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Limit Theorems for Random Evolutions with Explicit Error Estimates*

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1. Introduction

Let $\Lambda=\{v_i; 1\leqq i\leqq N\}$ be N distinct real numbers and $\{v(t); t\geqq 0\}$ an irreducible jump Markov process with state space Λ. We define

$$x(t)=\int_0^t v(s)\,ds$$

and for any starting point y

$$x(t,y)=y+x(t).$$

We think of $x(t,y)$ as the position of a particle at time t when its velocity is $v(t)$. The process $x(t,y)$ is the simplest example of a *random evolution*: one-dimensional motion at a constant but random velocity determined by the state of the Markov chain associated with $v(t)$. We denote by $P_{(y,v_i)}\{\cdot\}$, y real, $v_i\in\Lambda$, the probability laws of the joint process $(x(t,y), v(t))$, where $v(0)=v_i$. $E_{(y,v_i)}$ will denote integration with respect to $P_{(y,v_i)}$. The purpose of this paper is to prove the following two theorems, which correspond respectively to the weak law of large numbers and the central limit theorem for $x(t)$.

Theorem 1. *For any* $-\infty<a<b<\infty$, *y real*, $y\neq a,b$, $v_i\in\Lambda$, $t>0$, $\varepsilon>0$,

$$P_{(y,v_i)}\{a<x_\varepsilon(t)-\bar{v}\,t<b\}=1_{(a,b)}(y)+O(\varepsilon), \tag{1.1}$$

where $x_\varepsilon(t)=y+\varepsilon\, x(t/\varepsilon)$, $1_{(a,b)}$ *is the characteristic function of the interval* (a,b), *and* $\bar{v}$, *a real determined by* $v(t)$, *is given by formula* (2.5).

Theorem 2. *For any* $-\infty<a<b<\infty$, *y real*, $v_i\in\Lambda$, $t>0$, $\varepsilon>0$,

$$P_{(y,v_i)}\{a<x^\varepsilon(t)-\bar{v}\,t/\varepsilon<b\}=\frac{1}{\sqrt{2\pi t}\,\sigma}\int_a^b \exp\left(-(y-z)^2/2\sigma^2 t\right)dz+O(\varepsilon), \tag{1.2}$$

where $x^\varepsilon(t)=y+\varepsilon\, x(t/\varepsilon^2)$ *and where* σ^2, *a positive number determined by* $v(t)$, *is given by formula* (2.6).

The error $O(\varepsilon)$ in (1.1) is uniform over $\{y\colon |y-a|\geqq\delta,\ |y-b|\geqq\delta\}$, any $\delta>0$. The error in (1.2) is uniform over all y.

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These theorems are proved by noticing that if each function $f(x, v_i)$, x real, $v_i \in A$, is sufficiently smooth, then the N functions

$$p_i(t, x) = p(t, x, v_i) = E_{(x, v_i)}\{f(x(t), v(t))\}, \quad t > 0, \tag{1.3}$$

are the solution of the system of equations (backward equation of the $(x(t), v(t))$ process)

$$\begin{gathered}\frac{\partial p_i}{\partial t} = v_i \frac{\partial p_i}{\partial x} + \sum_{j=1}^{N} q_{ij} p_j, \\ \lim_{t \downarrow 0} p_i = f(\cdot, v_i), \quad 1 \leqq i \leqq N.\end{gathered} \tag{1.4}$$

In (1.4), $Q = (q_{ij})$ is the infinitesimal generator of the process $v(t)$. See [7] for a derivation of (1.4). Q has a simple eigenvalue at zero corresponding to the irreducibility of $v(t)$. In the sequel we write (1.4) in matrix form as

$$\frac{\partial p}{\partial t} = v \frac{\partial p}{\partial x} + Q p, \quad \lim_{t \downarrow 0} p = f, \tag{1.5}$$

where v is the diagonal matrix with entries v_i.

