

ASYMPTOTIC NONUNIQUENESS  
OF THE NAVIER-STOKES EQUATIONS  
IN KINETIC THEORY<sup>1</sup>

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We consider the linearized Boltzmann equation

$$(1) \quad \partial p / \partial t + \xi \cdot \text{grad } p = Qp / \epsilon,$$

whose solution  $p = p_\epsilon(t, x, \xi)$ ,  $t > 0$ ,  $x \in R^3$ ,  $\xi \in R^3$ ,  $\epsilon > 0$ .  $Q$  is the linearized collision operator corresponding to a spherically symmetric hard potential, and  $\epsilon$  is a parameter which represents the mean free path.

In a series of basic papers, Grad [6], [7], [8] studied the existence and asymptotic behavior of the solution of the initial value problem for (1), where the initial data  $p_\epsilon(0^+, x, \xi) = f(x, \xi)$  satisfies mild growth and smoothness conditions. Grad's method begins with the decomposition

$$(2) \quad Q = -\nu + K,$$

where  $\nu$  is the operator of multiplication by the collision frequency  $\nu(\xi)$ , a strictly positive function of  $|\xi|$ , and  $K$  is a compact operator on the Hilbert space  $H_0$  of functions  $f(\xi)$  which satisfy

$$\langle f, f \rangle \equiv \left( \frac{1}{\sqrt{2\pi}} \right)^3 \int |f(\xi)|^2 \exp(-|\xi|^2/2) d\xi < \infty.$$

Using (2), Grad wrote (1) as an integral equation and then derived *a priori* estimates for the solution in the Hilbert space

$$H \equiv L^2(R^6, (1/\sqrt{2\pi})^3 \exp(-|\xi|^2/2) dx d\xi).$$

Grad also related the asymptotic behavior of  $p_\epsilon$  to the solutions of the linear Euler and Navier-Stokes equations. Given  $f \in H$ , define

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$$\begin{aligned} f_0(x) &= \langle f(x, \cdot), 1 \rangle; \\ f_i(x) &= \langle f(x, \cdot), \xi_i \rangle, \quad i = 1, 2, 3; \\ f_4(x) &= \langle f(x, \cdot), (|\xi|^2 - 3)/\sqrt{6} \rangle, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product on  $H_0$ . The Navier-Stokes equations are written

$$\begin{aligned} (3) \quad & \partial n_0 / \partial t + \operatorname{div} n = 0, \\ & \partial n / \partial t + \operatorname{grad} n_0 + \sqrt{2/3} \operatorname{grad} n_4 = \epsilon \eta [\Delta n + (1/3) \operatorname{grad} \operatorname{div} n], \\ & \partial n_4 / \partial t + \sqrt{2/3} \operatorname{div} n = \epsilon \lambda \Delta n_4, \\ & n_i(0^+, \cdot) = f_i. \end{aligned}$$

In (3),  $\epsilon > 0$ ,  $n_i = n_i^\epsilon(t, x)$  ( $i = 0, \dots, 4$ ),  $n = (n_1, n_2, n_3)$ , and  $\eta > 0$  and  $\lambda > 0$  are physical constants. The Euler equations are obtained from (3) by putting  $\epsilon = 0$ . Setting

$$\begin{aligned} p_\epsilon &= T_\epsilon(t)f, \\ N_\epsilon(t)f &= n_0^\epsilon + \sum_{i=1}^3 n_i^\epsilon \xi_i + n_4^\epsilon \frac{|\xi|^2 - 3}{\sqrt{6}}, \\ E(t)f &= N_0(t)f, \end{aligned}$$

Grad proved the following asymptotic results:

$$\begin{aligned} (4) \quad & T_\epsilon(t)f = E(t)f + O(\epsilon), \quad (\epsilon \downarrow 0) \\ (5) \quad & T_\epsilon(t/\epsilon)f = N_\epsilon(t/\epsilon)f + O(\epsilon). \end{aligned}$$

In physical terms, (4) describes the nonviscous fluid approximation at a fixed time  $t > 0$ ; (5) describes the viscous effects when  $t \rightarrow \infty$ . Our aim is to show that (5) is only one of a large variety of possible refinements of (4). This is accomplished by the following two results.

**BOLTZMANN LIMIT THEOREM.** *Let  $f(x, \xi)$  be sufficiently regular.*

Then

$$(6) \quad E(-t/\epsilon)T_\epsilon(t/\epsilon)f = \bar{N}(t)f + O(\epsilon) \quad (\epsilon \downarrow 0),$$

where  $\bar{N}(t)$  is a contraction semigroup on  $H$  whose generator is given by the differential equations

