < 0$ sequence 3a satisfies all the hypotheses of Theorem 4.1.

Sequence 4a

Definition. Given $\alpha > 0$, a curvature parameter $\ell \in \mathbb{R}$, and another parameter $\tilde{\ell} \in \mathbb{R}$, the sequence 4 is defined by

$$\beta_n = \beta_c + 1/n^\alpha \text{ and } K_n = K(\beta_c) + K'(\beta_c)/n^\alpha + \ell/(2n^\alpha) + \tilde{\ell}/(6n^{3\alpha}). \tag{5.8}$$

This sequence converges from the right to the tricritical point $(\beta_c, K(\beta_c))$ along the curve $(\beta, \tilde{K}(\beta))$, where for $\beta > \beta_c$

$$\tilde{K}(\beta) = K(\beta_c) + K'(\beta_c)(\beta - \beta_c) + \ell(\beta - \beta_c)^2/2 + \tilde{\ell}(\beta - \beta_c)^3/6.$$

The first-order curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$ is shown in Figure 1 in the introduction. In order to determine a condition on the coefficients guaranteeing that sequence 4 satisfies the hypotheses of Theorem 3.1, we must study $K_1(\beta)$ more closely.

Since $\lim_{\beta \to \beta_c^+} K_1(\beta) = K(\beta_c)$ [13, Sects. 3.1, 3.3], by continuity we extend the definition of $K_1(\beta)$ from $\beta > \beta_c$ to $\beta = \beta_c$ by define $K_1(\beta_c) = K(\beta_c)$. In addition we must assume other properties of K_1 that are stated in conjectures 1 and 2 on page 119 of [9]. As a preliminary to stating these conjectures, we assume that the first three right-hand derivatives of $K_1(\beta)$ exist at β_c and denote them by $K_1'(\beta_c)$, $K_1''(\beta_c)$, and $K_1'''(\beta_c)$. We also define $\ell_c = K''(\beta_c) - 5/(4\beta_c)$. Conjectures 1 and 2 state the following: (1) $K_1'(\beta_c) = K'(\beta_c)$ and (2) $K_1''(\beta_c) = \ell_c < 0 < K''(\beta_c)$. These conjectures are discussed in detail in section 5 of [10] and are supported by properties of the Ginzburg-Landau polynomials and numerical calculations.

We assume that $\ell > \ell_c$, which by conjecture 1 equals $K_1''(\beta_c)$. Under this assumption it is proved in Theorem 4.4 in [9] that sequence 4 satisfies the hypotheses of Theorem 3.1 with $\alpha_0 = 1/3$ and $\theta = 1/2$. When $\ell > \ell_c$, we refer to sequence 4 as sequence 4a.

Hypothesis (iii′) *in Theorem* 4.1 *for sequence* 4a. Since $\alpha_0 = 1/3$ and $\theta = 1/2$, $\theta\alpha_0$ lies in the interval $[1/6, 1/2)$ as required by hypothesis (iii′). The limits in (5.2) for $j = 2, 3, 4, 5, 6$ are proved in appendix A. Define

$$y = \left(1 + \frac{3}{5}\beta_c(\ell - K''(\beta_c))\right)^{1/2}. \tag{5.9}$$

Since $\ell > \ell_c = K''(\beta_c) - 5/(4\beta_c)$, we have $y > 1/2$. We now prove that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4, 5, 6$ using the formulas for g and $\bar{x}$ in Theorem 4.4 in [9]. Let $c_4 = 3/16$ and $c_6 = 9/40$. These formulas yield

$$g^{(2)}(\bar{x}) = \beta_c(K''(\beta_c) - \ell) - 3 \cdot 4 \cdot 4c_4\bar{x}^2 + 5 \cdot 6c_6\bar{x}^4 = \frac{20}{3}y^2 + \frac{20}{3}y > 0,$$

$$g^{(3)}(\bar{x}) = -4! \cdot 4c_4\bar{x} + 4 \cdot 5 \cdot 6c_6\bar{x}^3 = 9\bar{x}\left(\frac{4}{3} + \frac{10}{3}y\right) > 0,$$

$$g^{(4)}(\bar{x}) = -4! \cdot 4c_4 + 3 \cdot 4 \cdot 5 \cdot 6c_6\bar{x}^2 = -18 + 90(1+y) > 0,$$

$$g^{(5)}(\bar{x}) = 6!c_6\bar{x} > 0, \text{ and } g^{(6)}(\bar{x}) = 6!c_6 > 0.$$

Thus under the condition $\ell > \ell_c = K_1''(\beta_c)$ sequence 4a satisfies all the hypotheses of Theorem 4.1.

Sequence 5a

Definition. Given $\alpha > 0$ and a real number $\ell \neq K''(\beta_c)$, the sequence 5 is defined by

$$\beta_n = \beta_c - 1/n^\alpha \text{ and } K_n = K(\beta_c) - K'(\beta_c)/n^\alpha + \ell/(2n^\alpha). \tag{5.10}$$

This sequence converges to the tricritical point $(\beta_c, K(\beta_c))$ from the left along the curve that coincide with the second-order curve to order 2 in powers of $\beta - \beta_c$. We assume that $\ell > K''(\beta_c)$. Under this assumption it is proved in Theorem 4.5 in [9] that sequence 5 satisfies the hypotheses of Theorem 3.1 with $\alpha_0 = 1/3$ and $\theta = 1/2$. When $\ell > K''(\beta_c)$, we refer to sequence 5 as sequence 5a.

Hypothesis (iii′) *in Theorem* 4.1 *for sequence* 5a. Since $\alpha_0 = 1/3$ and $\theta = 1/2$, $\theta\alpha_0$ lies in the interval $[1/6, 1/2)$ as required by hypothesis (iii′). The limits in (5.2) for $j = 2, 3, 4, 5, 6$ are proved in appendix A. Define y as in (5.9). Since $\ell > K''(\beta_c)$, we have $y > 1$. We now prove that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4, 5, 6$ using the formulas for g and $\bar{x}$ in Theorem 4.5 in [9]. Let $c_4 = 3/16$ and $c_6 = 9/40$. These formulas yield

$$g^{(2)}(\bar{x}) = \beta_c(K''(\beta_c) - \ell) + 3 \cdot 4 \cdot 4c_4\bar{x}^2 + 5 \cdot 6c_6\bar{x}^4 = \frac{20}{3}y(y-1) > 0,$$

$$g^{(3)}(\bar{x}) = 4! \cdot 4c_4\bar{x} + 4 \cdot 5 \cdot 6c_6\bar{x}^3 > 0, \; g^{(4)}(\bar{x}) = 4! \cdot 4c_4 + 3 \cdot 4 \cdot 5 \cdot 6c_6 \cdot \bar{x}^2 > 0,$$

$$g^{(5)}(\bar{x}) = 6!c_6\bar{x} > 0, \text{ and } g^{(6)}(\bar{x}) = 6!c_6 > 0.$$

Thus under the condition $\ell > K''(\beta_c)$ sequence 5a satisfies all the hypotheses of Theorem 4.1.

We have completed the discussion of the verification of the hypotheses of Theorem 4.1 for sequences 1a–5a in Table 5.1. This is the content of part (b) of Theorem 4.1. Part (a) of that theorem is proved in the next section.

6 Proof of Part (a) of Theorem 4.1

Theorem 6.1, a new theorem stated in this section, has two parts. Under the same hypotheses as Theorem 4.1, part (a) of Theorem 6.1 states a conditional central limit theorem: conditioned on the event $\{S_n/n > \delta m(\beta_n, K_n)\}$ for $\delta \in (0,1)$ sufficiently close to 1, the P_{n,β_n,K_n}-distributions of $n^\kappa(S_n/n - m(\beta_n, K_n))$ converge weakly to an $N(0, 1/g^{(2)}(\bar{x}))$-random variable with mean 0 and variance $1/g^{(2)}(\bar{x})$. Under the same hypotheses as Theorem 4.1, part (b) of Theorem 6.1 states the related conditional limit

$$\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\{|S_n/n - m(\beta_n, K_n)| \mid S_n/n > \delta m(\beta_n, K_n)\}$$
$$= E\{|N(0, 1/g^{(2)}(\bar{x}))|\} = (2/(\pi g^{(2)}(\bar{x})))^{1/2} = \bar{z}.$$

We now sketch the proof of part (a) of Theorem 4.1 from part (b) of Theorem 6.1. In Lemma 6.3, we show that the moderate deviation principle in Theorem 6.2 and the asymptotic behavior of $m(\beta_n, K_n)$ in Theorem 3.1 imply that the event $\{S_n/n > \delta m(\beta_n, K_n)\}$ and the symmetric event $\{S_n/n < -\delta m(\beta_n, K_n)\}$ have large probability and that the event $\{\delta m(\beta_n, K_n) > S_n/n > -\delta m(\beta_n, K_n)\}$ has an exponentially small probability. As we show at the end of this section, combining part (b) of Theorem 6.1 with Lemma 6.3 and using symmetry yield

$$\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\} = \bar{z}.$$

This is part (a) of Theorem 4.1.

The proofs of parts (a) and (b) of Theorem 6.1 are long and technical. Part (b) is proved in subsections 8a, 8b, and 8c using a number of preparatory lemmas in section 7. At the end of section 7 in [8] we outline the proof of part (a), which follows the pattern of proof of part (b) but is more straightforward. The weak convergence result proved in Lemma 7.7 is the seed that yields both the conditional central limit theorem in part (a) of Theorem 6.1 and the conditional limit in part (b) of Theorem 6.1.

The hypotheses of Theorem 6.1 coincide with the hypotheses of Theorem 4.1. Part (c) of Theorem 6.1 states that for $\alpha \in (0, \alpha_0)$, $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ lies in the interval $(\theta\alpha_0, 1/2)$. This fact is needed in the proofs of Lemmas 7.2, 7.5, and 8.4. The proof that $\kappa \in (\theta\alpha_0, 1/2)$ is elementary. By hypothesis (iii′) of Theorem 4.1, we have $\theta\alpha_0 < 1/2$, which gives $\theta < 1/(2\alpha_0)$. Therefore

$$\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha = \frac{1}{2} + \alpha(\theta - 1/(2\alpha_0)) < 1/2.$$

Since $0 < \alpha < \alpha_0$ and $\theta < 1/(2\alpha_0)$, we have $\kappa > \frac{1}{2} + \alpha_0(\theta - 1/(2\alpha_0)) = \theta\alpha_0$. This completes the proof of part (c) of Theorem 6.1.

Concerning part (d) of Theorem 6.1, the hypotheses of this theorem coincide with the hypotheses of Theorem 4.1. Thus, as shown in section 5 and appendix A, these hypotheses are satisfied by sequences 1a–5a in Table 5.1.

Theorem 6.1. *Let* (β_n, K_n) *be a positive sequence converging either to a second-order point* $(\beta, K(\beta))$, $0 < \beta < \beta_c$, *or to the tricritical point* $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. *We assume that for all* $0 < \alpha < \alpha_0$, (β_n, K_n) *satisfies the hypotheses of Theorem* 4.1, *which coincide with the hypotheses of Theorem* 3.1 *together with hypothesis* (iii′). *For* $\alpha \in (0, \alpha_0)$ *we define* $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. *Then for any* $0 < \alpha < \alpha_0$ *there exists* $\Delta \in (0, 1)$ *such that for any* $\delta \in (\Delta, 1)$ *the following conclusions hold.*

(a) *When conditioned on the event* $\{S_n/n > \delta m(\beta_n, K_n)\}$, *the* P_{n,β_n,K_n}*-distributions of* $n^\kappa(S_n/n - m(\beta_n, K_n))$ *converge weakly to a normal random variable* $N(0, 1/g^{(2)}(\bar{x}))$ *with mean* 0 *and variance* $1/g^{(2)}(\bar{x})$*; in symbols,*

$$P_{n,\beta_n,K_n}\{n^\kappa(S_n/n - m(\beta_n, K_n)) \in dx \mid S_n/n > \delta m(\beta_n, K_n)\} \Longrightarrow N(0, 1/g^{(2)}(\bar{x})).$$

(b) *We have the conditional limit*

$$\begin{aligned} &\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\{|S_n/n - m(\beta_n, K_n)| \mid S_n/n > \delta m(\beta_n, K_n)\} \\ &\quad = \lim_{n\to\infty} E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m(\beta_n, K_n)| \mid S_n/n > \delta m(\beta_n, K_n)\} = \bar{z}, \end{aligned}$$

where

$$\begin{aligned} \bar{z} &= E\{|N(0, 1/g^{(2)}(\bar{x}))|\} \\ &= \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2} g^{(2)}(\bar{x}) x^2] dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2} g^{(2)}(\bar{x}) x^2] dx = \left(\frac{2}{\pi g^{(2)}(\bar{x})}\right)^{1/2}. \end{aligned}$$

(c) *For* $\alpha \in (0, \alpha_0)$, $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ *lies in the interval* $(\theta\alpha_0, 1/2)$.

(d) *The hypotheses of this theorem are satisfied by sequences* 1a–5a *in Table* 5.1.

In part (a) of Theorem 6.2 we state a moderate deviation principle (MDP) for the mean-field B-C model. This MDP will be used to prove Lemma 6.3, which in turn will be used to prove part (a) of Theorem 4.1 from part (b) of Theorem 6.1. The rate function in the MDP is the continuous function $\Gamma(x) = g(x) - \inf_{y\in\mathbb{R}} g(y)$, where g is the associated Ginzburg-Landau polynomial. Γ satisfies $\Gamma(x) \to \infty$ as $|x| \to \infty$. For A a subset of $\mathbb{R}$ define $\Gamma(A) = \inf_{x\in A} \Gamma(x)$.

Theorem 6.2. *Let* (β_n, K_n) *be a positive sequence converging either to a second-order point* $(\beta, K(\beta))$, $0 < \beta < \beta_c$, *or to the tricritical point* $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. *We assume that*

(β_n, K_n) *satisfies the hypotheses of Theorem* 4.1 *for all* $0 < \alpha < \alpha_0$. *The following conclusions hold.*

(a) *For all* $0 < \alpha < \alpha_0$, $S_n/n^{1-\theta\alpha}$ *satisfies the MDP with respect to* P_{n,β_n,K_n} *with exponential speed* $n^{1-\alpha/\alpha_0}$ *and rate function* $\Gamma(x) = g(x) - \inf_{y\in\mathbb{R}} g(y)$; *i.e., for any closed set* F *in* $\mathbb{R}$

$$\limsup_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} \log P_{n,\beta_n,K_n}\{S_n/n^{1-\theta\alpha} \in F\} \leq -\Gamma(F)$$

and for any open set G *in* $\mathbb{R}$

$$\liminf_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} \log P_{n,\beta_n,K_n}\{S_n/n^{1-\theta\alpha} \in G\} \geq -\Gamma(G).$$

(b) *The hypotheses of this theorem are satisfied by sequences* 1a–5a *in Table* 5.1 .

The MDP in part (a) of Theorem 6.2 is proved like the MDP in part (a) of Theorem 8.1 in [6] with only changes in notation. Because of the importance of the MDP in part (a) of Theorem 6.2, the proof is given in appendix B. Concerning part (b) of Theorem 6.2, the hypotheses of this theorem coincide with the hypotheses of Theorem 4.1. Thus, as shown in section 5 of this paper and in appendix A, these hypotheses are satisfied by sequences 1a–5a in Table 5.1.

After proving the next lemma, we use it to derive part (a) of Theorem 4.1 from part (b) of Theorem 6.1.

Lemma 6.3. *We assume that* (β_n, K_n) *satisfies the hypotheses of Theorem* 4.1 *for all* $0 < \alpha < \alpha_0$. *Then for any* $0 < \alpha < \alpha_0$ *and any* $\delta \in (0, 1)$ *there exists* $c > 0$ *such that for all sufficiently large n*

$$P_{n,\beta_n,K_n}\{\delta m(\beta_n, K_n) \geq S_n/n \geq -\delta m(\beta_n, K_n)\} \leq \exp[-cn^{1-\alpha/\alpha_0}] \to 0 \text{ as } n \to \infty.$$

In addition,

$$\lim_{n\to\infty} P_{n,\beta_n,K_n}\{S_n/n > \delta m(\beta_n, K_n)\} = \lim_{n\to\infty} P_{n,\beta_n,K_n}\{S_n/n < -\delta m(\beta_n, K_n)\} = 1/2.$$

Proof. By hypothesis (iii)(b) of Theorem 3.1 the global minimum points of g are $\pm\bar{x}$, and by Theorem 3.1, $n^{\theta\alpha} m(\beta_n, K_n) \to \bar{x}$ as $n \to \infty$. Thus we can choose $\varepsilon > 0$ satisfying $(1+\varepsilon)\delta < 1$ such that $n^{\theta\alpha} m(\beta_n, K_n) \leq (1+\varepsilon)\bar{x}$ for large n. Let F be the closed set $[-(1+\varepsilon)\delta\bar{x}, \ (1+\varepsilon)\delta\bar{x}]$. Since $(1+\varepsilon)\delta\bar{x} < \bar{x}$ and $-(1+\varepsilon)\delta\bar{x} > -\bar{x}$, we have

$$\inf_{y\in F} g(y) > \inf_{z\in\mathbb{R}} g(z) = g(\bar{x}),$$

which implies

$$\Gamma(F) = \inf_{y\in F}\{g(y) - \inf_{z\in\mathbb{R}} g(z)\} > 0.$$

We write $m_n = m(\beta_n, K_n)$. The moderate deviation upper bound in part (a) of Theorem 6.2 implies that for all sufficiently large n

$$\begin{aligned} & P_{n,\beta_n,K_n}\{\delta m_n \geq S_n/n \geq -\delta m_n\} \\ & \quad = P_{n,\beta_n,K_n}\{\delta n^{\theta\alpha} m_n \geq S_n/n^{1-\theta\alpha} \geq -\delta n^{\theta\alpha} m_n\} \\ & \quad \leq P_{n,\beta_n,K_n}\{(1+\varepsilon)\delta\bar{x} \geq S_n/n^{1-\theta\alpha} \geq -(1+\varepsilon)\delta\bar{x}\} \\ & \quad \leq \exp[-n^{1-\alpha/\alpha_0}\Gamma(F)/2] \to 0 \text{ as } n \to \infty. \end{aligned}$$

This yields the first assertion in the lemma.

To prove the second assertion, we write

$$\begin{aligned} 1 = P_{n,\beta_n,K_n}\{S_n/n \in \mathbb{R}\} \; = \; & P_{n,\beta_n,K_n}\{\delta m_n \geq S_n/n \geq -\delta m_n\} \\ & + P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\} \\ & + P_{n,\beta_n,K_n}\{S_n/n < -\delta m_n\}, \end{aligned}$$

Symmetry and the first assertion imply that

$$\lim_{n\to\infty} P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\} = \lim_{n\to\infty} P_{n,\beta_n,K_n}\{S_n/n < -\delta m_n\} = 1/2.$$

This completes the proof of the lemma. $\square$

Now we are ready to prove part (a) of Theorem 4.1.