To prove the first theorem, we use the fact that

$$p_\varepsilon = E_{(x, v_i)}\{f(x_\varepsilon(t), v(t/\varepsilon))\} \tag{1.6}$$

solves

$$\frac{\partial p_\varepsilon}{\partial t} = v \frac{\partial p_\varepsilon}{\partial x} + \frac{1}{\varepsilon} Q p_\varepsilon, \quad \lim_{t \downarrow 0} p_\varepsilon = f,$$

which can be solved explicitly by Fourier transforms in terms of the eigenvalue and component matrices of $Q + i\gamma v$, γ real. Denoting by $\alpha_1(i\gamma)$ that eigenvalue which satisfies $\alpha_1(0) = 0$, we observe that the limiting behavior of p_ε is essentially determined by the behavior of $\alpha_1(i\gamma)$ and the associated component matrix for γ near zero, and this can be studied by hand because of the explicit representation of p_ε. The proof of Theorem 2 proceeds along the same lines since

$$p^\varepsilon = E_{(x, v_i)}\{f(x^\varepsilon(t), v(t/\varepsilon^2))\} \tag{1.7}$$

solves

$$\frac{\partial p^\varepsilon}{\partial t} = \frac{v}{\varepsilon} \frac{\partial p^\varepsilon}{\partial x} + \frac{1}{\varepsilon^2} Q p^\varepsilon, \quad \lim_{t \downarrow 0} p^\varepsilon = f.$$

The errors $0(\varepsilon)$ in the two results fall out with a little extra work.

These theorems, with a weaker error estimate, were first proved by Pinsky [7], who used a boundary layer expansion to study p_ε and p^ε. They have been generalized by Hersh-Papanicolaou [3], Hersh-Pinsky [4], Kurtz [5], and the author [2]. It is noteworthy that the idea for the present proofs arose from the study of approximation properties of certain special solutions of the equation

$$\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial x} = Q p, \tag{1.8}$$

which is fully treated in [1] as a Markovian model of the Boltzmann equation. The present methods have also been fruitful in yielding new limit theorems (see [2]) for scaled versions of (1.8), where Q is allowed to have a d-dimensional nullspace ($1 \leqq d < N$).

As for the organization of this paper, in Section 2 we summarize certain facts about Q, the eigenvalues of $Q+\lambda v$ (λ complex), and solutions of (1.5). In Section 3 we first present simple proofs of the limit theorems without error estimates followed by the more detailed proofs of the latter.

2. Summary of Facts Needed for Proofs

Fact 1. The matrix Q has a unique left eigenvector $\tilde{e}_0 = (\tilde{e}_0^{(k)})$ satisfying

$$\tilde{e}_0 Q = 0 \tag{2.1}$$

and normalized so that

$$\sum_{k=1}^{N} \tilde{e}_0^{(k)} = 1. \tag{2.2}$$

The components $\tilde{e}_0^{(k)}$ have the property $\lim_{t\to\infty} P_{v_i}\{v(t)=v_k\} = \tilde{e}_0^{(k)}$ independently of i. We write $e_0 = (1, \ldots, 1)$ for the right eigenvector of Q corresponding to the eigenvalue 0. Now (2.2) implies

$$\langle \tilde{e}_0, e_0 \rangle = 1, \tag{2.3}$$

where $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathscr{R}^N$.

Fact 2. Let $\{\alpha_j(\lambda)\}$ denote the eigenvalues of $Q+\lambda v$, λ complex, where $\alpha_1(\lambda)$ is that eigenvalue satisfying $\alpha_1(0)=0$. The $\{\alpha_j(\lambda)\}$ are branches of the algebraic function defined by $\det(Q+\lambda v-\alpha)=0$. Each $\alpha_j(\lambda)$ is continuous in λ. Because zero is a simple eigenvalue of Q, $\alpha_1(\lambda)$ and the associated (right) eigenvector $e(\lambda)$ have analytic expansions in λ for λ sufficiently small ([6], p. 241):

$$\alpha_1(\lambda) = \bar{v}\lambda + \tfrac{1}{2}\sigma^2\lambda^2 + O(\lambda^3), \tag{2.4}$$
$$e(\lambda) = \sum_{j \geqq 0} e_j \lambda^j \qquad (|\lambda| < \mu, \text{ some } \mu > 0).$$

