We consider the linearized Boltzmann equation

\[ \frac{\partial p}{\partial t} + \xi \cdot \text{grad } p = Qp/e, \]

whose solution \( p = p_\epsilon(t, x, \xi), t > 0, x \in \mathbb{R}^3, \xi \in \mathbb{R}^3, \epsilon > 0. \) \( Q \) is the linearized collision operator corresponding to a spherically symmetric hard potential, and \( e \) is a parameter which represents the mean free path.

In a series of basic papers, Grad \([6], [7], [8]\) studied the existence and asymptotic behavior of the solution of the initial value problem for (1), where the initial data \( p_\epsilon(0^+, x, \xi) = f(x, \xi) \) satisfies mild growth and smoothness conditions. Grad's method begins with the decomposition

\[ Q = -\nu + K, \]

where \( \nu \) is the operator of multiplication by the collision frequency \( \nu(\xi), \) a strictly positive function of \( |\xi|, \) and \( K \) is a compact operator on the Hilbert space \( H_\nu \) of functions \( f(\xi) \) which satisfy

\[ \langle f, f \rangle = \left( \frac{1}{\sqrt{2\pi}} \right)^3 \int |f(\xi)|^2 \exp \left( -|\xi|^2/2 \right) d\xi < \infty. \]

Using (2), Grad wrote (1) as an integral equation and then derived \textit{a priori} estimates for the solution in the Hilbert space

\[ H \equiv L^2(R^6, (1/\sqrt{2\pi})^3 \exp \left( -|\xi|^2/2 \right) dx d\xi). \]

Grad also related the asymptotic behavior of \( p_\epsilon \) to the solutions of the linear Euler and Navier-Stokes equations. Given \( f \in H, \) define
Navier-Stokes Equations in Kinetic Theory

\[ f_0(x) = (f(x, \cdot ), 1); \]
\[ f_i(x) = (f(x, \cdot ), \xi_i), \quad i = 1, 2, 3; \]
\[ f_4(x) = (f(x, \cdot ), (\|\xi\|^2 - 3)/\sqrt{6}), \]

where \((\cdot, \cdot)\) denotes the inner product on \(H_0\). The Navier-Stokes equations are written

\[ \frac{\partial n_0}{\partial t} + \text{div} n = 0, \]
\[ \frac{\partial n}{\partial t} + \text{grad} n_0 + \sqrt{2/3} \text{ grad } n_4 = \epsilon \eta [\Delta n + (1/3) \text{ grad } \text{div } n], \]
\[ \frac{\partial n_4}{\partial t} + \sqrt{2/3} \text{ div } n = \epsilon \lambda \Delta n_4, \]
\[ n_4(0^+, \cdot) = f_4. \]

In (3), \(\epsilon > 0, n_i = n_i^\epsilon(t, x) (i = 0, \cdots, 4), n = (n_1, n_2, n_3), \) and \(\eta > 0\) and \(\lambda > 0\) are physical constants. The Euler equations are obtained from (3) by putting \(\epsilon = 0\). Setting

\[ p_\epsilon = T_\epsilon(t)f, \]
\[ N_\epsilon(t)f = n_0^\epsilon + \sum_{i=1}^{3} n_i^\epsilon \xi_i + n_4^\epsilon \frac{\|\xi\|^2 - 3}{\sqrt{6}}, \]
\[ E(t)f = N_0(t)f, \]

Grad proved the following asymptotic results:

\[ T_\epsilon(t)f = E(t)f + O(\epsilon), \quad (\epsilon \downarrow 0) \]  
\[ T_\epsilon(t/e)f = N_\epsilon(t/e)f + O(\epsilon). \]

In physical terms, (4) describes the nonviscous fluid approximation at a fixed time \(t > 0\); (5) describes the viscous effects when \(t \to \infty\). Our aim is to show that (5) is only one of a large variety of possible refinements of (4). This is accomplished by the following two results.

**Boltzmann Limit Theorem.** Let \(f(x, \xi)\) be sufficiently regular. Then

\[ E(-t/e)T_\epsilon(t/e)f = \bar{N}(t)f + O(\epsilon) \quad (\epsilon \downarrow 0), \]

where \(\bar{N}(t)\) is a contraction semigroup on \(H\) whose generator is given by the differential equations
\[ \frac{\partial n_0}{\partial t} = \left( \frac{9}{25} \lambda + \frac{2}{5} \eta \right) \Delta n_0 + \sqrt{\frac{2}{3}} \left( -\frac{6}{25} \lambda + \frac{2}{5} \eta \right) \Delta n_4, \]
\[ \frac{\partial n}{\partial t} = \eta \Delta n + \left( \frac{\lambda}{5} - \frac{\eta}{3} \right) \text{grad div } n, \]
\[ \frac{\partial n_4}{\partial t} = \sqrt{\frac{2}{3}} \left( -\frac{6}{25} \lambda + \frac{2}{5} \eta \right) \Delta n_0 + \left( \frac{11}{25} \lambda + \frac{4}{15} \eta \right) \Delta n_4, \]
(7)
\[ n(x, 0^+) = f(x); \]
i.e.,
\[ \tilde{N}(t)f = n_0 + \sum_{i=1}^{3} n_i \xi_i + n_4 \frac{|\xi|^2 - 3}{\sqrt{6}}. \]

The semigroup \( \{ \tilde{N}(t), t \geq 0 \} \) commutes with the Euler semigroup \( \{ E(t), t \geq 0 \} \).

