

THE PROJECTION OF THE NAVIER-STOKES EQUATIONS UPON THE EULER EQUATIONS (*)

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1. INTRODUCTION

In this paper we formulate a new limit theorem for the linear Navier-Stokes equations in any number of dimensions. By combining this result with a companion limit theorem [1] for the linearized Boltzmann equation, we obtain a new proof of the known asymptotic relations between the Boltzmann and Navier-Stokes descriptions of a gas.

We denote by $\{\exp(tA); -\infty < t < \infty\}$ ⁽¹⁾ the solution operators for the Euler equations and by $\{\exp(t(A+\varepsilon B)); 0 \leq t < \infty\}$ ⁽²⁾ the solution operators for the Navier-Stokes equations with viscosity and heat conduction coefficients proportional to $\varepsilon > 0$. A and B are matrices whose entries are, respectively, first-order and second-order (spatial) different operators with constant coefficients. Our main result states that

$$(1.1) \quad \exp\left(-\frac{t}{\varepsilon}A\right)\exp\left(\frac{t}{\varepsilon}(A+\varepsilon B)\right)f = \exp(t\Pi_A B)f + O(\varepsilon) \quad \text{as } \varepsilon \downarrow 0,$$

for $t \geq 0$ and a wide class of initial data f . The operator $\Pi_A B$ is the projection, in the class of matrices, of B onto the set of matrices which commute with A . Furthermore, $\{\exp(t\Pi_A B); 0 \leq t < \infty\}$ are the solution operators for a parabolic system of partial differential equations.

In order to apply this result to the Boltzmann equation, we recall the main results of [1]. Denoting by $\{T_\varepsilon(t); 0 \leq t < \infty, 0 < \varepsilon < \infty\}$ the solution operators for the linearized Boltzmann equation, we showed that if f is suitably smooth, then

$$(1.2) \quad T_\varepsilon(t)f = \exp(tA)f + O(\varepsilon) \quad \text{as } \varepsilon \downarrow 0,$$

$$(1.3) \quad \exp\left(-\frac{t}{\varepsilon}A\right)T_\varepsilon\left(\frac{t}{\varepsilon}\right)f = \bar{N}(t)f + O(\varepsilon) \quad \text{as } \varepsilon \downarrow 0,$$

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⁽¹⁾ These operators are denoted by $E(t)$ in [1].

⁽²⁾ These operators are denoted by $N_\varepsilon(t)$ in [1].

where $\bar{N}(t)$ are the solution operators for a certain parabolic system of partial differential equations. We shall prove that these operators coincide with the operators $\exp(t \Pi_A B)$ in (1.1). Hence, we can combine (1.3) with the limit result (1.1) to obtain the asymptotic result

$$(1.4) \quad T_\varepsilon\left(\frac{t}{\varepsilon}\right)f = \exp\left(\frac{t}{\varepsilon}(A + \varepsilon B)\right)f + O(\varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

This has already been proved by Grad ([3], [6]), but he requires smoother initial data than we do. We also prove the existence of infinitely many solution operators for systems of partial differential equations such that (1.1) and (1.4) remain true when $\exp(t(A + \varepsilon B)/\varepsilon)$ is replaced by any of these. This illustrates the asymptotic non-uniqueness of the Navier-Stokes equations.

2. Statement of Main Results

Fix $N \geq 1$ and define the measure

$$\rho(d\xi) = (2\pi)^{-N/2} \exp(-|\xi|^2/2) d\xi, \quad \xi \in \mathbf{R}^N.$$

We denote by \mathcal{H}_0 the complex Hilbert space $L^2(\rho(d\xi))$ with inner product $\langle \cdot, \cdot \rangle$ and by \mathcal{H} the complex Hilbert space $L^2(\rho(d\xi) \times dx)$, $x \in \mathbf{R}^N$, with norm $\|\cdot\|$ ⁽³⁾.

We write the linearized Boltzmann equation in the form

$$(2.1) \quad \frac{\partial p}{\partial t} + \xi \cdot \nabla p = \frac{1}{\varepsilon} Q p,$$

$$\lim_{\varepsilon \downarrow 0} p = f.$$

In (2.1), $p = p(t, x, \xi)$ ($t > 0$, $x, \xi \in \mathbf{R}^N$), $\varepsilon > 0$, and $f \in \mathcal{H}$. Q , the collision operator, is a symmetric, negative semi-definite integral operator on \mathcal{H}_0 which operates on p as a function of ξ only. Q commutes with rotations of \mathbf{R}^N . The kernel of Q is determined by the intermolecular force law, which later on we shall restrict. For a discussion of these facts, see Uhlenbeck and Ford ([8]; chapter IV).

Corresponding to the conservation of number, momentum, and energy in an individual collision, the nullspace \mathcal{N}_0 of Q is spanned by 1 , ξ_j ($j = 1, \dots, N$), and $|\xi|^2$. We denote by $\{h_j, j = 0, 1, \dots, N+1\}$ the following orthonormal basis of \mathcal{N}_0 :

$$h_0 = 1, \quad h_j = \xi_j \quad (j = 1, \dots, N), \quad h_{N+1} = \frac{|\xi|^2 - N}{\sqrt{2N}}.$$

⁽³⁾ This norm is denoted $\|\cdot\|$ in [1].