By matching coefficients of λ and λ^2 in

$$(Q+\lambda v)\, e(\lambda) = \alpha_1(\lambda)\, e(\lambda)$$

and using (2.1) and (2.3), one can show

$$\bar{v} = \langle \tilde{e}_0, v e_0 \rangle, \tag{2.5}$$

$$\tfrac{1}{2}\sigma^2 = -\langle \tilde{e}_0, (v-\bar{v})\, Q^{-1} (v-\bar{v})\, e_0 \rangle. \tag{2.6}$$

In (2.6), Q^{-1} denotes the inverse of Q off its null space. That $\sigma^2 > 0$ has been proved in ([7], p. 103).

Fact 3. For any $\gamma_0 > 0$, there is a $\beta < 0$ such that

$$\begin{aligned} &\operatorname{Re}\alpha_j(i\gamma) \leqq \beta,\ 2 \leqq j \leqq N \text{ and } \gamma \text{ real},\\ &\operatorname{Re}\alpha_1(i\gamma) \leqq 0 \text{ for } \gamma \text{ real},\\ &\operatorname{Re}\alpha_1(i\gamma) \leqq \beta \text{ for } \gamma \text{ real},\ |\gamma| \geqq \gamma_0. \end{aligned} \tag{2.7}$$

This was proved in ([7], pp. 104–105).

Fact 4. Let p solve (1.5) with $f\in C_{\downarrow}^{\infty}$, i.e., for each $v_i\in\Lambda$, $f(\cdot, v_i)$ is a C^∞, rapidly decreasing function on $\mathscr{R}^1$. Then for each $t>0$, we have $p\in C_{\downarrow}^{\infty}$. We can write p in the form

$$p=\int_{-\infty}^{\infty}\exp[t(Q+i\gamma v)]\hat{f}e^{i\gamma x}\,d\gamma, \tag{2.8}$$

where

$$\hat{f}=\frac{1}{2\pi}\int_{-\infty}^{\infty} f e^{-i\gamma x}\,dx$$

is the Fourier transform of f and where

$$\exp[t(Q+\lambda v)]=\sum_{j=1}^{N}\{Z_{j1}(\lambda)+tZ_{j2}(\lambda)+\cdots+t^{N-1}Z_{j,N-1}(\lambda)\}\exp\big(t\,\alpha_j(\lambda)\big). \tag{2.9}$$

The $\{Z_{ji}\}$ are the *component matrices* of $Q+\lambda v$ ([6], p. 174). If $\alpha_j(\lambda)$ is distinct from the other eigenvalues, then $Z_{j2}=\cdots=Z_{j,N-1}=0$, and Z_{j1} is the eigenprojection onto the eigenspace corresponding to $\alpha_j(\lambda)$. Z_{j1} is then given by the formula

$$Z_{j1}(\lambda)=\prod_{k\neq j}\frac{Q+\lambda v-\alpha_k(\lambda)}{\alpha_j(\lambda)-\alpha_k(\lambda)}. \tag{2.10}$$

Also, p_ε and p^ε, defined by (1.6) and (1.7), are given by

$$p_\varepsilon=\int_{-\infty}^{\infty}\exp[t(Q+i\varepsilon\gamma v)/\varepsilon]\hat{f}e^{i\gamma x}\,d\gamma, \tag{2.11}$$

$$p^\varepsilon=\int_{-\infty}^{\infty}\exp[t(Q+i\varepsilon\gamma v)/\varepsilon^2]\hat{f}e^{i\gamma x}\,d\gamma. \tag{2.12}$$