In order to make connection with (5) we also need the following.

**Navier-Stokes Limit Theorem.** Let \( f(x, \xi) \) be sufficiently regular.

Then
\[ E(-t/e) N_e(t/e)f = \tilde{N}(t)f + O(e) \quad (e \downarrow 0). \]
(8)

The proof of (8) proceeds by means of Fourier transformation from the following purely algebraic result, of independent interest.

**Matrix Limit Theorem.** Let \( A, B \) be real, symmetric \( m \times m \) matrices and assume that \( B \) is negative semidefinite. Then
\[ \exp(-itA/e) \exp(t(iA + \epsilon B)/e) = \exp(t \pi_A B) + O(e) \quad (e \downarrow 0), \]
where \( \pi_A B \) is the orthogonal projection, in the space of \( m \times m \) matrices, of \( B \) onto the linear subspace of matrices which commute with \( A \).

In particular, we show that \( \tilde{N}(t) \) is obtained by a projection, in the space of operators, of \( N_e(t) \) upon the set of operators which commute with \( \{ E(t), t \geq 0 \} \).

Using (6), we have
\[ T_e(t/e)f = E(t/e) \tilde{N}(t)f + O(e) \quad (e \downarrow 0). \]
(9)

This is the simplest of an infinite number of alternatives to (5). Indeed, if \( \tilde{N}(t) \) is any operator whose projection is \( N(t) \), then we may substitute \( \tilde{N}(t) \) for \( \tilde{N}(t) \) in (9).

The proof of (6) depends on a careful spectral analysis of the operator \( Q - i(\gamma \cdot \xi) \), where \( \gamma \in \mathbb{R}^3 \) is a parameter. We prove the existence and
differentiability, for $|\gamma|$ sufficiently small, of the hydrodynamical eigenvalues and eigenfunctions $\{\alpha_j(\gamma), e_j(\gamma); j = 1, \cdots, 5\}$ which satisfy $\alpha_j(0) = 0, e_j(0) \in \text{span} \{1, \xi_1, \xi_2, \xi_3, |\xi|^2\}$. We then prove a contour integral representation

$$\exp \left[ i(Q - i(\gamma \cdot \xi))f \right] = \sum_{j=1}^{5} \alpha_j(\gamma) e_j(\gamma) e_j(-\gamma) + \frac{1}{2\pi i} \int_C e^{\alpha R(\alpha, \gamma)} \frac{(Q - i(\gamma \cdot \xi))^2}{\alpha^2} f(d\alpha),$$

where $C$ is a vertical contour in the half plane $\Re \alpha < 0$ and $R(\alpha, \gamma) \equiv (Q - i(\gamma \cdot \xi) - \alpha)^{-1}$. The first term of (10) corresponds to the Hilbert solution and gives the connection with hydrodynamics. The second term is negligible in the hydrodynamic limit. In case $\nu(\xi) \sim |\xi|^\alpha$ as $|\xi| \to \infty$ ($\alpha > 0$), the contour integral may be replaced by $\int_C e^{\alpha R(\alpha, \gamma)} f(d\alpha)$, where the contour $C$ is such that $\Re \alpha \to -\infty$ when $\Im \alpha \to \pm \infty$. The existence of the eigenvalues $\alpha_j(\gamma)$ follows by applying the implicit function theorem to the exact hydrodynamical dispersion laws. Previously, exact dispersion laws were obtained [11] only for hard sphere potentials, i.e., $\nu(\xi) \sim |\xi|$ as $|\xi| \to \infty$. In this case, the $\alpha_j(\gamma)$ are analytic functions and can also be obtained from Rellich's perturbation theorem [9], [10]. In case $\nu(\xi) \sim |\xi|^\alpha$ as $|\xi| \to \infty$, $0 \leq \alpha < 1$, the $\alpha_j(\gamma)$ will not be analytic around $\gamma = 0$. Nevertheless, we obtain an asymptotic development

$$\alpha_j(\gamma) \sim \sum_{n=1}^{\infty} a_n(\gamma)|\gamma|^n \quad (1 \leq j \leq 5),$$

where $a_1(\gamma)$ is imaginary and $a_2(\gamma) < 0$. These constants can be computed by formal perturbation theory. They correspond to the adiabatic sound speed and absorption coefficients for low frequency sound waves [5].

The results (6) and (8) extend known results on finite-state velocity models in one dimension [1], [2] to the full three-dimensional linearized Boltzmann equation. These theorems are valid in any number of dimensions. Their proofs and related matters will appear in full detail in [3], [4].

REFERENCES.


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