Proof of part (a) of Theorem 4.1 from part (b) of Theorem 6.1 and Lemma 6.3. We write $m_n = m(\beta_n, K_n)$. Define

$$p^+_{n,\delta} = P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\},\; p^-_{n,\delta} = P_{n,\beta_n,K_n}\{S_n/n < -\delta m_n\},$$
$$q_{n,\delta} = P_{n,\beta_n,K_n}\{\delta m_n \geq S_n/n \geq -\delta m_n\}.$$

Since by symmetry $p^+_{n,\delta} = p^-_{n,\delta}$ and

$$\begin{aligned} & E_{n,\beta_n,K_n}\{||S_n/n| - m_n| \bigm| S_n/n < -\delta m_n\} \\ & \quad = E_{n,\beta_n,K_n}\{||S_n/n| - m_n| \bigm| S_n/n > \delta m_n\} \\ & \quad = E_{n,\beta_n,K_n}\{|S_n/n - m_n| \bigm| S_n/n > \delta m_n\}, \end{aligned}$$

we have

$$\begin{aligned} E_{n,\beta_n,K_n}\{||S_n/n| - m_n|\} \; = \; & E_{n,\beta_n,K_n}\{||S_n/n| - m_n| \bigm| S_n/n > \delta m_n\} \cdot p^+_{n,\delta} \\ & + E_{n,\beta_n,K_n}\{||S_n/n| - m_n| \bigm| S_n/n < -\delta m_n\} \cdot p^-_{n,\delta} \\ & + E_{n,\beta_n,K_n}\{||S_n/n| - m_n| \bigm| \delta m_n \geq S_n/n \geq -\delta m_n\} \cdot q_{n,\delta} \\ = \; & 2 \cdot E_{n,\beta_n,K_n}\{|S_n/n - m_n| \bigm| S_n/n > \delta m_n\} \cdot p^+_{n,\delta} \\ & + E_{n,\beta_n,K_n}\{||S_n/n| - m_n| \bigm| \delta m_n \geq S_n/n \geq -\delta m_n\} \cdot q_{n,\delta}. \end{aligned}$$

By part (b) of Theorem 6.1 and Lemma 6.3

$$\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\{|S_n/n - m_n| \bigm| S_n/n > \delta m_n\} \cdot p^+_{n,\delta} = \frac{1}{2}\bar{z}.$$

Since $|S_n/n| \leq 1$ and $0 \leq m_n \leq 1$, Lemma 6.3 implies that there exists $c > 0$ such that for all sufficiently large n

$$\begin{aligned} &n^\kappa E_{n,\beta_n,K_n}\{||S_n/n| - m_n| \bigm| \delta m_n \geq S_n/n \geq -\delta m_n\} \cdot q_{n,\delta} \\ &\quad \leq 2n^\kappa q_{n,\delta} \leq 2n^\kappa \exp[-cn^{1-\alpha/\alpha_0}] \to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\{||S_n/n| - m_n|\} = \frac{1}{2}\bar{z} + \frac{1}{2}\bar{z} = \bar{z}.$$

Part (a) of Theorem 4.1 is proved. □

In the next section we prove a number of lemmas that will be used in section 8 to prove part (b) of Theorem 6.1.

7 Preparatory Lemmas for Proof of Part (b) of Theorem 6.1

Let (β_n, K_n) be a positive sequence. Throughout this section we work with $0 < \alpha < \alpha_0$ and denote $m(\beta_n, K_n)$ by m_n. Let W_n be a sequence of normal random variables with mean 0 and variance $(2\beta_n K_n)^{-1}$ defined on a probability space $(\Omega, \mathcal{F}, Q)$. We denote by $\tilde{E}_{n,\beta_n,K_n}$ expectation with respect to the product measure $P_{n,\beta_n,K_n} \times Q$; P_{n,β_n,K_n} is defined in (2.1)–(2.2). Because the proof of part (b) of Theorem 6.1 is long and technical, we start by explaining the logic. The hypotheses of this theorem coincide with those of Theorem 4.1.

Part (b) of Theorem 6.1 states that there exists $\Delta \in (0,1)$ such that for any $\delta \in (\Delta, 1)$

$$\begin{aligned} &\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\{|S_n/n - m_n| \bigm| S_n/n > \delta m_n\} \\ &\quad = \lim_{n\to\infty} E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \bigm| S_n/n > \delta m_n\} \\ &\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx = \bar{z}. \end{aligned} \tag{7.1}$$

$\Delta \in (0,1)$ is determined in Lemma 7.5. The key idea in proving (7.1) is to show that adding suitably scaled versions of the normal random variables W_n yields a quantity with the following two properties: its limit equals the last line of (7.1) and the second line of (7.1) has the same limit; specifically,

$$\begin{aligned}
&\lim_{n\to\infty} E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\} \qquad (7.2)\\
&= \lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n + W_n/n^{1/2} > \delta m_n\}\\
&= \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx.
\end{aligned}$$

Formula (7.2) is proved in two steps.

Step 1. Prove the second limit in (7.2). This is done in part (b) of Lemma 8.1 in subsection 8a.

Step 2. Prove the first limit in (7.2). This is done in two substeps, which we now explain.

Substep 2a. Define

$$C_n = \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\}$$

and

$$D_n = \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\}.$$

Thus D_n is obtained from C_n by replacing $S_n/n^{1-\kappa}$ by $S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa}$. Substep 2a is to prove that $\lim_{n\to\infty} |C_n - D_n| = 0$. This is done in Lemma 8.3 in subsection 8b.

Substep 2b. Define

$$F_n = \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n + W_n/n^{1/2} > \delta m_n\}.$$

Thus F_n is obtained from D_n by replacing S_n/n in the conditioned event $\{S_n/n > \delta m_n\}$ by $S_n/n + W_n/n^{1/2}$. Substep 2b is to prove that

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty} F_n = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx.$$

The limit of F_n as $n \to \infty$ is calculated in Step 1. Substep 2b is proved in part (b) of Lemma 8.4 in subsection 8c. The explanation of the logic of the proof of part (b) of Theorem 6.1 is now complete.

We next state and prove the preparatory lemmas needed to carry out Step 1, Substep 2a, and Substep 2b in the proof of part (b) of Theorem 6.1.

Lemma 7.1 is a representation formula that will be used to study the limit of the conditional expectation in the second line of (7.2). This lemma can be proved like Lemma 3.3 in [11], which applies to the Curie-Weiss model, or like Lemma 3.2 in [14], which applies to the Curie-Weiss-Potts model. It will also be used to prove Lemma 7.2 and Lemma 7.6.

Lemma 7.1. *Given a positive sequence (β_n, K_n), let W_n be a sequence of normal random variables with mean 0 and variance $(2\beta_n K_n)^{-1}$ defined on a probability space $(\Omega, \mathcal{F}, Q)$. Then for any $\bar{\gamma} \in [0, 1)$ and any bounded, measurable function φ*

$$\begin{aligned}&\int_{\Lambda^n \times \Omega} \varphi(S_n/n^{1-\bar{\gamma}} + W_n/n^{1/2-\bar{\gamma}}) d(P_{n,\beta_n,K_n} \times Q) \\ &\quad = \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^{\bar{\gamma}})]dx} \cdot \int_{\mathbb{R}} \varphi(x) \exp[-nG_{\beta_n,K_n}(x/n^{\bar{\gamma}})]dx.\end{aligned}$$

In this formula G_{β_n,K_n} is the free energy function defined in (2.4)*.*

Lemma 7.2 uses the representation formula in the preceding lemma to rewrite the conditional expectation in the second line of (7.2).

Lemma 7.2. *We assume that (β_n, K_n) satisfies the hypotheses of Theorem* 4.1 *for all $0 < \alpha < \alpha_0$. For any $\bar{\delta} \in (0, 1)$ define*

$$A_n(\bar{\delta}) = \{S_n/n + W_n/n^{1/2} > \bar{\delta} m_n\},$$

where $m_n = m(\beta_n, K_n)$. Given any $\alpha \in (0, \alpha_0)$, define $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. The following conclusions hold.

(a) *For any bounded, measurable function h*

$$\begin{aligned}&\tilde{E}_{n,\beta_n,K_n}\{h(S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n) \cdot 1_{A_n(\bar{\delta})}\} \\ &\quad = \frac{1}{Z_{n,\kappa}} \int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty} h(x) \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n)]dx,\end{aligned} \tag{7.3}$$

where $Z_{n,\kappa} = \int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\kappa)]dx$. In particular, if $h \equiv 1$, then

$$\begin{aligned}\tilde{E}_{n,\beta_n,K_n}\{1_{A_n(\bar{\delta})}\} &= (P_{n,\beta_n,K_n} \times Q)\{A_n(\bar{\delta})\} \\ &= \frac{1}{Z_{n,\kappa}} \int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n)]dx.\end{aligned} \tag{7.4}$$

(b) *We have the representation*

$$\begin{aligned}
&\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \big| A_n(\bar{\delta})\} \qquad (7.5)\\
&= \frac{1}{\int_{-n^\kappa(1-\bar{\delta})m_n}^\infty \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx} \\
&\quad \cdot \int_{-n^\kappa(1-\bar{\delta})m_n}^\infty |x| \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx.
\end{aligned}$$

Proof. (a) By part (c) of Theorem 6.1, $\kappa \in (\theta\alpha_0, 1/2)$. We apply Lemma 7.1 with $\varphi(x) = h(x - n^\kappa m_n) \cdot 1_{(n^\kappa \bar{\delta} m_n, \infty)}(x)$ and $\bar{\gamma} = \kappa$, obtaining

$$\begin{aligned}
&\tilde{E}_{n,\beta_n,K_n}\{h(S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n) \cdot 1_{\{S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa} > n^\kappa \bar{\delta} m_n\}}\} \\
&\quad = \int_{\Lambda^n \times \Omega} h(S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n) \cdot 1_{\{S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa} > n^\kappa \bar{\delta} m_n\}} d(P_{n,\beta_n,K_n} \times Q) \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\kappa)]dx} \cdot \int_{\mathbb{R}} h(x - n^\kappa m_n) \cdot 1_{(n^\kappa \bar{\delta} m_n, \infty)}(x) \exp[-nG_{\beta_n,K_n}(x/n^\kappa)]dx \\
&\quad = \frac{1}{Z_{n,\kappa}} \cdot \int_{-n^\kappa(1-\bar{\delta})m_n}^\infty h(x) \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n)]dx.
\end{aligned}$$

This yields (7.3). Formula (7.4) follows by taking $h \equiv 1$.

(b) We apply part (a) to the sequence of bounded, measurable functions $h_j(x) = |x| \wedge j$, $j \in \mathbb{N}$. By the monotone convergence theorem we obtain (7.3) with $h(x)$ replaced by $|x|$. Part (b) now follows by using the definition of conditional expectation and multiplying the numerator and denominator of the resulting fraction by $\exp[nG_{\beta_n,K_n}(m_n)]$. The proof of Lemma 7.2 is complete. □

Lemma 7.3 gives the asymptotic behavior of $G_{\beta_n,K_n}(m_n)$. This lemma is used to prove Lemma 7.4 and part (a) of Lemma 8.1.

Lemma 7.3. *We assume that* (β_n, K_n) *satisfies the hypotheses of Theorem* 4.1 *for all* $0 < \alpha < \alpha_0$. *Let* $m_n = m(\beta_n, K_n)$. *Then for all* $0 < \alpha < \alpha_0$,

$$\lim_{n\to\infty} n^{\alpha/\alpha_0} G_{\beta_n,K_n}(m_n) = g(\bar{x}) < 0.$$

Proof. We have

$$\begin{aligned}
|n^{\alpha/\alpha_0} G_{\beta_n,K_n}(m_n) - g(\bar{x})| \;\leq\; & |n^{\alpha/\alpha_0} G_{\beta_n,K_n}(n^{\theta\alpha} m_n / n^{\theta\alpha}) - g(n^{\theta\alpha} m_n)| \qquad (7.6)\\
& + |g(n^{\theta\alpha} m_n) - g(\bar{x})|.
\end{aligned}$$

The hypotheses of Theorem 3.1 for all $0 < \alpha < \alpha_0$ consist of a subset of the hypotheses of Theorem 4.1. By hypothesis (iii)(a) of Theorem 3.1

$$\lim_{n\to\infty} n^{\alpha/\alpha_0} G_{\beta_n,K_n}(x/n^{\theta\alpha}) = g(x)$$

uniformly for x in compact subsets of $\mathbb{R}$. According to Theorem 3.1, $n^{\theta\alpha} m_n \to \bar{x}$, and so for any $\varepsilon > 0$ the sequence $n^{\theta\alpha} m_n$ lies in the compact set $[\bar{x} - \varepsilon, \bar{x} + \varepsilon]$ for all sufficiently large n. Setting $x = n^{\theta\alpha} m_n$, we see that the first term on the right-hand side of (7.6) has the limit 0. Because of the limit $n^{\theta\alpha} m_n \to \bar{x}$ and the continuity of g, the second term on the right-hand side of (7.6) also converges to 0 as $n \to \infty$. It follows that

$$\lim_{n\to\infty} |n^{\alpha/\alpha_0} G_{\beta_n,K_n}(m_n) - g(\bar{x})| = 0.$$

By hypothesis (iii)(b) of Theorem 3.1, $\bar{x} > 0$ is the unique nonnegative, global minimum point of g. Thus

$$g(\bar{x}) < g(0) = \lim_{n\to\infty} n^{\alpha/\alpha_0} G_{\beta_n,K_n}(0) = 0.$$

The proof of lemma is complete. □

Lemma 7.4 gives an inequality involving $nG_{\beta_n,K_n}(m_n)$ and the quantity $Z_{n,\kappa}$ defined in part (a) of Lemma 7.2. This inequality is used in the proof of Lemma 7.6 and the proof of part (a) of Lemma 8.4.

Lemma 7.4. *We assume that* (β_n, K_n) *satisfies the hypotheses of Theorem* 4.1 *for all* $0 < \alpha < \alpha_0$. *For any* $0 < \alpha < \alpha_0$ *define* $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ *and*

$$Z_{n,\kappa} = \int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^{\kappa})]dx.$$

Let $m_n = m(\beta_n, K_n)$. *Then for any* $\varepsilon > 0$ *and all sufficiently large* n

$$\exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \leq \exp[\varepsilon n^{1-\alpha/\alpha_0}].$$

Proof. For any $0 < \alpha < \alpha_0$ define

$$Z_{n,\theta\alpha} = \int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^{\theta\alpha})]dx.$$

Changing variables shows that $Z_{n,\kappa} = n^{\kappa-\theta\alpha} Z_{n,\theta\alpha}$. The MDP stated in Theorem 6.2 is proved in Theorem 8.1 in [6] via an associated Laplace principle. A key step in this proof is the limit

$$\lim_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} \log \int_{\mathbb{R}} \exp[n^{1-\alpha/\alpha_0}\psi(x) - nG_{\beta_n,K_n}(x/n^{\theta\alpha})]dx = \sup_{x\in\mathbb{R}}\{\psi(x) - g(x)\},$$

where ψ is any bounded, continuous function mapping $\mathbb{R}$ to $\mathbb{R}$. This is proved on page 546 of [6] with $v = -(1 - \alpha/\alpha_0)$ and $\gamma = \theta\alpha$. Setting $\psi = 0$ gives the limit

$$\lim_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} \log Z_{n,\theta\alpha} = \lim_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} \log \int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^{\theta\alpha})]dx = -\inf_{y\in\mathbb{R}} g(y).$$

Since $Z_{n,\kappa} = n^{\kappa-\theta\alpha} Z_{n,\theta\alpha}$ and g has a unique positive, global minimum point at $\bar{x}$,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} \log Z_{n,\kappa} &= \lim_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} \log(n^{\kappa-\theta\alpha} Z_{n,\theta\alpha}) \\ &= -\inf_{y\in\mathbb{R}} g(y) = -g(\bar{x}).\end{aligned}$$

By Lemma 7.3 $\lim_{n\to\infty} n^{\alpha/\alpha_0} G_{\beta_n,K_n}(m_n) = g(\bar{x})$. Hence the asymptotic behaviors of $\log Z_{n,\kappa}$ and $nG_{\beta_n,K_n}(m_n)$ are related by

$$\lim_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} \log Z_{n,\kappa} = -\lim_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} nG_{\beta_n,K_n}(m_n).$$

Thus for any $\varepsilon > 0$ and all sufficiently large n

$$\frac{1}{n^{1-\alpha/\alpha_0}} \log Z_{n,\kappa} + \frac{1}{n^{1-\alpha/\alpha_0}} nG_{\beta_n,K_n}(m_n) \le \varepsilon,$$

or equivalently

$$\exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \le \exp(\varepsilon n^{1-\alpha/\alpha_0}).$$

The proof of Lemma 7.4 is complete. $\square$

We recall that Step 1 in the proof of part (b) of Theorem 6.1 is to prove the second limit in (7.2):

$$\begin{aligned}&\lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \bigm| S_n/n + W_n/n^{1/2} > \delta m_n\} \\ &\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx.\end{aligned}$$

By part (b) of Lemma 7.2 the limit of the conditional expectation equals the limit of the product in the last two lines of (7.5) with $\bar{\delta} = \delta$. For $\delta \in (0,1)$ this product has the form

$$\begin{aligned}&\frac{1}{\int_{-n^\kappa(1-\delta)m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx} \\ &\quad \cdot \int_{-n^\kappa(1-\delta)m_n}^{\infty} |x| \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx.\end{aligned}$$

The calculation of the limit of this product depends in part on Lemma 7.7, which will be proved via the Dominated Convergence Theorem (DCT). Two key estimates are given in the next lemma. Part (b) of the next lemma also removes an error term that arises in the proof of part (a) of Lemma 8.1. The proof of Lemma 7.5 is postponed until the end of this section.

Lemma 7.5. *We assume that* (β_n, K_n) *satisfies the hypotheses of Theorem* 4.1 *for all* $0 < \alpha < \alpha_0$. *For any* $0 < \alpha < \alpha_0$ *define* $\kappa = \frac{1}{2}(1-\alpha/\alpha_0) + \theta\alpha$, *and let* $m_n = m(\beta_n, K_n)$. *The following conclusions hold.*

(a) *For all* $x \in \mathbb{R}$

$$\lim_{n\to\infty} (nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n)) = \frac{1}{2} g^{(2)}(\bar{x})x^2.$$

(b) *There exists* $\Delta \in (0,1)$ *such that for any* $\bar{\delta} \in (\Delta, 1)$ *there exists* $R > 0$ *such that for all sufficiently large n and all* $x \in \mathbb{R}$ *satisfying* $|x/n^\kappa| < R$ *and* $x/n^\kappa > -(1-\bar{\delta})m_n$

$$nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \geq \frac{1}{8} g^{(2)}(\bar{x})x^2.$$

The next lemma removes an error term that arises in applying the DCT to prove Lemma 7.7. The next lemma also removes an error term that arises in the proof of part (a) of Lemma 8.1.

Lemma 7.6. *We assume that* (β_n, K_n) *satisfies the hypotheses of Theorem* 4.1 *for all* $0 < \alpha < \alpha_0$. *Then there exist a constant* $c_2 > 0$ *such that for all sufficiently large* n

$$\int_{Rn^\kappa}^\infty \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx \leq \exp[-c_2 n] \to 0 \ \textit{ as } \ n \to \infty,$$

where R *is chosen as in part* (b) *of Lemma* 7.5 *and* $m_n = m(\beta_n, K_n)$.

Proof. We start by applying Lemma 7.1 with $\varphi(x) = 1_{(Rn^\kappa + n^\kappa m_n, \infty)}(x)$ and $\bar{\gamma} = \kappa$, obtaining

$$\begin{aligned}
&(P_{n,\beta_n,K_n} \times Q)\{S_n/n + W_n/n^{1/2} \geq R + m_n\} \\
&= (P_{n,\beta_n,K_n} \times Q)\{S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} \geq Rn^\kappa + m_n n^\kappa\} \\
&= \int_{\Lambda^n \times \Omega} 1_{\{S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} \geq Rn^\kappa + m_n n^\kappa\}} d(P_{n,\beta_n,K_n} \times Q) \\
&= \frac{1}{Z_{n,\kappa}} \cdot \int_{\mathbb{R}} 1_{[Rn^\kappa + m_n n^\kappa, \infty)}(x) \exp[-nG_{\beta_n,K_n}(x/n^\kappa)]dx \\
&= \frac{1}{Z_{n,\kappa}} \cdot \int_{Rn^\kappa}^\infty \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n)]dx,
\end{aligned}$$

where $Z_{n,\kappa} = \int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\kappa)]dx$. Thus we have

$$\begin{aligned}
&\int_{Rn^\kappa}^\infty \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx \\
&\quad = \exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \cdot (P_{n,\beta_n,K_n} \times Q)\{S_n/n + W_n/n^{1/2} \geq R + m_n\} \\
&\quad \leq \exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \cdot (P_{n,\beta_n,K_n} \times Q)\{S_n/n + W_n/n^{1/2} \geq R\}.
\end{aligned}$$

By part (b) of Lemma 4.4 in [6], with respect to $P_{n,\beta_n,K_n} \times Q$, $S_n/n + W_n/n^{1/2}$ satisfies the large deviation principle on $\mathbb{R}$ with exponential speed n and rate function $G_{\beta,K(\beta)}$. In particular, for the closed set $[R,\infty)$ we have the large deviation upper bound

$$\limsup_{n\to\infty} \frac{1}{n} \log(P_{n,\beta_n,K_n} \times Q)\{S_n/n + W_n/n^{1/2} \geq R\} \leq -\inf_{x \geq R} G_{\beta,K(\beta)}(x).$$

By part (a) of Theorem 2.1, since $0 < \beta \leq \beta_c$, we have $\mathcal{M}_{\beta,K(\beta)} = \{0\}$. Thus $G_{\beta,K(\beta)}$ has a unique global minimum point at 0. Since $R > 0$, it follows that

$$\inf_{x \geq R} G_{\beta,K(\beta)}(x) > \inf_{x \in \mathbb{R}} G_{\beta,K(\beta)}(x) = 0.$$

Therefore for all sufficiently large n

$$(P_{n,\beta_n,K_n} \times Q)\{S_n/n + W_n/n^{1/2} \geq R\} \leq \exp[-c_1 n],$$

where $c_1 = \inf_{x \geq R} G_{\beta,K(\beta)}(x)/2 > 0$. We now appeal to Lemma 7.4, which states that for any $\varepsilon > 0$ and all sufficiently large n

$$\exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \leq \exp[\varepsilon n^{1-\alpha/\alpha_0}].$$

Since $0 < 1 - \alpha/\alpha_0 < 1$, it follows that for all sufficiently large n

$$\begin{aligned}
&\int_{Rn^\kappa}^\infty \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx \\
&\quad \leq \exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \cdot (P_{n,\beta_n,K_n} \times Q)\{S_n/n + W_n/n^{1/2} \geq R\} \\
&\quad \leq \exp[\varepsilon n^{1-\alpha/\alpha_0}] \cdot \exp[-c_1 n] \\
&\quad \leq \exp[-c_1 n/2].
\end{aligned}$$

This gives the conclusion of Lemma 7.6 with $c_2 = c_1/2$. The proof of the lemma is complete. $\square$

Lemma 7.7 is a key result in the proof of the conditional limit stated in part (b) of Theorem 6.1. The lemma deals with the weak convergence of certain measures needed in the proof of part (a) of Lemma 8.1. Lemma 7.7 is also used with $f \equiv 1$ in the proof of part (b) of Lemma 8.1 and part (a) of Lemma 8.4.