3. Proofs of Theorems 1 and 2

3 A. Without Error Estimates

Take $f\in C_{\downarrow}^{\infty}$ independent of v_i. We prove the theorems by showing for each $t>0$, x real, $v_i\in\Lambda$, that

$$\lim_{\varepsilon\downarrow 0} p_\varepsilon(t,x,v_i)=f(x+\bar{v}\,t), \tag{3 A.1}$$

$$\lim_{\varepsilon\downarrow 0} p^\varepsilon(t,x-\bar{v}\,t/\varepsilon,v_i)=\frac{1}{\sqrt{2\pi t}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-y)^2/2t\sigma^2}f(y)\,dy. \tag{3 A.2}$$

To do this we need the fact that for γ real

$$|\hat{p}_\varepsilon(t,\gamma,v)|,\quad |\hat{p}^\varepsilon(t,\gamma,v)|\leqq N^2|\hat{f}(\gamma)|. \tag{3 A.3}$$

But from (2.7) and (2.9), if $Q+i\gamma v$ is semisimple and hence diagonable, then

$$\|\exp[t(Q+i\gamma v)]\|\leqq N.$$

Now say γ_0 is such that $Q+i\gamma_0 v$ is not semisimple. Since there are at most finitely many points at which this can occur, there exist points $\gamma_n\to\gamma_0$ such that

$$\|\exp[t(Q+i\gamma_n v)]\|\leqq N.$$

This implies the bound for γ_0. Since $\|\hat{f}(\gamma,\cdot)\|=N\,|\hat{f}(\gamma)|$ when f is independent of v_i, we have proven (3 A.3).

We now prove (3 A.1). By the dominated convergence theorem, it suffices to show

$$\lim_{\varepsilon\downarrow 0}\hat{p}_\varepsilon=\hat{f}e^{i\gamma\bar{v}t}. \tag{3 A.4}$$

But by (2.11)

$$\hat{p}_\varepsilon=\exp[t(Q+i\,\varepsilon\,\gamma\,v)/\varepsilon]\,\hat{f},$$

and by (2.7) we can find $\beta<0$ such that

$$\exp[t(Q+i\,\varepsilon\,\gamma\,v)/\varepsilon]=\left(Z_{11}(i\,\varepsilon\,\gamma)+\cdots+\left(\frac{t}{\varepsilon}\right)^{N-1}Z_{1,N-1}(i\,\varepsilon\,\gamma)\right)$$
$$\cdot\exp[t\,\alpha_1(i\,\varepsilon\,\gamma)/\varepsilon]+O(e^{\beta t/\varepsilon})$$

as $\varepsilon\downarrow 0$. For ε suitably small, $Z_{12}(i\,\varepsilon\,\gamma)=\cdots=Z_{1,N-1}(i\,\varepsilon\,\gamma)=0$, since zero is a simple eigenvalue of Q. Since $Q\hat{f}=0$,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}Z_{11}(i\,\varepsilon\,\gamma)\hat{f}&=\lim_{\varepsilon\downarrow 0}\prod_{j\geqq 2}\frac{Q+i\,\varepsilon\,\gamma\,v-\alpha_j(i\,\varepsilon\,\gamma)}{\alpha_1(i\,\varepsilon\,\gamma)-\alpha_j(i\,\varepsilon\,\gamma)}\hat{f}\\&=\prod_{j\geqq 2}\frac{\alpha_j(0)}{\alpha_j(0)}\hat{f}\\&=\hat{f},\end{aligned}$$

where we have used (2.10). Also,

$$\lim_{\varepsilon\downarrow 0}\exp(t\,\alpha_1(i\,\varepsilon\,\gamma)/\varepsilon)=e^{i\gamma\bar{v}t},$$

by the expansion (2.4). This proves (3 A.4) and thus (3 A.1). To prove (3 A.2), we show

$$\lim_{\varepsilon\downarrow 0}\hat{p}^\varepsilon\,e^{-i\gamma\bar{v}t/\varepsilon}=e^{-\sigma^2\gamma^2t/2}\hat{f}.$$

This follows exactly as above once one notices that

$$\lim_{\varepsilon\downarrow 0}\exp\left[t(\alpha_1(i\,\gamma\,\varepsilon)-i\,\varepsilon\,\gamma\,\bar{v})/\varepsilon^2\right]=e^{-\sigma^2\gamma^2t/2}.$$

3 B. Error Estimates

We first prove the error estimate in Theorem 2, then sketch the proof for Theorem 1. For notational convenience, we assume $\bar{v}=0$.

Denoting by $T_t f$ the right-hand side of (3 A.2), we first prove that for $f\in C_\downarrow^\infty$ independent of v_i

$$\|p^\varepsilon-T_t f\|_\infty=O(\varepsilon)\|f\|_1+O(e^{\beta t/\varepsilon^2})\,\|\hat{f}\|_1, \tag{3 B.1}$$

where p^ε is given by (2.12). As in the proof in Section 3 A, p^ε is first compared with

$$p_0^\varepsilon=\int_{-\infty}^{\infty}\left(Z_{11}(i\,\varepsilon\,\gamma)+\cdots+\left(\frac{t}{\varepsilon}\right)^{N-1}Z_{1,N-1}(i\,\varepsilon\,\gamma)\right)\hat{f}$$
$$\cdot\exp[t\,\alpha_1(i\,\varepsilon\,\gamma)/\varepsilon^2]\,e^{i\gamma x}\,d\gamma \tag{3 B.2}$$

to give

$$\|p^\varepsilon - p_0^\varepsilon\|_\infty = O(e^{\beta t/\varepsilon^2})\,\|\hat{f}\|_1. \tag{3B.3}$$

Now pick M such that $\operatorname{Re}\alpha_1(i\gamma)$ is monotonically decreasing for $|\gamma| \leqq M$ and $\alpha_1(i\gamma)$ equals no other eigenvalue in this interval. The first of these is possible because of (2.4) and the positivity of σ^2. Denoting by Y the integrand in (3B.2), we write

$$\begin{aligned} p_0^\varepsilon &= \int\limits_{|\gamma|>M/\varepsilon} + \int\limits_{M/\varepsilon \geqq |\gamma| \geqq 1/\sqrt[3]{\varepsilon}} + \int\limits_{1/\sqrt[3]{\varepsilon}>|\gamma|} Y\,d\gamma \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{3B.4}$$

We note that for I_2 and I_3, $Z_{12} = \cdots = Z_{1,N-1} = 0$. For I_1, we have

$$\|I_1\|_\infty = O(e^{\beta t/\varepsilon^2})\,\|\hat{f}\|_1 \tag{3B.5}$$

by (2.7). For I_2, we find

$$\begin{aligned} \|I_2\|_\infty &\leqq \frac{\text{const}}{\varepsilon}\,\|\hat{f}\|_\infty \sup_{M/\varepsilon \geqq |\gamma| \geqq 1/\sqrt[3]{\varepsilon}} \exp\left[t\left(\operatorname{Re}\alpha_1(i\varepsilon\gamma)\right)/\varepsilon^2\right] \\ &= \frac{\text{const}}{\varepsilon}\,\|\hat{f}\|_\infty \max_{\pm} \exp\left[t\left(\operatorname{Re}\alpha_1(\pm i\varepsilon^{2/3})\right)/\varepsilon^2\right] \\ &= \frac{\text{const}}{\varepsilon}\,\|\hat{f}\|_\infty \exp(-t\sigma^2/2\varepsilon^{2/3}), \quad \text{as } \varepsilon \downarrow 0, \end{aligned}$$

where we have used the decrease of $\alpha_1(i\gamma)$ and the expansion

$$\frac{1}{\varepsilon^2}\alpha_1(i\varepsilon\gamma) = -\tfrac{1}{2}\sigma^2\gamma^2 + O(\varepsilon\gamma^3), \quad \text{as } \varepsilon\gamma \to 0. \tag{3B.6}$$