Lemma 7.7. *We assume that* (β_n, K_n) *satisfies the hypotheses of Theorem* 4.1 *for all* $0 < \alpha < \alpha_0$. *Given any* $\alpha \in (0, \alpha_0)$, *define* $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ *and*

$$Z_{n,\kappa} = \int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\kappa)]dx.$$

For $\bar{\delta} \in (0,1)$ *define*

$$A_n(\bar{\delta}) = \{S_n/n + W_n/n^{1/2} > \bar{\delta} m_n\},$$

where $m_n = m(\beta_n, K_n)$. *Let* f *be any bounded, continuous function and let* $\Delta \in (0,1)$ *be the number determined in part* (b) *of Lemma* 7.5. *Then for any* $0 < \alpha < \alpha_0$ *and any* $\bar{\delta} \in (\Delta, 1)$ *we have the limit*

$$\begin{aligned}
&\lim_{n\to\infty} \exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \cdot \tilde{E}_{n,\beta_n,K_n}\{f(S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n) \cdot 1_{A_n(\bar{\delta})}\} \\
&\quad = \lim_{n\to\infty} \int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty} f(x) \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx \\
&\quad = \int_{\mathbb{R}} f(x) \exp[-\tfrac{1}{2} g^{(2)}(\bar{x}) x^2]dx. \qquad (7.7)
\end{aligned}$$

Proof. The first equality follows by applying part (a) of Lemma 7.2 to $h = f$. Concerning the second equality, we denote by I_n the integral in the second line of (7.7). We write $I_n = I_{n_1} + I_{n_2}$, where

$$I_{n_1} = \int_{-n^\kappa(1-\bar{\delta})m_n}^{Rn^\kappa} f(x) \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx$$

and

$$I_{n_2} = \int_{Rn^\kappa}^{\infty} f(x) \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx.$$

The number R is chosen as in part (b) of Lemma 7.5. Since f is bounded, Lemma 7.6 implies that there exists $c_2 > 0$ such that for all sufficiently large n

$$\begin{aligned}
I_{n_2} &\leq \|f\|_\infty \int_{Rn^\kappa}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx \\
&\leq \|f\|_\infty \exp[-c_2 n] \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus $I_{n_2} \to 0$ as $n \to \infty$.

Define

$$h_n(x) = f(x) \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]$$

and

$$h(x) = f(x)\exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2].$$

By part (a) of Lemma 7.5, $h_n(x) \to h(x)$ for all $x \in \mathbb{R}$. In addition, by part (b) of Lemma 7.5, if $x \in (-(1-\bar{\delta})m_n n^\kappa, Rn^\kappa)$, then for all sufficiently large n

$$nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \geq H(x) = \frac{1}{8}g^{(2)}(\bar{x})x^2.$$

Since $\exp[-H(x)]$ is integrable, the Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} I_{n_1} = \lim_{n\to\infty}\int_{-n^\kappa(1-\bar{\delta})m_n}^{Rn^\kappa} h_n(x)dx = \int_{\mathbb{R}} h(x)dx = \int_{\mathbb{R}} f(x)\exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx.$$

We conclude that

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} I_{n_1} + \lim_{n\to\infty} I_{n_2} = \int_{\mathbb{R}} f(x)\exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx.$$

This completes the proof of Lemma 7.7. $\square$

The next lemma collects several elementary but useful facts concerning the normal random variables W_n.

Lemma 7.8. *Let (β_n, K_n) be a positive sequence that converges either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. Let W_n be a sequence of normal random variables with mean 0 and variance $\sigma_n^2 = (2\beta_n K_n)^{-1}$ defined on a probability space $(\Omega, \mathcal{F}, Q)$. The following conclusions hold.*

(a) *For $b > 0$ and $\zeta > 0$ there exists a constant $c > 0$ such that for all n, $Q\{|W_n| > bn^\zeta\} \leq \exp[-cn^{2\zeta}]$.*

(b) *There exist a constant $c_1 > 0$ such that for all n*

$$\int_\Omega |W_n|^2 dQ \leq c_1 \;\; \textit{and} \;\; \int_\Omega |W_n| dQ \leq \sqrt{c_1}.$$

Proof. (a) We have the bound

$$Q\{|W_n| > bn^\zeta\} = \frac{\sqrt{2}}{\sqrt{\pi}\sigma_n}\int_{bn^\zeta}^\infty \exp[-x^2/2\sigma_n^2]dx \leq \frac{\sqrt{2}\sigma_n}{\sqrt{\pi}bn^\zeta}\exp[-b^2n^{2\zeta}/2\sigma_n^2].$$

Part (a) now follows from the fact that since (β_n, K_n) is a positive sequence converging to $(\beta, K(\beta))$ for $0 < \beta \leq \beta_c$, the positive sequences σ_n and σ_n^2 are bounded.

(b) Since $\int_\Omega |W_n|^2 dQ = \sigma_n^2$ and $\int_\Omega |W_n| dQ \leq (\int_\Omega |W_n|^2 dQ)^{1/2} = \sigma_n$, this follows from the fact that the positive sequences σ_n^2 and σ_n are bounded. The proof of the lemma is complete. $\square$

The next lemma is used in the proof of part (a) of Lemma 8.4. Under the hypotheses of Theorem 4.1, for any $0 < \alpha < \alpha_0$ the interval $(0, \frac{1}{2} - \theta\alpha)$ appearing in the next lemma is nonempty because by hypothesis (iii′) $\frac{1}{2} - \theta\alpha > \frac{1}{2} - \theta\alpha_0 > 0$.

Lemma 7.9. *We assume that (β_n, K_n) satisfies the hypotheses of Theorem 4.1 for all $0 < \alpha < \alpha_0$. For $\bar{\delta} \in (0,1)$ define*

$$A_n(\bar{\delta}) = \{S_n/n + W_n/n^{1/2} > \bar{\delta} m_n\},$$

where $m_n = m(\beta_n, K_n)$. Let $\Delta \in (0,1)$ be the number determined in part (b) of Lemma 7.5. Assume that $0 < \alpha < \alpha_0$ and choose any numbers $\delta_1, \delta, \delta_2$ and ζ satisfying $\Delta < \delta_1 < \delta < \delta_2 < 1$ and $\zeta \in (0, \frac{1}{2} - \theta\alpha)$. Then there exist constants $c > 0$ and $c_2 > 0$ such that the following conclusions hold.

(a) *For all sufficiently large n*

$$\begin{aligned} (P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\} + e^{-cn^{2\zeta}} &\geq P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\} \\ &\geq (P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\} - e^{-cn^{2\zeta}}. \end{aligned}$$

(b) *For all sufficiently large n*

$$\begin{aligned} &\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{A_n(\delta_1)}\} \\ &\qquad + 2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2} \\ &\geq \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{\{S_n/n > \delta m_n\}}\} \\ &\geq \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{A_n(\delta_2)}\} \\ &\qquad - 2n^\kappa e^{-cn^{2\zeta}} - c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}. \end{aligned}$$

Proof of part (a) of Lemma 7.9. We choose $\zeta \in (0, \frac{1}{2} - \theta\alpha)$. The proof is based on the following two claims, which are proved later.

Claim 1. For all sufficiently large n, $\{S_n/n > \delta m_n\} \subset A_n(\delta_1) \cup \{|W_n| > \frac{1}{2} n^\zeta\}$.

Claim 2. For all sufficiently large n, $\{S_n/n > \delta m_n\} \supset A_n(\delta_2) \backslash \{|W_n| > \frac{1}{2} n^\zeta\}$.

By Claims 1 and 2, for all sufficiently large n

$$\begin{aligned}
&(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\} + Q\{\{|W_n| > \tfrac{1}{2}n^\zeta\} \\
&\quad = (P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\} + (P_{n,\beta_n,K_n} \times Q)\{\{|W_n| > \tfrac{1}{2}n^\zeta\} \\
&\quad \geq (P_{n,\beta_n,K_n} \times Q)\{S_n/n > \delta m_n\} = P\{S_n/n > \delta m_n\} \\
&\quad \geq (P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\} - Q\{|W_n| > \tfrac{1}{2}n^\zeta\}.
\end{aligned}$$

Part (a) of Lemma 7.8 completes the proof. Thus, given Claims 1 and 2, the proof of part (a) is complete.

Proof of part (b) of Lemma 7.9. We use Claim 1 to prove the first inequality in part (b). For all sufficiently large n

$$\begin{aligned}
&\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{\{S_n/n > \delta m_n\}}\} \qquad (7.8)\\
&= \int_{\Lambda^n \times \Omega} |S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{\{S_n/n>\delta m_n\}} d(P_{n,\beta_n,K_n} \times Q) \\
&\leq \int_{\Lambda^n \times \Omega} |S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{A_n(\delta_1)} d(P_{n,\beta_n,K_n} \times Q) \\
&\quad + \int_{\Lambda^n \times \Omega} |S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{\{|W_n| > \frac{1}{2}n^\zeta\}} d(P_{n,\beta_n,K_n} \times Q) \\
&\leq \int_{\Lambda^n \times \Omega} |S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{A_n(\delta_1)} d(P_{n,\beta_n,K_n} \times Q) \\
&\quad + \int_{\Lambda^n \times \Omega} |S_n/n^{1-\kappa} - n^\kappa m_n| \cdot 1_{\{|W_n| > \frac{1}{2}n^\zeta\}} d(P_{n,\beta_n,K_n} \times Q) \\
&\quad + \int_\Omega |W_n/n^{1/2-\kappa}| \cdot 1_{\{|W_n| > \frac{1}{2}n^\zeta\}} dQ.
\end{aligned}$$

Since $|S_n/n| \leq 1$ and $m_n \in (0,1)$, we have $|S_n/n^{1-\kappa} - n^\kappa m_n| \leq 2n^\kappa$. Using part (a) of Lemma 7.8, for all sufficiently large n we bound the next to last integral in (7.8) by

$$2n^\kappa \cdot Q\{|W_n| > \tfrac{1}{2}n^\zeta\} \leq 2n^\kappa \exp(-cn^{2\zeta}),$$

where $c > 0$ is a constant. The next step is to apply the Cauchy-Schwartz inequality to the last integral in (7.8) and use parts (a) and (b) of Lemma 7.8. There exist constants $c > 0$ and $c_2 = \sqrt{c_1} > 0$ such that for all n

$$\begin{aligned}
&\int_\Omega |W_n/n^{1/2-\kappa}| \cdot 1_{\{|W_n| > \frac{1}{2}n^\zeta\}} dQ \\
&\quad \leq \left(\int_\Omega |W_n/n^{1/2-\kappa}|^2 dQ\right)^{1/2} \cdot \left(Q\{|W_n| > \tfrac{1}{2}n^\zeta\}\right)^{1/2} \leq c_2 n^{\kappa-1/2} \exp[-cn^{2\zeta}/2].
\end{aligned}$$

It follows that for all sufficiently large n

$$\begin{aligned}\tilde{E}_{n,\beta_n,K_n}&\{|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n|\cdot 1_{\{S_n/n>\delta m_n\}}\} \\ &\leq \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n|\cdot 1_{A_n(\delta_1)}\} \\ &\quad + 2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}.\end{aligned} \tag{7.9}$$

This completes the proof of the first inequality in part (b).

We now use Claim 2 to prove the second inequality in part (b). For all sufficiently large n

$$\begin{aligned}&\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n|\cdot 1_{\{S_n/n>\delta m_n\}}\} \\ &= \int_{\Lambda^n\times\Omega}|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n|\cdot 1_{\{S_n/n>\delta m_n\}} dP_{n,\beta_n,K_n} \\ &\geq \int_{\Lambda^n\times\Omega}|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n|\cdot 1_{A_n(\delta_2)} d(P_{n,\beta_n,K_n}\times Q) \\ &\quad - \int_{\Lambda^n\times\Omega}|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n|\cdot 1_{\{|W_n|>\frac{1}{2}n^\zeta\}} d(P_{n,\beta_n,K_n}\times Q) \\ &\geq \int_{\Lambda^n\times\Omega}|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n|\cdot 1_{A_n(\delta_2)} d(P_{n,\beta_n,K_n}\times Q) \\ &\quad - \int_{\Lambda^n\times\Omega}|S_n/n^{1-\kappa}-n^\kappa m_n|\cdot 1_{\{|W_n|>\frac{1}{2}n^\zeta\}} d(P_{n,\beta_n,K_n}\times Q) \\ &\quad - \int_{\Omega}|W_n/n^{1/2-\kappa}|\cdot 1_{\{|W_n|>\frac{1}{2}n^\zeta\}} dQ.\end{aligned} \tag{7.10}$$

The last two integrals in (7.10) coincide with the last two integrals in (7.8) and hence can be bounded the same way. For all sufficiently large n this yields

$$\begin{aligned}\tilde{E}_{n,\beta_n,K_n}&\{|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n|\cdot 1_{\{S_n/n>\delta m_n\}}\} \\ &\geq \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n|\cdot 1_{A_n(\delta_2)}\} \\ &\quad - 2n^\kappa e^{-cn^{2\zeta}} - c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2},\end{aligned} \tag{7.11}$$

where $c>0$ and $c_2>0$ are constants. In combination with (7.9), the last inequality yields part (b).

In order to complete the proofs of parts (a) and (b) we now turn to the proofs of Claims 1 and 2.

Proof of Claim 1. We write

$$\begin{aligned}\{S_n/n>\delta m_n\} &= (\{S_n/n>\delta m_n\}\cap\{|W_n|\leq \tfrac{1}{2}n^\zeta\})\cup(\{S_n/n>\delta m_n\}\cap\{|W_n|>\tfrac{1}{2}n^\zeta\}) \\ &\subset (\{S_n/n>\delta m_n\}\cap\{|W_n|\leq \tfrac{1}{2}n^\zeta\})\cup\{|W_n|>\tfrac{1}{2}n^\zeta\}.\end{aligned}$$

Claim 1 follows if we prove for all sufficiently large n

$$\{S_n/n > \delta m_n\} \cap \{|W_n| \leq \tfrac{1}{2}n^{\zeta}\} \subset A_n(\delta_1) = \{S_n/n + W_n/n^{1/2} > \delta_1 m_n\}. \qquad (7.12)$$

We have

$$\begin{aligned} \{S_n/n > \delta m_n\} \cap \{|W_n| \leq \tfrac{1}{2}n^{\zeta}\} &= (\{S_n/n > \delta m_n\} \cap \{0 \leq W_n \leq \tfrac{1}{2}n^{\zeta}\}) \\ &\quad \cup(\{S_n/n > \delta m_n\} \cap \{-\tfrac{1}{2}n^{\zeta} \leq W_n < 0\}). \end{aligned}$$

If $S_n/n > \delta m_n$ and $0 \leq W_n \leq \frac{1}{2}n^{\zeta}$, then $S_n/n + W_n/n^{1/2} \geq S_n/n > \delta m_n > \delta_1 m_n$. Thus

$$\{S_n/n > \delta m_n\} \cap \{0 \leq W_n \leq \tfrac{1}{2}n^{\zeta}\} \subset A_n(\delta_1). \qquad (7.13)$$

Now assume that $S_n/n > \delta m_n$ and $-\frac{1}{2}n^{\zeta} \leq W_n < 0$. Since $\zeta < \frac{1}{2} - \theta\alpha$, we have for all sufficiently large n

$$(\delta - \delta_1)\bar{x} > n^{\zeta - 1/2 + \theta\alpha}.$$

Since $\lim_{n\to\infty} n^{\theta\alpha} m_n = \bar{x}$ [Thm. 3.1], it follows that for all sufficiently large n

$$(\delta - \delta_1)m_n > \tfrac{1}{2}n^{\zeta - 1/2}.$$

Thus $\delta m_n - \frac{1}{2}n^{\zeta-1/2} > \delta_1 m_n$ for all sufficiently large n. Hence, if $S_n/n > \delta m_n$ and $W_n/n^{1/2} \geq -\frac{1}{2}n^{\zeta-1/2}$, then for all sufficiently large n

$$S_n/n + W_n/n^{1/2} > \delta m_n + W_n/n^{1/2} \geq \delta m_n - \tfrac{1}{2}n^{\zeta-1/2} > \delta_1 m_n.$$

It follows that for all sufficiently large n

$$\{S_n/n > \delta m_n\} \cap \{-n^{\zeta} \leq W_n < 0\} \subset A_n(\delta_1).$$

Therefore (7.12) follows from (7.13) and the last display. This completes the proof of Claim 1.

Proof of Claim 2. It suffices to prove that $A_n(\delta_2) \subset \{S_n/n > \delta m_n\} \cup \{|W_n| > \frac{1}{2}n^{\zeta}\}$. We write

$$\begin{aligned} A_n(\delta_2) &= \{S_n/n + W_n/n^{1/2} > \delta_2 m_n\} \\ &= (\{S_n/n + W_n/n^{1/2} > \delta_2 m_n\} \cap \{|W_n| > \tfrac{1}{2}n^{\zeta}\}) \\ &\quad \cup(\{S_n/n + W_n/n^{1/2} > \delta_2 m_n\} \cap \{|W_n| \leq \tfrac{1}{2}n^{\zeta}\}). \\ &\subset \{|W_n| > \tfrac{1}{2}n^{\zeta}\} \cup (\{S_n/n + W_n/n^{1/2} > \delta_2 m_n\} \cap \{|W_n| \leq \tfrac{1}{2}n^{\zeta}\}. \end{aligned}$$

Hence Claim 2 follows if we prove for all sufficiently large n

$$\{S_n/n + W_n/n^{1/2} > \delta_2 m_n\} \cap \{|W_n| \leq \tfrac{1}{2} n^\zeta\} \subset \{S_n/n > \delta m_n\}.$$

We omit the proof, which is similar to the proof of (7.12). The proof of Lemma 7.9 is complete. $\Box$

We now prove Lemma 7.5, completing the preparatory lemmas that will be used in the next section to prove part (b) of Theorem 6.1.

Proof of Lemma 7.5. This is done when g has degree 4. We omit the analogous but more complicated proof when g has degree 6.