In I_3, (3B.6) can be used again to give

$$\begin{aligned} I_3 &= \int\limits_{1/\sqrt[3]{\varepsilon}>|\gamma|} Z_{11}(i\varepsilon\gamma)\hat{f}\exp[(-t\sigma^2\gamma^2/2) + O(\varepsilon\gamma^3)]\,e^{i\gamma x}\,d\gamma \\ &= \int\limits_{1/\sqrt[3]{\varepsilon}>|\gamma|} Z_{11}(i\varepsilon\gamma)\hat{f}\,e^{-t\sigma^2\gamma^2/2}(e^{O(\varepsilon\gamma^3)} - 1)\,e^{i\gamma x}\,d\gamma \\ &\quad + \int\limits_{1/\sqrt[3]{\varepsilon}>|\gamma|} Z_{11}(i\varepsilon\gamma)\hat{f}\,e^{-t\sigma^2\gamma^2/2}\,e^{i\gamma x}\,d\gamma \\ &\equiv J_1 + J_2. \end{aligned}$$

For J_1, use the fact that $e^{O(\varepsilon\gamma^3)} - 1 = O(\varepsilon\gamma^3)$ and pull out the ε. Hence

$$\begin{aligned} \|J_1\|_\infty &\leqq \text{const} \times \varepsilon \int\limits_{1/\sqrt[3]{\varepsilon}>|\gamma|} |Z_{11}(i\varepsilon\gamma)\hat{f}|\,e^{-t\sigma^2\gamma^2/2}\,|\gamma|^3\,d\gamma \\ &= O(\varepsilon)\,\|\hat{f}\|_\infty. \end{aligned} \tag{3B.7}$$

For J_2, we use the fact that

$$Z_{11}(i\varepsilon\gamma) = Z_{11}(0) + O(\varepsilon\gamma), \quad \text{as } \varepsilon\gamma \to 0. \tag{3B.8}$$

This follows from the fact that $Z_{11}(\lambda)$ is a symmetric function of the $\{\alpha_j(\lambda); 2 \leqq j \leqq N\}$ and so by the monodromy theorem can be shown to be analytic in any region of the complex λ-plane which does not contain any branch points of $\alpha_1(\lambda)$. Using (3B.8), we find

$$J_2 = \int_{1/\sqrt[3]{\varepsilon} > |\gamma|} Z_{11}(0) \hat{f} e^{-t\sigma^2\gamma^2/2} e^{i\gamma x} d\gamma + O(\varepsilon) \|\hat{f}\|_\infty. \tag{3B.9}$$

Since $Z_{11}(0)\hat{f} = \hat{f}$, the discrepancy between the integral in (3B.9) and $T_t f$ is

$$\Big\| \int_{|\gamma| > 1/\sqrt[3]{\varepsilon}} \hat{f} e^{-t\sigma^2\gamma^2/2} e^{i\gamma x} d\gamma \Big\|_\infty \leqq \|\hat{f}\|_\infty. \tag{3B.10}$$

From (3B.3), (3B.5), (3B.7), (3B.9), (3B.10), we conclude (3B.1).

To extend the result to $1_{(a,b)}$, take functions $f^\pm \in C_\downarrow^\infty$ such that

a) $f^- \leqq 1_{(a,b)} \leqq f^+$,

b) $f^-(y) = 1_{(a,b)}(y) = f^+(y)$ for $|y-a| \geqq \varepsilon$, $|y-b| \geqq \varepsilon$, (3B.11)

c) $\|(f^\pm)''\|_1 \leqq \dfrac{\text{const}}{\varepsilon}$.