Proof of part (a) of Lemma 7.5 when g has degree 4. By Taylor's theorem, for any $R > 0$, all $n \in \mathbb{N}$, and all $x \in \mathbb{R}$ satisfying $|x/n^\kappa| < R$

$$\begin{aligned} & nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \\ &= \sum_{j=1}^{4} \frac{1}{n^{j\kappa-1}} \cdot \frac{G^{(j)}_{\beta_n,K_n}(m_n)}{j!} x^j + \frac{1}{n^{5\kappa-1}} \cdot \frac{G^{(5)}_{\beta_n,K_n}(m_n + \tau x/n^\kappa)}{5!} x^5, \end{aligned}$$

where τ is a number in [0,1]. The quantity $m_n + \tau x/n^\kappa$ lies in the interval $[\, m_n - |x/n^\kappa|, m_n + |x/n^\kappa|\,]$. Since $m_n \in (0,1)$, $m_n \to 0$ and $|x/n^\kappa| < R$, we have $m_n + \tau x/n^\kappa \in (-R, R+1)$ for all n. Since the sequence (β_n, K_n) is bounded and positive, there exists $a \in (0,\infty)$ such that $0 \leq \beta_n \leq a$ and $0 \leq K_n \leq a$ for all n. As a continuous function of (β, K, y) on the compact set $[0,a] \times [0,a] \times [-R, R+1]$, it follows that $G^{(5)}_{\beta,K}(y)$ is uniformly bounded. Since $m_n + \tau x/n^\kappa \in (-R, R+1)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ satisfying $|x/n^\kappa| < R$, $G^{(5)}_{\beta_n,K_n}(m_n + \tau x/n^\kappa)$ is uniformly bounded for $n \in \mathbb{N}$ and $x \in (-Rn^\kappa, Rn^\kappa)$. We summarize the last display by writing

$$\begin{aligned} & nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \\ &= \sum_{j=1}^{4} \frac{1}{n^{j\kappa-1}} \cdot \frac{G^{(j)}_{\beta_n,K_n}(m_n)}{j!} x^j + \mathrm{O}\left(\frac{1}{n^{5\kappa-1}}\right) x^5, \end{aligned}$$

where the big-oh term is uniform for $x \in (-Rn^\kappa, Rn^\kappa)$.

Let ε_n denote a sequence that converges to 0 and that represents the various error terms arising in the proof. The same notation ε_n will be used to represent different error terms. To simplify the arithmetic, we introduce $u = 1 - \alpha/\alpha_0 > 0$. We have the following three properties:

(1) Since m_n is the unique positive, global minimum point of G_{β_n,K_n}, $G^{(1)}_{\beta_n,K_n}(m_n) = 0$.

(2) By hypothesis (iii′) of Theorem 4.1, for $j = 2, 3, 4$, we have

$$G^{(j)}_{\beta_n,K_n}(m_n) = (g^{(j)}(\bar{x}) + \varepsilon_n)/n^{\alpha/\alpha_0 - j\theta\alpha} = (g^{(j)}(\bar{x}) + \varepsilon_n)/n^{1-u-j\theta\alpha}.$$

(3) Since $\kappa = \frac{1}{2}u + \theta\alpha$, we have $j\kappa - u - j\theta\alpha = \left(\frac{j}{2} - 1\right)u$ for $j = 2, 3, 4$.

Using these properties, we obtain the following asymptotic formula, which is valid for any $R > 0$, all $n \in \mathbb{N}$, and all $x \in \mathbb{R}$ satisfying $|x/n^\kappa| < R$:

$$\begin{aligned} nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) &= \sum_{j=2}^{4} \frac{1}{n^{j\kappa-1}} \cdot \frac{g^{(j)}(\bar{x}) + \varepsilon_n}{n^{1-u-j\theta\alpha} \cdot j!} \cdot x^j + \mathrm{O}\left(\frac{1}{n^{5\kappa-1}}\right) x^5 \\ &= \sum_{j=2}^{4} \frac{1}{j!} \cdot \frac{g^{(j)}(\bar{x}) + \varepsilon_n}{n^{(j/2-1)u}} \cdot x^j + \mathrm{O}\left(\frac{1}{n^{5\kappa-1}}\right) x^5 \\ &= \frac{1}{2!} \cdot (g^{(2)}(\bar{x}) + \varepsilon_n)x^2 + \frac{1}{3!} \cdot \frac{(g^{(3)}(\bar{x}) + \varepsilon_n)}{n^{u/2}} x^3 \\ &\quad + \frac{1}{4!} \cdot \frac{(g^{(4)}(\bar{x}) + \varepsilon_n)}{n^u} x^4 + \mathrm{O}\left(\frac{1}{n^{5\kappa-1}}\right) x^5. \end{aligned}$$

By hypothesis (iii′) of Theorem 4.1 and part (c) of Theorem 6.1, we have $1/4 \leq \theta\alpha_0 < \kappa < 1/2$. Therefore $5\kappa - 1 > 5\theta\alpha_0 - 1 > 0$. Since $u > 0$ and $\varepsilon_n \to 0$, we have for all $x \in \mathbb{R}$

$$\lim_{n\to\infty} (nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n)) = \frac{1}{2} g^{(2)}(\bar{x})x^2.$$

This completes the proof of part (a) of Lemma 7.5 when g has degree 4.

Proof of part (b) of Lemma 7.5 when g has degree 4. Hypothesis (iii′) of Theorem 4.1 states that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4$. It follows that for all sufficiently large n and all $x \in \mathbb{R}$

$$\frac{1}{2!} \cdot (g^{(2)}(\bar{x}) + \varepsilon_n)x^2 \geq \frac{1}{2 \cdot 2!} \cdot g^{(2)}(\bar{x})x^2$$

and

$$\frac{1}{4!} \cdot \frac{(g^{(4)}(\bar{x}) + \varepsilon_n)}{n^u} x^4 \geq \frac{1}{2 \cdot 4!} \cdot \frac{g^{(4)}(\bar{x})}{n^u} x^4$$

and that for all sufficiently large n

$$\frac{1}{3!} \cdot \frac{g^{(3)}(\bar{x}) + \varepsilon_n}{n^{u/2}} x^3 \geq \frac{1}{2 \cdot 3!} \cdot \frac{g^{(3)}(\bar{x})}{n^{u/2}} x^3 \ \text{ for all } \ x \geq 0$$

and

$$\frac{1}{3!} \cdot \frac{g^{(3)}(\bar{x}) + \varepsilon_n}{n^{u/2}} x^3 \geq \frac{2}{3!} \cdot \frac{g^{(3)}(\bar{x})}{n^{u/2}} x^3 \text{ for all } x < 0.$$

We first consider $x \in [0, Rn^\kappa)$. Since $g^{(4)}(\bar{x}) > 0$, for all sufficiently large n and all such x we have

$$\begin{aligned}
& nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \qquad (7.14)\\
\geq\ & \frac{1}{2 \cdot 2!} \cdot g^{(2)}(\bar{x}) x^2 + \frac{1}{2 \cdot 3!} \cdot \frac{g^{(3)}(\bar{x})}{n^{u/2}} x^3 \\
& + \frac{1}{2 \cdot 4!} \cdot \frac{g^{(4)}(\bar{x})}{n^u} x^4 + \frac{1}{n^u} \mathrm{O}\Big(\frac{x}{n^{5\kappa-1-u}}\Big) x^4 \\
\geq\ & \frac{1}{2 \cdot 2!} \cdot g^{(2)}(\bar{x}) x^2 + \frac{1}{2 \cdot 3!} \cdot \frac{g^{(3)}(\bar{x})}{n^{u/2}} x^3 \\
& + \frac{1}{2 \cdot 4!} \cdot \frac{g^{(4)}(\bar{x})}{n^u} x^4 (1 + \mathrm{O}(x/n^{5\kappa-1-u})).
\end{aligned}$$

By hypothesis (iii′) of Theorem 4.1, $\theta\alpha_0 \in [1/4, 1/2)$. Hence $4\theta - 1/\alpha_0 \geq 0$, and so

$$5\kappa - 1 - u = \kappa + (4\theta - 1/\alpha_0)\alpha \geq \kappa.$$

Hence for all $0 < x < Rn^\kappa$ we have $0 \leq x/n^{5\kappa-1-u} \leq x/n^\kappa < R$. Thus the term $\mathrm{O}(x/n^{5\kappa-1-u})$ appearing in (7.14) can be made larger than -1 for all $0 \leq x/n^\kappa < R$ by choosing R to be sufficiently small. Since $g^{(3)}(\bar{x}) > 0$, $g^{(4)}(\bar{x}) > 0$, and $1 + \mathrm{O}(x/n^{5\kappa-1-u}) > 0$, we have that for all sufficiently large n and all $x \in [0, Rn^\kappa)$

$$nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \geq \frac{1}{2 \cdot 2!} g^{(2)}(\bar{x}) x^2 \geq \frac{1}{8} g^{(2)}(\bar{x}) x^2.$$

This is the conclusion of part (b) of Lemma 7.5 for all $0 \leq x < Rn^\kappa$ when g has degree 4.

We now consider $x \in (-Rn^\kappa, 0]$. Since $g^{(4)}(\bar{x}) > 0$, for all sufficiently large n and all such x we have

$$\begin{aligned}
nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \ \geq\ & \frac{1}{2 \cdot 2!} \cdot g^{(2)}(\bar{x}) x^2 + \frac{2}{3!} \cdot \frac{g^{(3)}(\bar{x})}{n^{u/2}} x^3 \qquad (7.15)\\
& + \frac{1}{2 \cdot 4!} \cdot \frac{g^{(4)}(\bar{x})}{n^u} x^4 + \frac{1}{n^u} \mathrm{O}\Big(\frac{x}{n^{5\kappa-1-u}}\Big) x^4 \\
\geq\ & \frac{1}{2 \cdot 2!} \cdot g^{(2)}(\bar{x}) x^2 + \frac{2}{3!} \cdot \frac{g^{(3)}(\bar{x})}{n^{u/2}} x^3 \\
& + \frac{1}{2 \cdot 4!} \cdot \frac{g^{(4)}(\bar{x})}{n^u} x^4 (1 + \mathrm{O}(x/n^{5\kappa-1-u})).
\end{aligned}$$

Since $5\kappa - 1 - u \geq \kappa$, for all $-Rn^\kappa < x < 0$ we have $-R < x/n^\kappa \leq x/n^{5\kappa-1-u} < 0$. Thus the term $\mathrm{O}(x/n^{5\kappa-1-u})$ appearing in (7.15) can be made larger than -1 for all $-R < x/n^\kappa < 0$ by choosing R to be sufficiently small. Since $g^{(4)}(\bar{x}) > 0$ and $1 + \mathrm{O}(x/n^{5\kappa-1-u}) > 0$, we have that for all sufficiently large n and all $x \in (-Rn^\kappa, 0)$

$$nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \geq \frac{1}{2 \cdot 2!} g^{(2)}(\bar{x})x^2 + \frac{2}{3!} \cdot \frac{g^{(3)}(\bar{x})}{n^{u/2}} x^3.$$

By Theorem 3.1 we have $m_n \sim \bar{x}/n^{\theta\alpha}$. Thus $n^{\theta\alpha} m_n = \bar{x} + \varepsilon_n$, and

$$n^\kappa m_n = n^{u/2} \cdot n^{\theta\alpha} m_n = n^{u/2}(\bar{x} + \varepsilon_n).$$

In part (b) of Lemma 7.5 we assume that $x/n^\kappa > -(1-\bar{\delta})m_n$ and $0 < \bar{\delta} < 1$. Thus for all sufficiently large n and all such x

$$x > -(1-\bar{\delta})n^\kappa m_n = -(1-\bar{\delta})n^{u/2}(\bar{x} + \varepsilon_n) \geq -(1-\bar{\delta})n^{u/2} \cdot 2\bar{x}.$$

Since $g^{(3)}(\bar{x}) > 0$, we see that for all sufficiently large n, all $x \in (-Rn^\kappa, 0)$, and all $x/n^\kappa > -(1-\bar{\delta})m_n$ there exists $\Delta \in (0,1)$ such that for any $\bar{\delta} \in (\Delta, 1)$ the following inequalities hold:

$$\begin{aligned}
& nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \\
\geq \;& \frac{1}{2 \cdot 2!} \cdot g^{(2)}(\bar{x})x^2 + \frac{2}{3!} \cdot \frac{g^{(3)}(\bar{x})}{n^{u/2}} x^2 \cdot [-(1-\bar{\delta})n^{u/2} \cdot 2\bar{x}] \\
= \;& \left(\frac{1}{2 \cdot 2!} \cdot g^{(2)}(\bar{x}) - \frac{2 \cdot 2}{3!} \cdot g^{(3)}(\bar{x})(1-\bar{\delta})\bar{x} \right) x^2 \\
\geq \;& \frac{1}{2} \cdot \frac{1}{2 \cdot 2!} g^{(2)}(\bar{x})x^2 = \frac{1}{8} g^{(2)}(\bar{x})x^2.
\end{aligned}$$

This is the conclusion of part (b) of Lemma 7.5 for $-Rn^\kappa < x < 0$ and $x/n^\kappa > -(1-\bar{\delta})m_n$ when g has degree 4.

We have shown that for any $\bar{\delta} \in (\Delta, 1)$ there exists $R > 0$ such that for all sufficiently large n and all $x \in \mathbb{R}$ satisfying $|x/n^\kappa| < R$ and $x/n^\kappa > -(1-\bar{\delta})m_n$

$$nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \geq \frac{1}{8} g^{(2)}(\bar{x})x^2.$$

This completes the proof of part (b) of Lemma 7.5 when g has degree 4. Because we are omitting the proof of part (b) when g has degree 6, the proof of Lemma 7.5 is complete. $\square$

We have completed the statements and proofs of the preparatory lemmas. We now turn to the proof of part (b) of Theorem 6.1.

8 Proof of Part (b) of Theorem 6.1

As we saw at the start of section 7, the proof of part (b) of Theorem 6.1 involves two steps. Step 1 is proved in the next subsection. Step 2 is subdivided into two substeps. Substep 2a is proved in subsection 8b, and Substep 2b is proved in subsection 8c.

8a Proof of Step 1 in Proof of Theorem 6.1 (b)

Part (b) of the following lemma states Step 1 in the proof of part (b) of Theorem 6.1. We recall that W_n is a sequence of normal random variables with mean 0 and variance $(2\beta_n K_n)^{-1}$ defined on a probability space $(\Omega, \mathcal{F}, Q)$. We denote by $\tilde{E}_{n,\beta_n,K_n}$ expectation with respect to the product measure $P_{n,\beta_n,K_n} \times Q$; P_{n,β_n,K_n} is defined in (2.1)–(2.2).

Lemma 8.1. *We assume that (β_n, K_n) satisfies the hypotheses of Theorem 4.1 for all $0 < \alpha < \alpha_0$. For $\bar{\delta} \in (0,1)$ define*

$$A_n(\bar{\delta}) = \{S_n/n + W_n/n^{1/2} > \bar{\delta} m_n\}$$

where $m_n = m(\beta_n, K_n)$. Let $\Delta \in (0,1)$ be the number determined in part (b) of Lemma 7.5. Then for any $0 < \alpha < \alpha_0$ and any $\bar{\delta} \in (\Delta, 1)$, the following conclusions hold.

(a) *We have the limit*

$$\begin{aligned}
&\lim_{n\to\infty} \int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty} |x| \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx \\
&\quad = \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2} g^{(2)}(\bar{x}) x^2] dx.
\end{aligned}$$

(b) *We have the limit*

$$\begin{aligned}
&\lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid A_n(\bar{\delta})\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2} g^{(2)}(\bar{x}) x^2] dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2} g^{(2)}(\bar{x}) x^2] dx \\
&\quad = \bar{z}.
\end{aligned}$$

Proof of part (a) of Lemma 8.1. Let Ψ_n and Ψ denote the measures on $\mathbb{R}$ defined by

$$\Psi_n(dx) = 1_{(-n^\kappa(1-\bar{\delta})m_n, \infty)}(x) \cdot \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx$$

and

$$\Psi(dx) = \exp[-\tfrac{1}{2} g^{(2)}(\bar{x}) x^2] dx.$$

According to Lemma 7.7, Ψ_n converges weakly to Ψ. The limit in part (a) of Lemma 8.1 can be expressed as

$$\lim_{n\to\infty} \int_{\mathbb{R}} |x| d\Psi_n = \int_{\mathbb{R}} |x| d\Psi.$$

As discussed in Theorem 4 in §II.6 of [15], this limit would follow from the weak convergence of Ψ_n to Ψ if one could prove the uniform integrability estimate

$$\lim_{j\to\infty} \sup_{n\in\mathbb{R}} \int_{\{|x|>j\}} |x| d\Psi_n = 0.$$

The next proposition shows that the limit $\lim_{n\to\infty} \int_{\mathbb{R}} |x| d\Psi_n = \int_{\mathbb{R}} |x| d\Psi$ is a consequence of a condition that is weaker than uniform integrability.

Proposition 8.2. *Let Ψ_n be a sequence of measures on $\mathbb{R}$ that converges weakly to a measure Ψ on $\mathbb{R}$. Assume in addition that $\int_{\mathbb{R}} |x| d\Psi < \infty$ and that*

$$\lim_{j\to\infty} \limsup_{n\to\infty} \int_{\{|x|>j\}} |x| d\Psi_n = 0.$$

It then follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |x| d\Psi_n = \int_{\mathbb{R}} |x| d\Psi.$$

Proof. Since $\Psi_n \Longrightarrow \Psi$, we have $\Psi_n(\mathbb{R}) \to \Psi(\mathbb{R})$. Hence the proposition is a consequence of Proposition 8.3 in [12] applied to the sequence of probability measures

$$\frac{1}{\Psi_n(\mathbb{R})} \cdot \Psi_n(dx) \Longrightarrow \frac{1}{\Psi(\mathbb{R})} \cdot \Psi(dx).$$

This completes the proof. □.

We now verify the following hypotheses of Proposition 8.2 for the measures Ψ_n and Ψ:

(1) $\Psi_n \Longrightarrow \Psi$.

(2) $\int_{\mathbb{R}} |x| d\Psi < \infty$.

(3) $\lim_{j\to\infty} \limsup_{n\to\infty} \int_{\{|x|>j\}} |x| d\Psi_n = 0$.

Item (1) is proved in Lemma 7.7, and item (2) is immediate from the definition of Ψ. We now prove item (3). Since

$$\int_{\{|x|>j\}} |x| d\Psi_n = \int_{\{|x|>j\}\cap\{x>-n^\kappa(1-\bar{\delta})m_n\}} |x| d\Psi_n,$$

we can prove item (3) by showing that

$$\lim_{j\to\infty} \limsup_{n\to\infty} \int_{\{|x|>j\}\cap\{x>-n^\kappa(1-\bar{\delta})m_n\}} |x| d\Psi_n = 0.$$

In order to do this we find, for any $j \in \mathbb{N}$ and all sufficiently large n, quantities A_j, B_n and C_n with the properties that

$$\int_{\{|x|>j\}\cap\{x>-n^\kappa(1-\bar{\delta})m_n\}} |x| d\Psi_n \leq A_j + B_n + C_n,$$

$A_j \to 0$ as $j \to \infty$, $B_n \to 0$ as $n \to \infty$, and $C_n \to 0$ as $n \to \infty$.

We now specify the quantities A_j, B_n and C_n. Given positive integers j and n, let R and K be positive numbers that satisfy $K > R$ and that will be specified below. Then

$$\begin{aligned}
&\{|x| > j\} \cap \{x/n^\kappa > -(1-\bar{\delta})m_n\} \\
&\quad = \quad [\{|x| > j\} \cap \{|x/n^\kappa| < R\} \cap \{x/n^\kappa > -(1-\bar{\delta})m_n\}] \\
&\qquad\quad \cup[\{|x| > j\} \cap \{R \leq |x/n^\kappa| < K\} \cap \{x/n^\kappa > -(1-\bar{\delta})m_n\}] \\
&\qquad\quad \cup[\{|x| > j\} \cap \{|x/n^\kappa| \geq K\} \cap \{x/n^\kappa > -(1-\bar{\delta})m_n\}] \\
&\quad \subset \quad [\{|x| > j\} \cap \{|x/n^\kappa| < R\} \cap \{x/n^\kappa > -(1-\bar{\delta})m_n\}] \\
&\qquad\quad \cup[\{R \leq |x/n^\kappa| < K\} \cap \{x/n^\kappa > -(1-\bar{\delta})m_n\}] \cup \{|x/n^\kappa| \geq K\}.
\end{aligned}$$

Since $m_n \to 0$, for all sufficiently large n

$$\{R \leq |x/n^\kappa| < K\} \cap \{x/n^\kappa > -(1-\bar{\delta})m_n\} = \{R \leq x/n^\kappa < K\}.$$

Hence for all sufficiently large n

$$\begin{aligned}
&\int_{\{|x|>j\}\cap\{x/n^\kappa>-(1-\bar{\delta})m_n\}} |x| d\Psi_n \qquad (8.1)\\
&\quad \leq \int_{\{|x|>j\}\cap\{|x/n^\kappa|<R\}\cap\{x>-n^\kappa(1-\bar{\delta})m_n\}} |x| d\Psi_n \\
&\qquad + \int_{\{R\leq x/n^\kappa<K\}} |x| d\Psi_n + \int_{\{|x/n^\kappa|\geq K\}} |x| d\Psi_n.
\end{aligned}$$

We next estimate each of these three integrals. By part (b) of Lemma 7.5, there exists $\Delta \in (0,1)$ such that for any $\bar{\delta} \in (\Delta, 1)$ there exists $R > 0$ such that for all sufficiently large n and all $x \in \mathbb{R}$ satisfying $|x/n^\kappa| < R$ and $x/n^\kappa > -(1-\bar{\delta})m_n$

$$nG_{\beta_n,K_n}(x/n^\kappa + m_n) - nG_{\beta_n,K_n}(m_n) \geq H(x) = \frac{1}{8} g^{(2)}(\bar{x})x^2.$$

Since $\exp[-H(x)]$ is integrable, for all sufficiently large n we estimate the first integral on the right hand side of equation (8.1) by

$$\begin{aligned} &\int_{\{|x|>j\}\cap\{|x/n^\kappa|<R\}\cap\{x>-n^\kappa(1-\bar{\delta})m_n\}} |x| d\Psi_n \\ &\qquad \leq A_j = \int_{\{|x|>j\}} |x| \cdot \exp[-H(x)]dx \to 0 \;\text{ as }\; j \to \infty. \end{aligned} \tag{8.2}$$

By part (a) of Lemma 4.4 in [6], there exists $K > 0$ and $D_1 > 0$ such that $G_{\beta_n,K_n}(x) \geq D_1 x^2$ for all $|x| > K$. Since $m_n \to 0$, it follows that for all sufficiently large n and all $x \in \mathbb{R}$ satisfying $|x/n^\kappa| \geq K$, there exists $D > 0$ such that

$$nG_{\beta_n,K_n}(x/n^\kappa + m_n) \geq nD_1(x/n^\kappa + m_n)^2 \geq nD(x/n^\kappa)^2.$$

Without loss of generality K can be chosen to be larger than the quantity R specified in the preceding paragraph. By Lemma 7.3, for all sufficiently large n there exists $\varepsilon_n \to 0$ such that

$$G_{\beta_n,K_n}(m_n) = \frac{g(\bar{x}) + \varepsilon_n}{n^{\alpha/\alpha_0}} \leq \frac{g(\bar{x})}{2n^{\alpha/\alpha_0}} < 0.$$

These bounds allow us to estimate the third integral on the right hand side of equation (8.1) by

$$\begin{aligned} &\int_{\{|x/n^\kappa| \geq K\}} |x| d\Psi_n \\ &\quad \leq \int_{\{|x/n^\kappa| \geq K\}} |x| \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx \\ &\quad \leq \int_{\{|x/n^\kappa| \geq K\}} |x| \exp[-nD(x/n^\kappa)^2]dx \\ &\quad \leq C_n = \frac{2}{D} \cdot n^{2\kappa-1} \exp[-nDK^2] \to 0 \;\text{ as }\; n \to \infty. \end{aligned} \tag{8.3}$$

With these choices of R and K, we use Lemma 7.6 to estimate the second integral on the right hand side of (8.1). There exists $c_2 > 0$ such that for all sufficiently large n

$$\begin{aligned}
&\int_{\{R \leq x/n^\kappa < K\}} |x| d\Psi_n \\
&\quad = \int_{\{R \leq x/n^\kappa < K\}} |x| \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx \\
&\quad \leq Kn^\kappa \int_{\{x/n^\kappa \geq R\}} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx \\
&\quad \leq B_n = Kn^\kappa \exp[-c_2 n] \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{8.4}$$

Together equations (8.2), (8.4) and (8.3) prove (8.1). This completes the proof of part (a) of Lemma 8.1.

Proof of part (b) of Lemma 8.1. Part (b) of Lemma 7.2 states that

$$\begin{aligned}
&\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid A_n(\bar{\delta})\} \\
&\quad = \frac{\int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty} |x| \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx}{\int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx}.
\end{aligned}$$

Hence by part (a) of Lemma 8.1 and Lemma 7.7 for $f(x) \equiv 1$, the last integral has the limit

$$\frac{\int_{\mathbb{R}} |x| \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} = \bar{z}.$$

This completes the proof of part (b) of Lemma 8.1 and hence the proof of the lemma. □

Having completed Step 1 in the proof of part (b) of Theorem 6.1, we now turn to Substep 2a.

8b Proof of Substep 2a in Proof of Theorem 6.1 (b)

Lemma 8.3 proves Substep 2a of part (b) of Theorem 6.1. We recall that W_n is a sequence of normal random variables with mean 0 and variance $(2\beta_n K_n)^{-1}$ defined on a probability space $(\Omega, \mathcal{F}, Q)$. We denote by $\tilde{E}_{n,\beta_n,K_n}$ expectation with respect to the product measure $P_{n,\beta_n,K_n} \times Q$.

Lemma 8.3. *We assume that* (β_n, K_n) *satisfies the hypotheses of Theorem* 4.1 *for all* $0 < \alpha < \alpha_0$. *Denote* $m_n = m(\beta_n, K_n)$. *For* $\delta \in (0,1)$ *define*

$$C_n = \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\}$$

and

$$D_n = \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \bigm| S_n/n > \delta m_n\}.$$

Then $\lim_{n\to\infty} |C_n - D_n| = 0$.

Proof. By part (b) of Lemma 7.8 there exists a constant $c_1 > 0$ such that for all n

$$\tilde{E}_{n,\beta_n,K_n}\{|W_n/n^{1/2-\kappa}|\} \leq \sqrt{c_1}/n^{1/2-\kappa}.$$

By Lemma 6.3

$$\lim_{n\to\infty} P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\} = 1/2.$$

It follows that there exists a constant $c_2 > 0$ such that for all sufficiently large n

$$\begin{aligned}
\tilde{E}_{n,\beta_n,K_n}\{|W_n/n^{1/2-\kappa}| \bigm| S_n/n > \delta m_n\} &= \frac{\tilde{E}_{n,\beta_n,K_n}\{|W_n/n^{1/2-\kappa}| \cdot 1_{\{S_n/n > \delta m_n\}}\}}{P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\}} \\
&\leq c_2/n^{1/2-\kappa}.
\end{aligned}$$

Since

$$\begin{aligned}
|S_n/n^{1-\kappa} - n^\kappa m_n| - |W_n/n^{1/2-\kappa}| &\leq |S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \\
&\leq |S_n/n^{1-\kappa} - n^\kappa m_n| + |W_n/n^{1/2-\kappa}|,
\end{aligned}$$

we have for all sufficiently large n

$$\begin{aligned}
&D_n + c_2/n^{1/2-\kappa} \qquad (8.5)\\
&= \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \bigm| S_n/n > \delta m_n\} + c_2/n^{1/2-\kappa} \\
&= \frac{\int_{\Lambda^n\times\Omega} |S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{\{S_n/n > \delta m_n\}} d(P_{n,\beta_n,K_n} \times Q)}{E_{n,\beta_n,K_n}\{1_{\{S_n/n > \delta m_n\}}\}} + c_2/n^{1/2-\kappa} \\
&\geq \frac{1}{E_{n,\beta_n,K_n}\{1_{\{S_n/n > \delta m_n\}}\}} \cdot \Big(\int_{\Lambda^n\times\Omega} |S_n/n^{1-\kappa} - n^\kappa m_n| \cdot 1_{\{S_n/n > \delta m_n\}} d(P_{n,\beta_n,K_n} \times Q) \\
&\quad - \int_{\Lambda^n\times\Omega} |W_n/n^{1/2-\kappa}| \cdot 1_{\{S_n/n > \delta m_n\}} d(P_{n,\beta_n,K_n} \times Q)\Big) \\
&\quad + \tilde{E}_{n,\beta_n,K_n}\{|W_n/n^{1/2-\kappa}| \bigm| S_n/n > \delta m_n\} \\
&= E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \bigm| S_n/n > \delta m_n\} = C_n
\end{aligned}$$

and

$$
\begin{aligned}
&D_n - c_2/n^{1/2-\kappa} \qquad\qquad (8.6)\\
&= \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\} - c_2/n^{1/2-\kappa} \\
&= \frac{\int_{\Lambda^n \times \Omega} |S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{\{S_n/n > \delta m_n\}} d(P_{n,\beta_n,K_n} \times Q)}{E_{n,\beta_n,K_n}\{1_{\{S_n/n > \delta m_n\}}\}} - c_2/n^{1/2-\kappa} \\
&\le \frac{1}{E_{n,\beta_n,K_n}\{1_{\{S_n/n > \delta m_n\}}\}} \cdot \Big(\int_{\Lambda^n \times \Omega} |S_n/n^{1-\kappa} - n^\kappa m_n| \cdot 1_{\{S_n/n > \delta m_n\}} d(P_{n,\beta_n,K_n} \times Q) \\
&\quad + \int_{\Lambda^n \times \Omega} |W_n/n^{1/2-\kappa}| \cdot 1_{\{S_n/n > \delta m_n\}} d(P_{n,\beta_n,K_n} \times Q) \Big) \\
&\quad - \tilde{E}_{n,\beta_n,K_n}\{|W_n/n^{1/2-\kappa}| \mid S_n/n > \delta m_n\} \\
&= E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\} = C_n.
\end{aligned}
$$

Thus we obtain for any $\delta \in (0,1)$ and all sufficiently large n

$$D_n + c_2/n^{1/2-\kappa} \ge C_n \ge D_n - c_2/n^{1/2-\kappa}.$$

Since $\kappa < 1/2$ [Thm. 6.1(c)], it follows that $\lim_{n\to\infty} |C_n - D_n| = 0$. This completes the proof of Lemma 8.3. $\square$

Having proved Substep 2a in the proof of part (b) of Theorem 6.1, we next turn to Substep 2b.

8c Proof of Substep 2b in Proof of Theorem 6.1 (b)

We recall that W_n is a sequence of normal random variables with mean 0 and variance $(2\beta_n K_n)^{-1}$ defined on a probability space $(\Omega, \mathcal{F}, Q)$. We denote by $\tilde{E}_{n,\beta_n,K_n}$ expectation with respect to the product measure $P_{n,\beta_n,K_n} \times Q$. Substep 2b in the proof of part (b) of Theorem 6.1 states the following:

$$
\begin{aligned}
&\lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\} \qquad (8.7)\\
&\quad = \lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n + W_n/n^{1/2} > \delta m_n\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2} g^{(2)}(\bar{x}) x^2] dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2} g^{(2)}(\bar{x}) x^2] dx = \bar{z}.
\end{aligned}
$$

This will be proved in Lemma 8.4.

Part (a) of Lemma 8.4 relates the expectation of $|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n|$ conditioned on the event $\{S_n/n > \delta m_n\}$ to the expectation of the same random variable conditioned on the event $A_n(\bar{\delta})$ for two choices of $\bar{\delta}$. Part (b) of the next lemma proves (8.7). The hypotheses of this lemma coincide with the hypotheses of Lemma 7.9 together with the additional condition $\zeta > \frac{1}{2}(1-\alpha/\alpha_0)$, which is used to prove $\tilde{\Theta}_{n,1} \to 0$, $\tilde{\Theta}_{n,3} \to 0$, $\tilde{\Gamma}_{n,1} \to 0$, and $\tilde{\Gamma}_{n,3} \to 0$ in part (a). According to part (c) of Theorem 6.1, $\frac{1}{2}(1-\alpha/\alpha_0)+\theta\alpha = \kappa < 1/2$, which implies $\frac{1}{2}(1-\alpha/\alpha_0) < \frac{1}{2}-\theta\alpha$. This additional condition on ζ is consistent with the hypothesis on ζ in Lemma 7.9, which is $0 < \zeta < \frac{1}{2}-\theta\alpha$.

Lemma 8.4. *We assume that (β_n, K_n) satisfies the hypotheses of Theorem 4.1 for all $0 < \alpha < \alpha_0$. For $\bar{\delta} \in (0,1)$ define*

$$A_n(\bar{\delta}) = \{S_n/n + W_n/n^{1/2} > \bar{\delta} m_n\}$$

where $m_n = m(\beta_n, K_n)$. Let $\Delta \in (0,1)$ be the number determined in part (b) of Lemma 7.5. Assume that $0 < \alpha < \alpha_0$ and choose any numbers $\delta_1, \delta, \delta_2$ and ζ satisfying $\Delta < \delta_1 < \delta < \delta_2 < 1$ and $\zeta \in (\frac{1}{2}(1-\alpha/\alpha_0), \frac{1}{2}-\theta\alpha)$. The following conclusions hold.

(a) *There exists sequences $\tilde{\Theta}_{n,1} \to 0$, $\tilde{\Theta}_{n,2} \to 1$, $\tilde{\Theta}_{n,3} \to 0$, $\tilde{\Gamma}_{n,1} \to 0$, $\tilde{\Gamma}_{n,2} \to 1$, and $\tilde{\Gamma}_{n,3} \to 0$ such that for all sufficiently large n*

$$\begin{aligned} &\frac{\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n| \bigm| A_n(\delta_1)\} + \tilde{\Theta}_{n,1}}{\tilde{\Theta}_{n,2}-\tilde{\Theta}_{n,3}} \\ &\quad \geq \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n| \bigm| S_n/n > \delta m_n\} \\ &\quad \geq \frac{\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n| \bigm| A_n(\delta_2)\} - \tilde{\Gamma}_{n,1}}{\tilde{\Gamma}_{n,2}+\tilde{\Gamma}_{n,3}}. \end{aligned}$$

(b) *We have the conditional limit*

$$\begin{aligned} &\lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n| \bigm| S_n/n > \delta m_n\} \\ &\quad = \lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n| \bigm| A_n(\delta)\} \\ &\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx = \bar{z}. \end{aligned}$$

Proof of part (a) of Lemma 8.4. The hypotheses of this lemma are a subset of the hypotheses of Lemma 7.9. We start by proving the first inequality in part (a). By the first inequality in part

(b) of Lemma 7.9 and the second inequality in part (a) of Lemma 7.9 we have for all sufficiently large n

$$\begin{aligned}
&\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \bigm| S_n/n > \delta m_n\} \\
&\quad = \frac{\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{\{S_n/n > \delta m_n\}}\}}{P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\}} \\
&\quad \leq \frac{\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{A_n(\delta_1)}\} + 2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\} - e^{-cn^{2\zeta}}} \\
&\quad = \frac{\frac{\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{A_n(\delta_1)}\}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}} + \frac{2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}}}{\frac{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}} - \frac{e^{-cn^{2\zeta}}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}}} \\
&\quad = \frac{\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \bigm| A_n(\delta_1)\} + \tilde{\Theta}_{n,1}}{\tilde{\Theta}_{n,2} \ - \ \tilde{\Theta}_{n,3}},
\end{aligned}$$

where $c > 0$ and $c_2 > 0$ are constants and

$$\tilde{\Theta}_{n,1} = \frac{2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}}, \tag{8.8}$$

$$\tilde{\Theta}_{n,2} = \frac{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}}, \tag{8.9}$$

and

$$\tilde{\Theta}_{n,3} = \frac{e^{-cn^{2\zeta}}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}}. \tag{8.10}$$

We prove the first inequality in part (a) of the present lemma by showing that, as $n \to \infty$, $\tilde{\Theta}_{n,1} \to 0$, $\tilde{\Theta}_{n,2} \to 1$ and $\tilde{\Theta}_{n,3} \to 0$. These limits hold for any $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$. By (7.4) in part (a) of Lemma 7.2 with $\bar{\delta} = \delta_1$

$$\begin{aligned}
\tilde{\Theta}_{n,1} &= \frac{2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}} \qquad (8.11)\\
&= \frac{2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}}{\int_{-n^\kappa(1-\delta_1)m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n)]dx / Z_{n,\kappa}} \\
&= \frac{\exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \cdot (2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2})}{\int_{-n^\kappa(1-\delta_1)m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx}.
\end{aligned}$$

We now use Lemma 7.7 with $f \equiv 1$ and $\bar{\bar{\delta}} = \delta_1$. This gives

$$\lim_{n\to\infty} \int_{-n^\kappa(1-\delta_1)m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx = \int_{\mathbb{R}} \exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx.$$

By Lemma 7.4, for any $\varepsilon > 0$ and all sufficiently large n

$$\begin{aligned}&\exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \cdot (2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}) \\ &\quad \le 2n^\kappa \exp[\varepsilon n^{1-\alpha/\alpha_0} - cn^{2\zeta}] + c_2 n^{\kappa-1/2} \exp[\varepsilon n^{1-\alpha/\alpha_0} - cn^{2\zeta}/2].\end{aligned}$$

Since by hypothesis $\zeta > \frac{1}{2}(1-\alpha/\alpha_0)$, it follows that

$$\lim_{n\to\infty} \exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \cdot (2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}) = 0.$$

It follows from the last line of (8.11) that

$$\begin{aligned}\lim_{n\to\infty} \tilde{\Theta}_{n,1} &= \lim_{n\to\infty} \frac{\exp[nG_{\beta_n,K_n}(m_n)] \cdot Z_{n,\kappa} \cdot (2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2})}{\int_{-n^\kappa(1-\delta_1)m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx} \\ &= \frac{0}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} = 0,\end{aligned}$$

as claimed.

We now prove that $\lim_{n\to\infty} \tilde{\Theta}_{n,2} = 0$. By (7.4) in part (a) of Lemma 7.2 with $\bar{\bar{\delta}} = \delta_1$ and $\bar{\delta} = \delta_2$

$$\begin{aligned}\tilde{\Theta}_{n,2} &= \frac{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}} \\ &= \frac{\int_{-n^\kappa(1-\delta_2)m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n)]dx/Z_{n,\kappa}}{\int_{-n^\kappa(1-\delta_1)m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n)]dx/Z_{n,\kappa}} \\ &= \frac{\int_{-n^\kappa(1-\delta_2)m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)]dx}{\int_{-n^\kappa(1-\delta_1)m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG\beta_n, K_n(m_n)]dx}.\end{aligned}$$

By Lemma 7.7 for $f \equiv 1$, $\bar{\bar{\delta}} = \delta_1$, and $\bar{\bar{\delta}} = \delta_2$, both the numerator and denominator have the same limit $\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx$. It follows that $\lim_{n\to\infty} \tilde{\Theta}_{n,2} = 1$, as claimed.

We now prove that $\lim_{n\to\infty} \tilde{\Theta}_{n,3} = 0$. Since $\tilde{\Theta}_{n,1} \ge \tilde{\Theta}_{n,3} > 0$,

$$\lim_{n\to\infty} \tilde{\Theta}_{n,1} = 0 \ \text{ implies } \lim_{n\to\infty} \tilde{\Theta}_{n,3} = 0,$$

as claimed. This completes the proof of the first inequality in part (a) of the present lemma.

We now prove the second inequality in part (a) of the present lemma. By the second inequality in part (b) of Lemma 7.9 and the first inequality in part (a) of Lemma 7.9 we have for all sufficiently large n

$$\begin{aligned}
&\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \bigm| S_n/n > \delta m_n\} \\
&\quad = \frac{\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{\{S_n/n > \delta m_n\}}\}}{P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\}} \\
&\quad \geq \frac{\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{A_n(\delta_2)}\} - 2n^\kappa e^{-cn^{2\zeta}} - c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\} + e^{-cn^{2\zeta}}} \\
&\quad = \frac{\frac{\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \cdot 1_{A_n(\delta_2)}\}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\}} - \frac{2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\}}}{\frac{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\}} + \frac{e^{-cn^{2\zeta}}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\}}} \\
&\quad = \frac{\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \bigm| A_n(\delta_2)\} - \tilde{\Gamma}_{n,1}}{\tilde{\Gamma}_{n,2} \ + \ \tilde{\Gamma}_{n,3}},
\end{aligned}$$

where $c > 0$ and $c_2 > 0$ are constants and

$$\tilde{\Gamma}_{n,1} = \frac{2n^\kappa e^{-cn^{2\zeta}} + c_2 n^{\kappa-1/2} e^{-cn^{2\zeta}/2}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\}},$$

$$\tilde{\Gamma}_{n,2} = \frac{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_1)\}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\}},$$

and

$$\tilde{\Gamma}_{n,3} = \frac{e^{-cn^{2\zeta}}}{(P_{n,\beta_n,K_n} \times Q)\{A_n(\delta_2)\}}.$$

The sequences $\tilde{\Gamma}_{n,1}$, $\tilde{\Gamma}_{n,2}$, and $\tilde{\Gamma}_{n,3}$ are obtained from $\tilde{\Theta}_{n,1}$, $\tilde{\Theta}_{n,2}$, and $\tilde{\Theta}_{n,3}$ in (8.8)–(8.10) by interchanging δ_1 and δ_2. Hence the limits $\tilde{\Gamma}_{n,1} \to 0$, $\tilde{\Gamma}_{n,2} \to 1$, and $\tilde{\Gamma}_{n,3} \to 0$ follow from the limits $\tilde{\Theta}_{n,1} \to 0$, $\tilde{\Theta}_{n,2} \to 1$, and $\tilde{\Theta}_{n,3} \to 0$, which hold for any $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$. The proof of part (a) of Lemma 8.4 is complete.