Then

$$|\hat{f}^\pm(\gamma)| \leqq \frac{\text{const}}{\varepsilon\gamma^2}, \qquad |\gamma| \geqq 1,$$

so that

$$\|\hat{f}^\pm\|_1 \leqq \frac{\text{const}}{\varepsilon}.$$

If $C_t^\varepsilon f$ denotes the right-hand side of (1.7), then

$$\begin{aligned} \|C_t^\varepsilon 1_{(a,b)} - T_t 1_{(a,b)}\|_\infty &\leqq \max_\pm \|C_t^\varepsilon f^\pm - T_t f^\mp\|_\infty \\ &\leqq \max_\pm \|C_t^\varepsilon f^\pm - T_t f^\pm\|_\infty + \|T_t f^+ - T_t f^-\|_\infty \\ &= O(\varepsilon) \end{aligned}$$

by (3B.1). This completes the proof of Theorem 2.

Concerning Theorem 1, we denote by $B_t^\varepsilon f$ the right-hand side of (1.6). For $f \in C_\downarrow^\infty$, one proves that

$$\|B_t^\varepsilon f - f\|_\infty = O(e^{\beta t/\varepsilon}) \|\hat{f}\|_1 + O(\varepsilon)(\|\gamma \hat{f}\|_1 + \|\gamma^2 \hat{f}\|_1). \tag{3B.12}$$

Given $\delta > 0$, take y such that $|y-a| \geqq \delta$, $|y-b| \geqq \delta$, and pick $f^\pm \in C_\downarrow^\infty$ satisfying a) and b) in (3B.11) but with δ instead of ε. Then

$$|B_t^\varepsilon 1_{(a,b)}(y) - 1_{(a,b)}(y)| \leqq \max_\pm \|B_t^\varepsilon f^\pm - f^\pm\|_\infty + |f^+(y) - f^-(y)| = O(\varepsilon)$$

by (3B.12) and the fact that $f^+(y) = f^-(y)$. The error $O(\varepsilon)$ depends on $1/\delta$. This takes care of Theorem 1.

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Note Added in Proof. For certain special cases, the error in Theorem 2 can be strengthened to $O(\varepsilon^2)$.

Theorem. *Assume that Q is bisymmetric (symmetric about both its diagonals) and that the velocities $\{v_i\}$ satisfy $v_i = -v_{N+1-i}$, $i=1,\dots,N$. Denote by π the invariant distribution of $v(t)$. Then for $-\infty<a<b<\infty$, y real, $t>0$, $\varepsilon>0$,*

$$P_{(y,\pi)}\{a<x^\varepsilon(t)-\bar{v}\,t/\varepsilon<b\}=T_t 1_{(a,b)}(y)+O(\varepsilon^2),$$

uniformly in y.

Assume $\bar{v}=0$. The sources of the $O(\varepsilon)$ error in proof of Theorem 2 in Section 3B are (3B.6) and (3B.8). The assumptions on Q and $\{v_i\}$ can be shown to imply that the eigenvalues $\alpha_j(i\gamma)$ of $Q+i\gamma v$ are analytic functions of γ^2, $|\gamma|$ sufficiently small. Hence in this case the $O(\varepsilon\gamma^3)$ term in (3B.6) becomes $O(\varepsilon^2\gamma^4)$. We are finished if we can prove that

$$\sum_{k=1}^{N}\tilde{e}_0^{(k)}\left(Z_{11}(i\varepsilon\gamma)\hat{f}(\gamma)\right)(v_k)=\hat{f}(\gamma)+O(\varepsilon^2\gamma^2),\quad \text{as } \varepsilon\gamma\to 0, \tag{3B.13}$$

where $\tilde{e}_0^{(k)}=1/N$, $k=1,\dots,N$, is the invariant distribution of $v(t)$. But from (2.10),

$$\left(\frac{dZ(i\gamma)}{d\gamma}\bigg|_{\gamma=0}\hat{f}\right)(v_k)=c\,v_k\hat{f}+(Qg)(\gamma,v_k),$$

where c is a constant and g is some function of v_i and γ. Since $N^{-1}\Sigma v_k=\bar{v}=0$ and $\tilde{e}_0 Q=0$, (3B.13) follows.

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