Proof of part (b) of Lemma 8.4. We know from part (b) of Lemma 8.1 that, as $n \to \infty$, for any $\bar{\delta} = \delta_1$ and $\bar{\delta} = \delta_2$, $\tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \bigm| A_n(\bar{\delta})\}$ has the same limit

$$\frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2} g^{(2)}(\bar{x}) x^2] dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2} g^{(2)}(\bar{x}) x^2] dx = \bar{z}.$$

By sending $n \to \infty$ in the inequality in part (a), we have

$$\begin{aligned}
\bar{z} &= \lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid A_n(\delta_1)\} \\
&\geq \limsup_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\} \\
&\geq \liminf_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\} \\
&= \lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid A_n(\delta_2)\} \\
&= \bar{z}.
\end{aligned}$$

Because the first and last terms in this display are the same, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx = \bar{z}.
\end{aligned}$$

On the other hand, by part (b) of Lemma 8.1 with $\bar{\delta} = \delta$

$$\begin{aligned}
&\lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid A_n(\delta)\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx = \bar{z}.
\end{aligned}$$

The proof of part (b) of Lemma 8.4 is complete. □

We now put together the pieces to complete the proof of part (b) of Theorem 6.1. Let δ be any number satisfying $\Delta < \delta < 1$, where $\Delta \in (0, 1)$ is determined in part (b) of Lemma 7.5. The proof of part (b) of Theorem 6.1 is divided into Step 1, Substep 2a, and Substep 2b. Step 1 is done in part (b) of Lemma 8.1. There we prove that with $\bar{\delta} = \delta$

$$\begin{aligned}
&\lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n + W_n/n^{1/2} > \delta m_n\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx = \bar{z}.
\end{aligned}$$

Substep 2a is done in Lemma 8.3. There we prove that $\lim_{n\to\infty} |C_n - D_n| = 0$, where

$$C_n = E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\}$$

and

$$D_n = \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\}.$$

Substep 2b is done in part (b) of Lemma 8.4. There we prove that

$$\begin{aligned}
&\lim_{n\to\infty} D_n \\
&\quad = \lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n + W_n/n^{1/2} > \delta m_n\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx = \bar{z}.
\end{aligned}$$

Combining these limits yields

$$\begin{aligned}
&\lim_{n\to\infty} E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\} \\
&\quad = \lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n + W_n/n^{1/2} > \delta m_n\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tfrac{1}{2}g^{(2)}(\bar{x})x^2]dx = \bar{z}.
\end{aligned}$$

This gives the conditional limit stated in part (b) of Theorem 6.1:

$$\begin{aligned}
&\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\{|S_n/n - m(\beta_n,K_n)| \mid S_n/n > \delta m(\beta_n,K_n)\} \\
&\quad = \lim_{n\to\infty} E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m(\beta_n,K_n)| \mid S_n/n > \delta m(\beta_n,K_n)\} = \bar{z}.
\end{aligned}$$

The proof of part (b) of Theorem 6.1 is complete. □

Appendix

A Proof That Sequences 1a–5a Satisfy the Limits in Hypothesis (iii′) of Theorem 4.1

In this appendix we prove that sequences 1a–5a satisfy the limits in hypothesis (iii′) of Theorem 4.1. These limits take the following form.

(a) Assume that g has degree 4. For $\forall \alpha \in (0, \alpha_0)$ and for $j = 2, 3, 4$

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}(m(\beta_n,K_n)) = g^{(j)}(\bar{x}).$$

(b) Assume that g has degree 6. For $\forall \alpha \in (0, \alpha_0)$ and for $j = 2, 3, 4, 5, 6$

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}(m(\beta_n,K_n)) = g^{(j)}(\bar{x}).$$

We do this by verifying the limits (A.1) and (A.3) in Lemma A.1. Let ε_n denote a sequence that converges to 0 and that represents the various error terms arising in the proof. We use the same notation ε_n to represent different error terms.

Lemma A.1. *We assume the hypotheses of Theorem* 3.1. *We also assume* (A.1) *when the degree of the Ginzburg-Landau polynomial g is 4 and* (A.3) *when the degree of g is 6.*

(a) *Assume that g has degree 4 and that for $\alpha \in (0, \alpha_0)$ and for $j = 2, 3, 4$*

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n, K_n}(x/n^{\theta\alpha}) = g^{(j)}(x) \tag{A.1}$$

uniformly for x in compact subsets of $\mathbb{R}$. Then we have

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n, K_n}(m(\beta_n, K_n)) = g^{(j)}(\bar{x}). \tag{A.2}$$

(b) *Assume that g has degree 6 and that for $\alpha \in (0, \alpha_0)$ and for $j = 2, 3, 4, 5, 6$*

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n, K_n}(x/n^{\theta\alpha}) = g^{(j)}(x) \tag{A.3}$$

uniformly for x in compact subsets of $\mathbb{R}$. Then we have

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n, K_n}(m(\beta_n, K_n)) = g^{(j)}(\bar{x}). \tag{A.4}$$

Proof. We write $m_n = m(\beta_n, K_n)$. When g has degree 4, we have for $j = 2, 3, 4$, and when g has degree 6, we have for $j = 2, 3, 4, 5, 6$

$$\begin{aligned} |n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n K_n}(m_n) - g^{(j)}(\bar{x})| \;\; &\leq \;\; |n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n K_n}(n^{\theta\alpha} m_n / n^{\theta\alpha}) - g^{(j)}(n^{\theta\alpha} m_n)| \\ &\quad + |g^{(j)}(n^{\theta\alpha} m_n) - g^{(j)}(\bar{x})|. \end{aligned} \tag{A.5}$$

Let Ξ be any compact subset of $\mathbb{R}$. By hypothesis (A.1) for $j = 2, 3, 4$ when g has degree 4 and by hypothesis (A.3) for $j = 2, 3, 4, 5, 6$ when g has degree 6

$$\lim_{n\to\infty} \sup_{x\in\Xi} |n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n, K_n}(x/n^{\theta\alpha}) - g^{(j)}(x)| = 0.$$

According to Theorem 3.1, $n^{\theta\alpha} m_n \to \bar{x}$, and so for any $\varepsilon > 0$ the sequence $n^{\theta\alpha} m_n$ lies in the compact set $[\bar{x} - \varepsilon, \bar{x} + \varepsilon]$ for all sufficiently large n. It follows that the first term on the right-hand side of (A.5) converges to 0 as $n \to \infty$. Because of the limit $n^{\theta\alpha} m_n \to \bar{x}$ and the continuity of $g^{(j)}$, the second term on the right-hand side of (A.5) also converges to 0 as

$n \to \infty$. We conclude that for $j = 2, 3, 4$ when g has degree 4 and for $j = 2, 3, 4, 5, 6$ when g has degree 6
$$|n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n, K_n}(m_n) - g^{(j)}(\bar{x})| \to 0 \text{ as } n \to \infty.$$
This completes the proof of the lemma. $\square$

The main point of this section is to justify rigorously the limits in (A.1) and (A.3) for sequences 1a–5a. We start by doing some preparatory work involving the Taylor expansion of $G^{(j)}_{\beta_n, K_n}(x/n^\gamma)$ for $\gamma > 0$.

Case 1: g has degree 4, *j* = 2, 3, 4. This case arises for sequences 1 and 2, which converge to a second-order point $(\beta, K(\beta))$ for $0 < \beta < \beta_c$. We consider the Taylor expansions of $G^{(j)}_{\beta_n, K_n}(x/n^\gamma)$ to order 4 with error terms. Since $K(\beta) = (4\beta + 2)/4\beta$ is continuous and (β_n, K_n) converges to $(\beta, K(\beta))$, we have $\beta_n K_n / K(\beta_n) \to \beta$. Thus the coefficients in Taylor expansion of $G^{(j)}_{\beta_n, K_n}(x/n^\gamma)$ are given by

$$G^{(2)}_{\beta_n, K_n}(0) = 2\beta_n K_n - \frac{8\beta_n^2 K_n^2}{e^{\beta_n} + 2} = \frac{2\beta_n K_n (K(\beta_n) - K_n)}{K(\beta_n)} = 2\beta(K(\beta_n) - K_n)(1 + \varepsilon_n),$$

$$G^{(3)}_{\beta_n, K_n}(0) = 0,$$
$$G^{(4)}_{\beta_n, K_n}(0) = \frac{2(2\beta_n K_n)^4 (4 - e^{\beta_n})}{(e^{\beta_n} + 2)^2}.$$
Let $c_4(\beta) = (e^\beta + 2)^2 (4 - e^\beta)/(8 \cdot 4!)$. Since $2\beta_n K_n \to 2\beta K(\beta) = (e^\beta + 2)/2$, we have

$$G^{(4)}_{\beta_n, K_n}(0) = (e^\beta + 2)^2 (4 - e^\beta)(1 + \varepsilon_n)/8 = c_4(\beta)(1 + \varepsilon_n) \cdot 4!.$$

Thus for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the Taylor expansion

$$G^{(2)}_{\beta_n, K_n}(x/n^\gamma) = G^{(2)}_{\beta_n, K_n}(0) + \frac{G^{(4)}_{\beta_n, K_n}(0)}{2!} \cdot \frac{x^2}{n^{2\gamma}} + \mathrm{O}\left(\frac{1}{n^{3\gamma}}\right) x^3.$$

Multiplying both sides by $n^{1-u-2\gamma}$ for $u > 0$ yields

$$\begin{aligned} n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1+u}} G^{(2)}_{\beta_n,K_n}(0) + \frac{1}{n^{4\gamma-1+u}} \cdot \frac{G^{(4)}_{\beta_n,K_n}(0)}{2!} x^2 \\ &\quad + \mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right) x^3 \\ &= \frac{1}{n^{2\gamma-1+u}} \cdot 2\beta(K(\beta_n) - K_n)(1+\varepsilon_n) \\ &\quad + \frac{1}{n^{4\gamma-1+u}} \cdot \frac{c_4(\beta)(1+\varepsilon_n)\cdot 4!}{2!} x^2 + \mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right) x^3. \end{aligned} \tag{A.6}$$

For all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the Taylor expansion

$$G^{(3)}_{\beta_n,K_n}(x/n^\gamma) = G^{(4)}_{\beta_n,K_n}(0) \cdot \frac{x}{n^\gamma} + \mathrm{O}\left(\frac{1}{n^{2\gamma}}\right) x^2.$$

Multiplying both sides by $n^{1-u-3\gamma}$ for $u > 0$ yields

$$\begin{aligned} n^{1-u-3\gamma} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{4\gamma-1+u}} \cdot G^{(4)}_{\beta_n,K_n}(0) \cdot x + \mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right) x^2 \\ &= \frac{1}{n^{4\gamma-1+u}} \cdot c_4(\beta)(1+\varepsilon_n) \cdot 4! \cdot x + \mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right) x^2. \end{aligned} \tag{A.7}$$

For all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the Taylor expansion

$$G^{(4)}_{\beta_n,K_n}(x/n^\gamma) = G^{(4)}_{\beta_n,K_n}(0) + \mathrm{O}\left(\frac{1}{n^\gamma}\right) x.$$

Multiplying both sides by $n^{1-u-4\gamma}$ for $u > 0$ yields

$$\begin{aligned} n^{1-u-4\gamma} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{4\gamma-1+u}} \cdot G^{(4)}_{\beta_n,K_n}(0) + \mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right) x \\ &= \frac{1}{n^{4\gamma-1+u}} \cdot c_4(\beta)(1+\varepsilon_n) \cdot 4! + \mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right) x. \end{aligned} \tag{A.8}$$

In formulas (A.6)–(A.8) the big-oh terms are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$. We will use (A.6)–(A.8) to verify hypothesis (A.1) for sequences 1a and 2a.

Case 2: g has degree 6*, j* = 2, 3, 4, 5, 6. This case arises for sequences 3, 4 and 5, which converge to the tricritical point $(\beta_c, K(\beta_c))$. We consider the Taylor expansions of $G^{(j)}_{\beta_n,K_n}(x/n^\gamma)$

to order 6 with error terms. Since $K(\beta) = (4\beta+2)/4\beta$ is continuous and (β_n, K_n) converges to $(\beta_c, K(\beta_c))$, we have $\beta_n K_n / K(\beta_n) \to \beta_c$. Thus the coefficient in Taylor expansion of $G^{(j)}_{\beta_n,K_n}(x/n^\gamma)$ are given by

$$G^{(2)}_{\beta_n,K_n}(0) = 2\beta_n K_n - \frac{8\beta_n^2 K_n^2}{e^{\beta_n}+2} = \frac{2\beta_n K_n (K(\beta_n) - K_n)}{K(\beta_n)} = 2\beta_c (K(\beta_n) - K_n)(1+\varepsilon_n),$$

$$G^{(3)}_{\beta_n,K_n}(0) = 0,$$

$$G^{(4)}_{\beta_n,K_n}(0) = \frac{2(2\beta_n K_n)^4 (4 - e^{\beta_n})}{(e^{\beta_n}+2)^2}.$$

Let $c_4 = 3/16$. Since $2\beta_n K_n \to 2\beta_c K(\beta_c) = (e^{\beta_c}+2)/2 = 3$ and $e^{\beta_n} + 2 \to e^{\beta_c} + 2 = 6$, we have

$$G^{(4)}_{\beta_n,K_n}(0) = 2 \cdot 3^4 (4 - e^{\beta_n})(1+\varepsilon_n)/6^2 = c_4 (4 - e^{\beta_n})(1+\varepsilon_n) \cdot 4!,$$

$$G^{(5)}_{\beta_n,K_n}(0) = 0.$$

Let $c_6 = 9/40$, since $G^{(6)}_{\beta_n,K_n}(0) \to G^{(6)}_{\beta_c,K(\beta_c)}(0) = 2 \cdot 3^4$, we have

$$G^{(6)}_{\beta_n,K_n}(0) = 2 \cdot 3^4 (1+\varepsilon_n) = c_6 (1+\varepsilon_n) \cdot 6!.$$

Thus for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the Taylor expansion

$$G^{(2)}_{\beta_n,K_n}(x/n^\gamma) = G^{(2)}_{\beta_n,K_n}(0) + \frac{G^{(4)}_{\beta_n,K_n}(0)}{2!} \cdot \frac{x^2}{n^{2\gamma}} + \frac{G^{(6)}_{\beta_n,K_n}(0)}{4!} \cdot \frac{x^4}{n^{4\gamma}} + \mathrm{O}\left(\frac{1}{n^{5\gamma}}\right) x^5.$$

Multiplying both sides by $n^{1-u-2\gamma}$ for $u > 0$ yields

$$\begin{aligned} n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1+u}} G^{(2)}_{\beta_n,K_n}(0) + \frac{1}{n^{4\gamma-1+u}} \cdot \frac{G^{(4)}_{\beta_n,K_n}(0)}{2!} x^2 \\ &\quad + \frac{1}{n^{6\gamma-1+u}} \cdot \frac{G^{(6)}_{\beta_n,K_n}(0)}{4!} x^4 + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x^5 \\ &= \frac{1}{n^{2\gamma-1+u}} \cdot 2\beta_c (K(\beta_n) - K_n)(1+\varepsilon_n) \\ &\quad + \frac{1}{n^{4\gamma-1+u}} \cdot \frac{c_4 (4 - e^{\beta_n})(1+\varepsilon_n) \cdot 4!}{2!} x^2 + \frac{c_6 (1+\varepsilon_n) \cdot 6!}{4!} x^4 \\ &\quad + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x^5. \end{aligned} \tag{A.9}$$

For all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the Taylor expansion

$$G^{(3)}_{\beta_n,K_n}(x/n^\gamma) = G^{(4)}_{\beta_n,K_n}(0) \cdot \frac{x}{n^\gamma} + \frac{G^{(6)}_{\beta_n,K_n}(0)}{3!} \cdot \frac{x^3}{n^{3\gamma}} + \mathrm{O}\!\left(\frac{1}{n^{4\gamma}}\right) x^4.$$

Multiplying both sides by $n^{1-u-3\gamma}$ for $u > 0$ yields

$$\begin{aligned} n^{1-u-3\gamma} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{4\gamma-1+u}} \cdot G^{(4)}_{\beta_n,K_n}(0) \cdot x + \frac{1}{n^{6\gamma-1+u}} \cdot \frac{G^{(6)}_{\beta_n,K_n}(0)}{3!} \cdot x^3 \\ &\quad + \mathrm{O}\!\left(\frac{1}{n^{7\gamma-1+u}}\right) x^4 \\ &= \frac{1}{n^{4\gamma-1+u}} \cdot c_4(4 - e^{\beta_n})(1+\varepsilon_n) \cdot 4! \cdot x + \frac{1}{n^{6\gamma-1+u}} \cdot \frac{c_6(1+\varepsilon_n)\cdot 6!}{3!} x^3 \\ &\quad + \mathrm{O}\!\left(\frac{1}{n^{7\gamma-1+u}}\right) x^4. \end{aligned} \tag{A.10}$$

For all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the Taylor expansion

$$G^{(4)}_{\beta_n,K_n}(x/n^\gamma) = G^{(4)}_{\beta_n,K_n}(0) + \frac{G^{(6)}_{\beta_n,K_n}(0)}{2!} \cdot \frac{x^2}{n^{2\gamma}} + \mathrm{O}\!\left(\frac{1}{n^{3\gamma}}\right) x^3.$$

Multiplying both sides by $n^{1-u-4\gamma}$ for $u > 0$ yields

$$\begin{aligned} n^{1-u-4\gamma} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{4\gamma-1+u}} \cdot G^{(4)}_{\beta_n,K_n}(0) + \frac{1}{n^{6\gamma-1+u}} \cdot \frac{G^{(6)}_{\beta_n,K_n}(0)}{2!} \cdot x^2 \\ &\quad + \mathrm{O}\!\left(\frac{1}{n^{7\gamma-1+u}}\right) x^3 \\ &= \frac{1}{n^{4\gamma-1+u}} \cdot c_4(4 - e^{\beta_n})(1+\varepsilon_n) \cdot 4! + \frac{1}{n^{6\gamma-1+u}} \cdot \frac{c_6(1+\varepsilon_n)\cdot 6!}{2!} \cdot x^2 \\ &\quad + \mathrm{O}\!\left(\frac{1}{n^{7\gamma-1+u}}\right) x^3. \end{aligned} \tag{A.11}$$

For all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the Taylor expansion

$$G^{(5)}_{\beta_n,K_n}(x/n^\gamma) = G^{(6)}_{\beta_n,K_n}(0) \cdot \frac{x}{n^\gamma} + \mathrm{O}\!\left(\frac{1}{n^{2\gamma}}\right) x^2.$$

Multiplying both sides by $n^{1-u-5\gamma}$ for $u > 0$ yields

$$\begin{aligned} n^{1-u-5\gamma} G^{(5)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{6\gamma-1+u}} \cdot G^{(6)}_{\beta_n,K_n}(0)x + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x^2 \\ &= \frac{1}{n^{6\gamma-1+u}} \cdot c_6(1+\varepsilon_n) \cdot 6! \cdot x + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x^2. \end{aligned} \tag{A.12}$$

For all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the Taylor expansion

$$G^{(6)}_{\beta_n,K_n}(x/n^\gamma) = G^{(6)}_{\beta_n,K_n}(0) + \mathrm{O}\left(\frac{1}{n^\gamma}\right) x.$$

Multiplying both sides by $n^{1-u-6\gamma}$ for $u > 0$ yields

$$\begin{aligned} n^{1-u-6\gamma} G^{(6)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{6\gamma-1+u}} \cdot G^{(6)}_{\beta_n,K_n}(0) + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x \\ &= \frac{1}{n^{6\gamma-1+u}} \cdot c_6(1+\varepsilon_n) \cdot 6! \cdot + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x. \end{aligned} \tag{A.13}$$

In formulas (A.9)–(A.13) the big-oh terms are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$. We will use (A.9)–(A.13) to verify assumption (A.3) for sequences 3a, 4a, 5a.

Sequence 1a

This sequence is defined in (5.5). For sequence 1a, g has degree 4. Since $K(\beta_n) - K_n = (K'(\beta)b - k)/n^\alpha + \mathrm{O}(1/n^{2\alpha})$, it follows from (A.6) that for all $n \in \mathbb{N}$, any $u > 0$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have

$$\begin{aligned} n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1+u+\alpha}} 2\beta(K'(\beta)b - k)(1+\varepsilon_n) \\ &\quad + \frac{1}{n^{4\gamma-1+u}} \cdot \frac{c_4(\beta)(1+\varepsilon_n)\cdot 4!}{2!} x^2 \\ &\quad + \mathrm{O}\left(\frac{1}{n^{2\gamma-1+u+2\alpha}}\right) + \mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right) x^3. \end{aligned} \tag{A.14}$$

We now define $\gamma = \theta\alpha$ and $u = 1 - \alpha/\alpha_0$, and we recall that $\alpha_0 = 1/2$, $\theta = 1/2$. With these choices of γ and u, the powers of n appearing in the first two terms in (A.14) are 0, and the powers of n appearing in the last two terms in (A.14) are positive. Letting $n \to \infty$ in (A.14), we have uniformly for x in compact subsets of $\mathbb{R}$

$$\begin{aligned} \lim_{n\to\infty} n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0 - 2\theta\alpha} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) \\ &= 2\beta(K'(\beta)b - k) + \frac{c_4(\beta)\cdot 4!}{2!} x^2 = g^{(2)}(x). \end{aligned}$$

The same choices of γ and u ensure that the powers of n appearing in the first term in (A.7) and (A.8) are 0, and the powers of n appearing in the last term in (A.7) and (A.8) are positive. Taking $n \to \infty$ in (A.7) and (A.8) gives

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-3\gamma} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0 - 3\theta\alpha} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) \\ &= c_4(\beta)\cdot 4! \cdot x = g^{(3)}(x).\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-4\gamma} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0 - 4\theta\alpha} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) \\ &= c_4(\beta)\cdot 4! = g^{(4)}(x).\end{aligned}$$

uniformly for x in compact subsets of $\mathbb{R}$. Thus sequence 1 satisfies hypothesis (A.1) in Lemma A.1, and so the conclusion (A.2) in Lemma A.1 follows for $j = 2, 3, 4$. This is the convergence in hypothesis (iii′) of Theorem 4.1.

Sequence 2a

This sequence is defined in (5.6). For sequence 2a, g has degree 4. Since $K(\beta_n) - K_n = (K^{(p)}(\beta) - \ell)b^p/p!n^{p\alpha} + \mathrm{O}(1/n^{\alpha(p+1)})$, it follows from (A.6) that for all $n \in \mathbb{N}$, any $u > 0$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have

$$\begin{aligned}n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1+u+\alpha}} \cdot \frac{1}{p!} \cdot 2\beta(K^{(p)}(\beta) - \ell)b^p(1+\varepsilon_n) \\ &\quad + \frac{1}{n^{4\gamma-1+u}} \cdot \frac{c_4(\beta)(1+\varepsilon_n)\cdot 4!}{2!} x^2 \\ &\quad + \mathrm{O}\left(\frac{1}{n^{2\gamma-1+u+(p+1)\alpha}}\right) + \mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right) x^3.\end{aligned} \tag{A.15}$$

We now define $\gamma = \theta\alpha$ and $u = 1 - \alpha/\alpha_0$, and we recall that $\alpha_0 = 1/2p$, $\theta = p/2$. With these choices of γ and u, the power of n appearing in the first two terms in (A.15) are 0, and the power of n appearing in the last two terms in (A.15) are positive. Letting $n \to \infty$ in (A.15), we have uniformly for x in compact subsets of $\mathbb{R}$

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0 - 2\theta\alpha} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) \\ &= \frac{1}{p!}\cdot 2\beta(K^{(p)}(\beta) - \ell)b^p + \frac{c_4(\beta)\cdot 4!}{2!}x^2 = g^{(2)}(x).\end{aligned}$$

The same choices of γ and u ensure that the powers of n appearing in the first term in (A.7) and (A.8) are 0, and the powers of n appearing in the last term in (A.7) and (A.8) are positive.

Taking $n \to \infty$ in (A.7) and (A.8) gives

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-3\gamma} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-3\theta\alpha} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) \\ &= c_4(\beta)\cdot 4!\cdot x = g^{(3)}(x)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-4\gamma} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-4\theta\alpha} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) \\ &= c_4(\beta)\cdot 4! = g^{(4)}(x).\end{aligned}$$

uniformly for x in compact subsets of $\mathbb{R}$. Thus sequence 2 satisfies hypothesis (A.1) in Lemma A.1, and so the conclusion (A.2) in Lemma A.1 follows for $j = 2, 3, 4$. This is the convergence in hypothesis (iii′) of Theorem 4.1.

Sequence 3a

This sequence is defined in (5.7). For sequence 3a, g has degree 6. Since $K(\beta_n) - K_n = (K'(\beta_c)b - k)/n^\alpha + \mathrm{O}(1/n^{2\alpha})$, it follows (A.9), (A.10), and (A.11) that for all $n \in \mathbb{N}$, any $u > 0$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the following:

$$\begin{aligned}n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1+u+\alpha}}\cdot 2\beta_c(K'(\beta_c)b-k)(1+\varepsilon_n) \\ &\quad + \frac{1}{n^{4\gamma-1+u+\alpha}}\cdot\frac{c_4(-4b)(1+\varepsilon_n)\cdot 4!}{2!}x^2 \\ &\quad + \frac{1}{n^{6\gamma-1+u}}\cdot\frac{c_6(1+\varepsilon_n)\cdot 6!}{4!}\cdot x^4 + \mathrm{O}\left(\frac{1}{n^{2\gamma-1+u+2\alpha}}\right) \\ &\quad + \mathrm{O}\left(\frac{1}{n^{4\gamma-1+u+2\alpha}}\right)x^2 + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right)x^5,\end{aligned} \tag{A.16}$$

$$\begin{aligned}n^{1-u-3\gamma} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{4\gamma-1+u+\alpha}}\cdot c_4(-4b)(1+\varepsilon_n)\cdot 4!\cdot x \\ &\quad + \frac{1}{n^{6\gamma-1+u}}\cdot\frac{c_6(1+\varepsilon_n)\cdot 6!}{3!}\cdot x^3 \\ &\quad + \mathrm{O}\left(\frac{1}{n^{4\gamma-1+u+2\alpha}}\right)x + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right)x^4,\end{aligned} \tag{A.17}$$

and

$$n^{1-u-4\gamma}G^{(4)}_{\beta_n,K_n}(x/n^{\gamma}) = \frac{1}{n^{4\gamma-1+u+\alpha}} \cdot c_4(-4b)(1+\varepsilon_n)\cdot 4! + \frac{1}{n^{6\gamma-1+u}} \cdot \frac{c_6(1+\varepsilon_n)\cdot 6!}{2!} \cdot x^2 + \mathrm{O}\left(\frac{1}{n^{4\gamma-1+u+2\alpha}}\right)x + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right)x^3. \quad \text{(A.18)}$$

We now define $\gamma = \theta\alpha$ and $u = 1 - \alpha/\alpha_0$, and we recall that $\alpha_0 = 2/3$, $\theta = 1/4$. With these choices of γ and u, the powers of n appearing in the first term and the third term in (A.16) are 0, and the powers of n appearing in the second term and the last three terms in (A.16) are positive. Letting $n \to \infty$ in (A.16), we have uniformly for x in compact subsets of $\mathbb{R}$

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-2\gamma}G^{(2)}_{\beta_n,K_n}(x/n^{\gamma}) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-2\theta\alpha}G^{(2)}_{\beta_n,K_n}(x/n^{\gamma}) \\ &= 2\beta_c(K'(\beta_c)b-k) + \frac{c_6\cdot 6!}{4!}x^4 = g^{(2)}(x).\end{aligned}$$

The same choices of γ and u ensure that the powers of n appearing in the second term in (A.17) and (A.18) are 0, and the powers of n appearing in the first term and last two terms in (A.17) and (A.18) are positive. Taking $n \to \infty$ in (A.17) and (A.18) gives that

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-3\gamma}G^{(3)}_{\beta_n,K_n}(x/n^{\gamma}) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-3\theta\alpha}G^{(3)}_{\beta_n,K_n}(x/n^{\gamma}) \\ &= \frac{c_6\cdot 6!}{3!}x^3 = g^{(3)}(x)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-4\gamma}G^{(4)}_{\beta_n,K_n}(x/n^{\gamma}) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-4\theta\alpha}G^{(4)}_{\beta_n,K_n}(x/n^{\gamma}) \\ &= \frac{c_6\cdot 6!}{2!}x^2 = g^{(4)}(x).\end{aligned}$$

uniformly for x in compact subsets of $\mathbb{R}$. Similarly, the powers of n appearing in the first term in the expansions (A.12) and (A.13) are 0, and the powers of n appearing in the last term in the expansions (A.12) and (A.13) are positive. Letting $n \to \infty$ in (A.12) and (A.13), we have uniformly for x in compact subsets of $\mathbb{R}$

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-5\gamma}G^{(5)}_{\beta_n,K_n}(x/n^{\gamma}) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-5\theta\alpha}G^{(5)}_{\beta_n,K_n}(x/n^{\gamma}) \\ &= c_6\cdot 6!x = g^{(5)}(x)\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} n^{1-u-6\gamma} G^{(6)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-6\theta\alpha} G^{(6)}_{\beta_n,K_n}(x/n^\gamma) \\
&= c_6 \cdot 6! = g^{(6)}(x).
\end{aligned}$$

Thus sequence 3 satisfies hypothesis (A.3) in Lemma A.1, and so the conclusion (A.4) in Lemma A.1 follows for $j = 2,3,4,5,6$. This is the convergence in hypothesis (iii′) of Theorem 4.1.

Sequence 4a

This sequence is defined in (5.8). For sequence 4a, g has degree 6. Since

$$\begin{aligned}
K(\beta_n) - K_n &= K(\beta_c + 1/n^\alpha) - K_n \\
&= K(\beta_c) + K'(\beta_c) \cdot 1/n^\alpha + K''(\beta_c) \cdot 1/2!n^{2\alpha} + K'''(\beta_c) \cdot 1/3!n^{3\alpha} \\
&\quad + \mathrm{O}(1/n^{4\alpha}) - K_n \\
&= (K''(\beta_c) - \ell)/2n^{2\alpha} + (K'''(\beta_c) - \tilde{\ell})/6n^{3\alpha} + \mathrm{O}(1/n^{4\alpha})
\end{aligned}$$

and

$$4 - e^{\beta_n} = -4/n^\alpha + \mathrm{O}(1/n^{2\alpha}),$$

it follows from (A.9), (A.10), and (A.11) that for all $n \in \mathbb{N}$, any $u > 0$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the following:

$$\begin{aligned}
n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1+u+2\alpha}} \cdot 2\beta_c \cdot (K''(\beta_c) - \ell)/2 \cdot (1+\varepsilon_n) \\
&\quad + \frac{1}{n^{4\gamma-1+u+\alpha}} \cdot \frac{c_4(-4)(1+\varepsilon_n)\cdot 4!}{2!} x^2 \\
&\quad + \frac{1}{n^{6\gamma-1+u}} \cdot \frac{c_6(1+\varepsilon_n)\cdot 6!}{4!} \cdot x^4 \\
&\quad + \mathrm{O}\left(\frac{1}{n^{2\gamma-1+u+3\alpha}}\right) + \mathrm{O}\left(\frac{1}{n^{2\gamma-1+u+4\alpha}}\right) \\
&\quad + \mathrm{O}\left(\frac{1}{n^{4\gamma-1+u+2\alpha}}\right) x^2 + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x^5,
\end{aligned} \tag{A.19}$$

$$\begin{aligned}
n^{1-u-3\gamma} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{4\gamma-1+u+\alpha}} \cdot c_4(-4)(1+\varepsilon_n)\cdot 4! \cdot x \\
&\quad + \frac{1}{n^{6\gamma-1+u}} \cdot \frac{c_6(1+\varepsilon_n)\cdot 6!}{3!} \cdot x^3 \\
&\quad + \mathrm{O}\left(\frac{1}{n^{4\gamma-1+u+2\alpha}}\right) x + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x^4,
\end{aligned} \tag{A.20}$$

and

$$n^{1-u-4\gamma} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) = \frac{1}{n^{4\gamma-1+u+\alpha}} \cdot c_4(-4)(1+\varepsilon_n)\cdot 4! + \frac{1}{n^{6\gamma-1+u}} \cdot \frac{c_6(1+\varepsilon_n)\cdot 6!}{2!} \cdot x^2 + \mathrm{O}\left(\frac{1}{n^{4\gamma-1+u+2\alpha}}\right) + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x^3. \tag{A.21}$$

We now define $\gamma = \theta\alpha$ and $u = 1 - \alpha/\alpha_0$, and we recall that $\alpha_0 = 1/3$, $\theta = 1/2$. With these choices of γ and u, the powers of n appearing in the first three terms in (A.19) are 0, and the powers of n appearing in the last four terms in (A.19) are positive. Letting $n \to \infty$ in (A.19), we have uniformly for x in compact subsets of $\mathbb{R}$

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-2\theta\alpha} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) \\ &= 2\beta_c(K''(\beta_c)-\ell)/2 + \frac{c_4(-4)\cdot 4!}{2!}x^2 + \frac{c_6\cdot 6!}{4!}x^4 = g^{(2)}(x).\end{aligned}$$

The same choices of γ and u ensure that the powers of n appearing in the first two terms in (A.20) and (A.21) are 0 and the powers of n appearing in the last two terms in (A.20) and (A.21) are positive. Taking $n \to \infty$ in (A.20) and (A.21) gives

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-3\gamma} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-3\theta\alpha} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) \\ &= c_4(-4)\cdot 4! x + \frac{c_6\cdot 6!}{3!}x^3 = g^{(3)}(x)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-4\gamma} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-4\theta\alpha} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) \\ &= c_4(-4)\cdot 4! + \frac{c_6\cdot 6!}{2!}x^2 = g^{(4)}(x).\end{aligned}$$

uniformly for x in compact subsets of $\mathbb{R}$. Similarly, the powers of n appearing in the first term in the expansions (A.12) and (A.13) are 0 and the powers of n appearing in the last term in the expansions (A.12) and (A.13) are positive. Letting $n \to \infty$ in (A.12) and (A.13), we have uniformly for x in compact subsets of $\mathbb{R}$

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-5\gamma} G^{(5)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-5\theta\alpha} G^{(5)}_{\beta_n,K_n}(x/n^\gamma) \\ &= c_6\cdot 6! x = g^{(5)}(x)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-6\gamma} G^{(6)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-6\theta\alpha} G^{(6)}_{\beta_n,K_n}(x/n^\gamma) \\ &= c_6\cdot 6! = g^{(6)}(x).\end{aligned}$$

Thus sequence 4 satisfies hypothesis (A.3) in Lemma A.1, and so the conclusion (A.4) in Lemma A.1 follows for $j = 2,3,4,5,6$. This is the convergence in hypothesis (iii′) of Theorem 4.1.

Sequence 5a

This sequence is defined in (5.10). For sequence 5a, g has degree 6. Since $K(\beta_n) - K_n = (K''(\beta_c)-\ell)/2n^{2\alpha} + \mathrm{O}(1/n^{3\alpha})$ and $4 - e^{\beta_n} = 4/n^\alpha + \mathrm{O}(1/n^{2\alpha})$, it follows from (A.9), (A.10), and (A.11) that for all $n\in\mathbb{N}$, any $u>0$, any $\gamma>0$, any $R>0$, and all $x\in\mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have the following:

$$\begin{aligned}n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1+u+2\alpha}}\cdot 2\beta_c\cdot(K''(\beta_c)-\ell)/2\cdot(1+\varepsilon_n) \\ &\quad+\frac{1}{n^{4\gamma-1+u+\alpha}}\cdot\frac{c_4\cdot 4\cdot(1+\varepsilon_n)\cdot 4!}{2!}x^2 \\ &\quad+\frac{1}{n^{6\gamma-1+u}}\cdot\frac{c_6(1+\varepsilon_n)\cdot 6!}{4!}\cdot x^4 \\ &\quad+\mathrm{O}\!\left(\frac{1}{n^{2\gamma-1+u+3\alpha}}\right) \\ &\quad+\mathrm{O}\!\left(\frac{1}{n^{4\gamma-1+u+2\alpha}}\right)x^2+\mathrm{O}\!\left(\frac{1}{n^{7\gamma-1+u}}\right)x^5,\end{aligned}\tag{A.22}$$

$$\begin{aligned}n^{1-u-3\gamma} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{4\gamma-1+u+\alpha}}\cdot c_4\cdot 4\cdot(1+\varepsilon_n)\cdot 4!\cdot x \\ &\quad+\frac{1}{n^{6\gamma-1+u}}\cdot\frac{c_6(1+\varepsilon_n)\cdot 6!}{3!}\cdot x^3 \\ &\quad+\mathrm{O}\!\left(\frac{1}{n^{4\gamma-1+u+2\alpha}}\right)x+\mathrm{O}\!\left(\frac{1}{n^{7\gamma-1+u}}\right)x^4,\end{aligned}\tag{A.23}$$

and

$$\begin{aligned}n^{1-u-4\gamma} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{4\gamma-1+u+\alpha}}\cdot c_4\cdot 4\cdot(1+\varepsilon_n)\cdot 4! \\ &\quad+\frac{1}{n^{6\gamma-1+u}}\cdot\frac{c_6(1+\varepsilon_n)\cdot 6!}{2!}\cdot x^2 \\ &\quad+\mathrm{O}\!\left(\frac{1}{n^{4\gamma-1+u+2\alpha}}\right)+\mathrm{O}\!\left(\frac{1}{n^{7\gamma-1+u}}\right)x^3.\end{aligned}\tag{A.24}$$

We now define $\gamma = \theta\alpha$ and $u = 1 - \alpha/\alpha_0$, and we recall that $\alpha_0 = 1/3$, $\theta = 1/2$. With these choices of γ and u, the powers of n appearing in the first three terms in (A.22) are 0, and the powers of n appearing in the last three terms in (A.22) are positive. Letting $n \to \infty$ in (A.22), we have uniformly for x in compact subsets of $\mathbb{R}$

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-2\gamma} G^{(2)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-2\theta\alpha} G^{(2)}_{\beta_n,K_n}(x/n^\gamma)\\ &= \beta_c(K''(\beta_c)-\ell)/2 + \frac{c_4\cdot 4\cdot 4!}{2!}x^2 + \frac{c_6\cdot 6!}{4!}x^4 = g^{(2)}(x).\end{aligned}$$

The same choices of γ and u ensure that the powers of n appearing in the first two terms in (A.23) and (A.24) are 0 and the powers of n appearing in the last two terms in (A.23) and (A.24) are positive. Taking $n \to \infty$ in (A.23) and (A.24) gives

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-3\gamma} G^{(3)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-3\theta\alpha} G^{(3)}_{\beta_n,K_n}(x/n^\gamma)\\ &= c_4\cdot 4\cdot 4! x + \frac{c_6\cdot 6!}{3!}x^3 = g^{(3)}(x)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-4\gamma} G^{(4)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-4\theta\alpha} G^{(4)}_{\beta_n,K_n}(x/n^\gamma)\\ &= c_4\cdot 4\cdot 4! + \frac{c_6\cdot 6!}{2!}x^2 = g^{(4)}(x).\end{aligned}$$

uniformly for x in compact subsets of $\mathbb{R}$. Similarly, the powers of n appearing in the first term in the expansions (A.12) and (A.13) are 0, and the powers of n appearing in the last term in the expansions (A.12) and (A.13) are positive. Letting $n \to \infty$ in (A.12) and (A.13), we have uniformly for x in compact subsets of $\mathbb{R}$

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-5\gamma} G^{(5)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-5\theta\alpha} G^{(5)}_{\beta_n,K_n}(x/n^\gamma)\\ &= c_6\cdot 6! x = g^{(5)}(x)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} n^{1-u-6\gamma} G^{(6)}_{\beta_n,K_n}(x/n^\gamma) &= \lim_{n\to\infty} n^{\alpha/\alpha_0-6\theta\alpha} G^{(6)}_{\beta_n,K_n}(x/n^\gamma)\\ &= c_6\cdot 6! = g^{(6)}(x).\end{aligned}$$

Thus sequence 5 satisfies hypothesis (A.3) in Lemma A.1, and so the conclusion (A.4) in Lemma A.1 follows for $j = 2, 3, 4, 5, 6$. This is the convergence in hypothesis (iii′) of Theorem 4.1.

B Proof of the MDP in Part (a) of Theorem 6.2

In this appendix we give the proof of part (a) of the MDP stated in Theorem 6.2. We restate the theorem here for easy reference. Concerning the proof of part (b) of Theorem 6.2, see the comment before Lemma 6.3.

Theorem 6.2. *Let (β_n, K_n) be a positive sequence converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. We assume that (β_n, K_n) satisfies the hypotheses of Theorem* 4.1 *for all* $0 < \alpha < \alpha_0$. *The following conclusions hold.*

(a) *For all* $0 < \alpha < \alpha_0$, $S_n/n^{1-\theta\alpha}$ *satisfies the MDP with respect to* P_{n,β_n,K_n} *with exponential speed* $n^{1-\alpha/\alpha_0}$ *and rate function* $\Gamma(x) = g(x) - \inf_{y\in\mathbb{R}} g(y)$*; in symbols*

$$P_{n,\beta_n,K_n}\{S_n/n^{1-\theta\alpha} \in dx\} \asymp \exp[-n^{1-\alpha/\alpha_0}\Gamma(x)]dx.$$

(b) *The hypotheses of this theorem are satisfied by sequence* 1a–5a *defined in Table* 5.1 .

We work with an arbitrary α satisfying $0 < \alpha < \alpha_0$. To ease the notation we write $\gamma = \theta\alpha$ and $u = 1 - \alpha/\alpha_0$. The hypotheses of Theorem 6.2 coincide with the hypotheses of Theorem 4.1, which in turn consist of hypothesis (iii′) and the hypotheses of Theorem 3.1 for all $0 < \alpha < \alpha_0$. Clearly we have $0 < u = 1 - \alpha/\alpha_0 < 1$ and by hypothesis (iii′) $0 < \gamma = \theta\alpha < \theta\alpha_0 < 1/2$. In addition, $1 - 2\gamma - u = (1 - 2\theta\alpha_0)\alpha/\alpha_0 > 0$, which implies $1 - 2\gamma > u$.

The proof of Theorem 6.2 is analogous to the proof of Theorem 8.1 in [6]. Let W_n be a sequence of normal random variables with mean 0 and variance $\sigma_n^2 = (2\beta_n K_n)^{-1}$ defined on a probability space $(\Omega, \mathcal{F}, Q)$. Theorem 6.2 is proved in two steps.

Step 1. $W_n/n^{1/2-\gamma}$ is superexponentially small relative to $\exp(n^{-v})$; i.e., for any $\delta > 0$

$$\limsup_{n\to\infty} \frac{1}{n^{-v}} \log Q\{|W_n/n^{1/2-\gamma}| > \delta\} = -\infty \tag{B.1}$$

Step 2. With respect to $P_{n,\beta_n,K_n} \times Q$, $S_n/n^{1-\gamma} + W_n/n^{1/2-\gamma}$ satisfies the Laplace principle with exponential speed n^{-v} and rate function Γ.

According to Theorem 1.3.3 in [7], if we prove Step 1 and Step 2, then with respect to P_{n,β_n,K_n}, $S_n/n^{1-\gamma}$ satisfies the Laplace principle with speed n^u and rate function Γ; i.e., for any bounded, continuous function ψ

$$\lim_{n\to\infty} \frac{1}{n^u} \log \int_{\Lambda^n} \exp[n^u \psi(S_n/n^{1-\gamma})] dP_{n,\beta_n,K_n} = \sup_{x\in\mathbb{R}} \{\psi(x) - \Gamma(x)\}.$$

Since the Laplace principle implies the MDP (Thm 1.2.3 in [7]), Theorem 6.2 follows.

Next, we prove Step 1 and Step 2.

Proof of Step 1. Since β_n and K_n are bounded and uniformly positive over n, the sequence σ_n^2 is bounded and uniformly positive over n. We have

$$\begin{aligned} Q\{|W_n/n^{1/2-\gamma}| > \delta\} &= Q\{|N(0,\sigma_n^2)| > n^{1/2-\gamma}\delta\} \\ &\leq \frac{\sqrt{2}\sigma_n}{\sqrt{\pi} n^{1/2-\gamma}\delta} \cdot \exp(-n^{1-2\gamma}\delta^2/(2\sigma_n^2)). \end{aligned}$$

$$\frac{1}{n^u} \log Q\{|W_n/n^{1/2-\gamma}| > \delta\} \leq \frac{1}{n^u} \left[\log \frac{\sqrt{2}\sigma_n}{\sqrt{\pi}\delta} + \log(n^{\gamma-1/2}) - \frac{n^{1-2\gamma}\delta^2}{2\sigma_n^2} \right].$$

The limit of the right hand side of the last inequality is $-\infty$ since $u > 0$ and $1 - 2\gamma > u$. Thus (B.1) follows. The proof of Step 1 is done.

Proof of Step 2. Let ψ be an arbitrary bounded, continuous function. Choosing $\varphi = \exp[n^u\psi]$ and $\bar{\gamma} = \gamma$ in Lemma 7.1 yields

$$\begin{aligned} &\int_{\Lambda^n\times\Omega} \exp\left[n^u \psi\left(\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}} \right) \right] d(P_{n,\beta_n,K_n} \times Q) \qquad \text{(B.2)} \\ &= \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]dx} \cdot \int_{\mathbb{R}} \exp[n^u\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx. \end{aligned}$$

The proof of Step 2 rests on the following three properties of $nG_{\beta_n,K_n}(x/n^\gamma)$.

1. By hypothesis (iv) of Theorem 3.1 for $0 < \alpha < \alpha_0$, there exists a polynomial H satisfying $H(x) \to \infty$ as $|x| \to \infty$ together with the following property: $\exists R > 0$ such that for $\forall n \in \mathbb{N}$ sufficiently large and $\forall x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$nG_{\beta_n,K_n}(x/n^\gamma) \geq n^u H(x).$$

2. Let $\Delta = \sup_{x\in\mathbb{R}}\{\psi(x) - g(x)\}$. Since $H(x) \to \infty$ and $g(x) \to \infty$ as $|x| \to \infty$, there exists $M > 0$ with the following three properties:

$$\sup_{|x|>M} \{\psi(x) - H(x)\} \le -|\Delta| - 1,$$

the supremum of $\psi - g$ on $\mathbb{R}$ is attained on the interval $[-M, M]$, and the supremum of $-g$ on $\mathbb{R}$ is attained on the interval $[-M, M]$. In combination with item 1, we have that for all $n \in \mathbb{N}$ satisfying $Rn^\gamma > M$

$$\begin{aligned} &\sup_{M<|x|<Rn^\gamma} \{n^u\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)\} \\ &\quad\le \sup_{M<|x|<Rn^\gamma} \{n^u\psi(x) - n^u H(x)\} \\ &\quad\le -n^u(|\Delta| + 1). \end{aligned} \tag{B.3}$$

3. Let M be the number selected in item 2. By hypothesis (iii)(a) of Theorem 3.1 for $0 < \alpha < \alpha_0$, for all $x \in \mathbb{R}$ satisfying $|x| \le M$, $n^{1-u}G_{\beta_n,K_n}(x/n^\gamma)$ converges uniformly to $g(x)$ as $n \to \infty$.

Item 3 implies that for any $\delta > 0$ and all sufficiently large n

$$\begin{aligned} &\exp(-n^u\delta)\int_{\{|x|\le M\}} \exp[n^u(\psi(x) - g(x))]dx \\ &\quad\le \int_{\{|x|\le M\}} \exp[n^u\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx \\ &\quad\le \exp(n^u\delta)\int_{\{|x|\le M\}} \exp[n^u(\psi(x) - g(x))]dx. \end{aligned}$$

In addition, item 2 implies that

$$\int_{\{M<|x|<Rn^\gamma\}} \exp[n^u\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx \le 2Rn^\gamma \exp[-n^u(|\Delta| + 1)].$$

Since ψ is bounded, the last two displays show that there exists $a_1 > 0$ and $a_2 \in \mathbb{R}$ such that for all sufficiently large n

$$\int_{\{|x|<Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]dx \le a_1 \exp(n^u a_2).$$

Since $u \in (0,1)$, by part (d) of Lemma 4.4 in [6] there exists $a_3 > 0$ such that for all sufficiently large n

$$\int_{\{|x| \geq Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]dx \leq 2a_1 \exp(-na_3).$$

Together these three estimates show that for all sufficiently large n

$$\begin{aligned}
&\int_{\mathbb{R}} \exp[n^u \psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx \\
&= \int_{\{|x| \leq M\}} \exp[n^u \psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx \\
&\quad + \int_{\{M < |x| < Rn^\gamma\}} \exp[n^u \psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx \\
&\quad + \int_{\{|x| \geq Rn^\gamma\}} \exp[n^u \psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx \\
&\leq \exp(n^u \delta) \int_{\{|x| \leq M\}} \exp[n^u(\psi(x) - g(x))]dx \\
&\quad + 2Rn^\gamma \exp[-n^u(|\Delta| + 1)] + 2a_1 \exp(-na_3 + n^u ||\psi||_\infty).
\end{aligned}$$

Hence for all sufficiently large n we have

$$\begin{aligned}
&\exp(-n^u \delta) \int_{\{|x| \leq M\}} \exp[n^u(\psi(x) - g(x))]dx \\
&\quad \leq \int_{\mathbb{R}} \exp[n^u \psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx \\
&\quad \leq \exp(n^u \delta) \int_{\{|x| \leq M\}} \exp[n^u(\psi(x) - g(x))]dx + \delta_n,
\end{aligned}$$

where

$$\begin{aligned}
\delta_n &\leq 2Rn^\gamma \exp[-n^u(|\Delta| + 1)] + 2a_1 \exp(-na_3 + n^u ||\psi||_\infty) \\
&\leq 4Rn^\gamma \exp[-n^u(|\Delta| + 1)].
\end{aligned} \tag{B.4}$$

It follows that

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n^u} \log \left[\exp(-n^u\delta) \int_{\{|x|\leq M\}} \exp[n^u(\psi(x) - g(x))]dx \right] \qquad \text{(B.5)} \\
&\leq \liminf_{n\to\infty} \frac{1}{n^u} \log \int_{\mathbb{R}} \exp[n^u\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx \\
&\leq \limsup_{n\to\infty} \frac{1}{n^u} \log \int_{\mathbb{R}} \exp[n^u\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx \\
&\leq \limsup_{n\to\infty} \frac{1}{n^u} \log \left[\exp(n^u\delta) \int_{\{|x|\leq M\}} \exp[n^u(\psi(x) - g(x))]dx + \delta_n \right].
\end{aligned}$$

By Laplace's method applied to the continuous function $\psi - g$ on $|x| \leq M$ and the fact that the supremum of $\psi - g$ on $\mathbb{R}$ is attained on the interval $[-M, M]$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^u} \log \int_{\{|x|\leq M\}} \exp[n^u(\psi(x) - g(x))]dx \qquad \text{(B.6)} \\
&= \sup_{|x|\leq M} \{\psi(x) - g(x)\} = \sup_{x\in\mathbb{R}} \{\psi(x) - g(x)\}.
\end{aligned}$$

Hence the first line of (B.5) equals

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n^u} \left[-n^u\delta + \log \int_{\{|x|\leq M\}} \exp[n^u(\psi(x) - g(x))]dx \right] \qquad \text{(B.7)} \\
&= -\delta + \liminf_{n\to\infty} \frac{1}{n^u} \log \int_{\{|x|\leq M\}} \exp[n^u(\psi(x) - g(x))]dx \\
&= -\delta + \sup_{x\in\mathbb{R}} \{\psi(x) - g(x)\}.
\end{aligned}$$

We have to work harder to evaluate the last line of (B.5). At the end of the proof we will show that the term δ_n can be neglected in evaluating the last line of (B.5); i.e.,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^u} \log \left[\exp(n^u\delta) \int_{\{|x|\leq M\}} \exp[n^u(\psi(x) - g(x))]dx + \delta_n \right] \qquad \text{(B.8)} \\
&= \limsup_{n\to\infty} \frac{1}{n^u} \log \left[\exp(n^u\delta) \int_{\{|x|\leq M\}} \exp[n^u(\psi(x) - g(x))]dx \right].
\end{aligned}$$

Under the assumption that this is true, by (B.6) the last line of (B.5) equals

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^u} \log \left[\exp(n^u\delta) \int_{\{|x|\le M\}} \exp[n^u(\psi(x) - g(x))]dx\right] \\
&= \limsup_{n\to\infty} \frac{1}{n^u} \left[n^u\delta + \log \int_{\{|x|\le M\}} \exp[n^u(\psi(x) - g(x))]dx\right] \\
&= \delta + \lim_{n\to\infty} \frac{1}{n^u} \log \int_{\{|x|\le M\}} \exp[n^u(\psi(x) - g(x))]dx \\
&= \delta + \sup_{x\in\mathbb{R}}\{\psi(x) - g(x)\}.
\end{aligned} \tag{B.9}$$

Since $\delta > 0$ is arbitrary, combining (B.5), (B.7), and (B.9) yields

$$\lim_{n\to\infty} \frac{1}{n^{-v}} \log \int_{\mathbb{R}} \exp[n^{-v}\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx = \sup_{x\in\mathbb{R}}\{\psi(x) - g(x)\}.$$

Using the fact that the supremum of g is attained on the interval $[-M, M]$ (see item 2 in the proof of Step 2), we apply the limit in the last display to $\psi = 0$. We conclude from (B.2) that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^u} \log \int_{\Lambda^n\times\Omega} \exp\left[n^u\psi\left(\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}}\right)\right] d(P_{n,\beta_n,K_n} \times Q) \\
&= \lim_{n\to\infty} \frac{1}{n^u} \log \int_{\mathbb{R}} \exp[n^u\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]dx \\
&\quad - \lim_{n\to\infty} \frac{1}{n^u} \log \int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]dx \\
&= \sup_{x\in\mathbb{R}}\{\psi(x) - g(x)\} + \sup_{x\in\mathbb{R}}\{-g(x)\} \\
&= \sup_{x\in\mathbb{R}}\{\psi(x) - g(x)\} + \inf_{y\in\mathbb{R}} g(y) \\
&= \sup_{x\in\mathbb{R}}\{\psi(x) - \Gamma(x)\}.
\end{aligned}$$

Except for the proof of (B.8) we have completed the proof of Step 2, which show that with respect to $P_{n,\beta_n,K_n} \times Q$, $S_n/n^{1-\gamma} + W_n/n^{1/2-\gamma}$ satisfies the Laplace principle with exponential speed n^{-v} and rate function Γ.

To prove (B.8) we define

$$A_n = \exp(n^u\delta) \int_{\{|x|\le M\}} \exp[n^u(\psi(x) - g(x))]dx.$$

It suffices to show that $\delta_n/A_n \to 0$. To see this, we rewrite (B.8) as follows:

$$\begin{aligned}
&\limsup \frac{1}{n^u} \log(A_n + \delta_n) \\
&\quad = \limsup \frac{1}{n^u} \log \left[A_n \left(1 + \frac{\delta_n}{A_n} \right) \right] \\
&\quad = \limsup \left[\frac{1}{n^u} \log A_n + \frac{1}{n^u} \log \left(1 + \frac{\delta_n}{A_n} \right) \right] \\
&\quad = \limsup \frac{1}{n^u} \log A_n.
\end{aligned}$$

Now we prove that $\delta_n/A_n \to 0$. By (B.6) we have

$$\begin{aligned}
\lim \frac{1}{n^u} \log A_n &= \delta + \lim_{n\to\infty} \frac{1}{n^u} \log \int_{\{|x|\le M\}} \exp[n^u(\psi(x) - g(x))]dx \\
&= \delta + \sup_{x\in\mathbb{R}} \{\psi(x) - g(x)\} \\
&= \delta + \Delta,
\end{aligned}$$

which implies that for all sufficiently large n

$$A_n \ge \exp\left[n^u \left(\frac{\delta}{2} + \Delta \right) \right].$$

Since by (B.4) we have for all sufficiently large n

$$\delta_n \le 4Rn^\gamma \exp[-n^u(|\Delta| + 1)],$$

it follows that for any $0 < \varepsilon < 1$ and all sufficiently large n

$$\delta_n \le \exp[n^u(-|\Delta| - 1 + \varepsilon)]$$

and thus

$$0 \le \frac{\delta_n}{A_n} \le \exp\left[n^u \left(-|\Delta| - 1 + \varepsilon - \frac{\delta}{2} - \Delta \right) \right]. \tag{B.10}$$

If $\Delta \ge 0$, then

$$-|\Delta| - 1 + \varepsilon - \frac{\delta}{2} - \Delta = -1 + \varepsilon - \frac{\delta}{2} - 2\Delta < 0.$$

If $\Delta < 0$, then

$$-|\Delta| - 1 + \varepsilon - \frac{\delta}{2} - \Delta = -1 + \varepsilon - \frac{\delta}{2} < 0.$$

Thus in all cases the limit of the right hand side of (B.10) is 0. This completes the proof of (B.8).

Together Step 1 and Step 2 prove that with respect to P_{n,β_n,K_n}, $S_n/n^{1-\gamma}$ satisfies the Laplace principle with speed n^{-v} and rate function Γ. The proof of Theorem 6.2 is complete.

References

[1] M. N. Barber. Finite-size scaling, in *Phase Transitions and Critical Phenomena*, Vol 8, 145–266. C. Domb and J. Lebowitz, editors, Academic Press, London, 1983.

[2] M. Blume. Theory of the first-order magnetic phase change in UO_2. *Phys. Rev.* **141** (1966) 517–524.

[3] H. W. Capel. On the possibility of first-order phase transitions in Ising systems of triplet ions with zero-field splitting. *Physica* **32**: 966–988, 1966.

[4] H. W. Capel. On the possibility of first-order phase transitions in Ising systems of triplet ions with zero-field splitting II. *Physica* **33**:295–331, 1967.

[5] H. W. Capel. On the possibility of first-order phase transitions in Ising systems of triplet ions with zero-field splitting III. *Physica* **37**:423–441, 1967.

[6] M. Costeniuc, R. S. Ellis, and P. T.-H. Otto. Multiple critical behavior of probabilistic limit theorems in the neighborhood of a tricritical point. *J. Stat. Phys.* **127** (2007) 495–552.

[7] P. Dupuis and R. S. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*, John Wiley & Sons, New York, 1997.

[8] R. S. Ellis and J. Li. Conditional Gaussian Fluctuations and Refined Asymptotics of the Spin in the Phase-Coexistence Region. 36-page Latex file. Submitted for publication.

[9] R. S. Ellis, J. Machta, and P. T. Otto. Asymptotic behavior of the magnetization near critical and tricritical points via Ginzburg-Landau polynomials. *J. Stat. Phys.* **133** (2008) 101–129.

[10] R. S. Ellis, J. Machta, and P. T. Otto. Ginzburg-Landau polynomials and the asymptotic behavior of the magnetization near critical and tricritical points. Unpublished. http://arxiv.org/abs/0803.0178

[11] R. S. Ellis and C. M. Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Z. Wahrsch. verw. Geb.* **44** (1979) 117–139.

[12] R. S. Ellis, J. Machta, and P. T. Otto. Asymptotic behavior of the finite-size magnetization as a function of the speed of approach to criticality. *Ann. Appl. Prob.* **20** (2010) 2118–2161.

[13] R. S. Ellis, P. T. Otto, and H. Touchette. Analysis of phase transitions in the mean-field Blume-Emery-Griffiths model. *Ann. Appl. Prob.* **15** (2005) 2203–2254.

[14] R. S. Ellis and K. Wang. Limit theorems for the empirical vector of the Curie-Weiss-Potts model. *Stoch. Proc. Appl.* **35** (1990) 59–79.

[15] A. N. Shiryaev, Probability, Second edition, translated by R. P. Boas, Springer, New York, 1995.