

Monotone Decrease of Characteristic Functions

Richard S. Ellis^{1,2}

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Let $F(x)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \geq 1$, be an n -dimensional distribution function and define $f(t) = \int \exp(it \cdot x) dF(x)$. We prove the following result, which is used in the work of Monroe and Siegert.⁽¹⁾

Theorem. We assume that

$$F(x) = F(-x), \quad \text{all } x \in \mathbb{R}^n \quad (1a)$$

$$\infty > \int (t \cdot x)^2 dF(x) > 0, \quad \text{all } t \in \mathbb{R}^n \quad (1b)$$

Then f is real and there exists $\delta > 0$ such that whenever $\epsilon > 0$, $t \in \mathbb{R}^n$, $0 < |t| < \delta/(1 + \epsilon)$, we have

$$f(t(1 + \epsilon)) < f(t) < 1 = f(0) \quad (2)$$

Thus in a suitable neighborhood of the origin, f is monotonically decreasing along rays through the origin.

Remark. Condition (1b) is equivalent to the nondegeneracy of F and the finiteness of the second moments $\int x_j^2 dF(x)$, $j = 1, \dots, n$.

Proof. We first prove (2) for $n = 1$ and then reduce the proof for general n to this case. Condition (1a) implies that f is real. Condition (1b) implies that $f \in C^2(\mathbb{R})$ and $f''(0) = -\int_{\mathbb{R}} x^2 dF(x) < 0$ (prime denotes d/dt). By continuity, there exists $\delta > 0$ such that $f''(t) < 0$ for $|t| < \delta$. By (1a), $f'(0) =$

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¹ Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts.

² On leave from Northwestern University, Evanston, Illinois.

$i \int_{\mathbb{R}} x dF(x) = 0$. Thus for $0 < t < \delta$, $f'(t) = f'(t) - f'(0) = \int_0^t f''(s) ds < 0$. For $-\delta < t < 0$, $f'(t) = -\int_t^0 f''(s) ds > 0$. Now take $\epsilon > 0$ and t such that $0 < t < \delta/(1 + \epsilon)$. Then $f(t(1 + \epsilon)) - f(t) = \int_0^{\epsilon t} f'(t + s) ds < 0$. If $-\delta/(1 + \epsilon) < t < 0$, then $f(t(1 + \epsilon)) - f(t) = -\int_0^{-\epsilon t} f'(t - s) ds < 0$. To show $f(t) < f(0)$ is similar. This completes the proof for $n = 1$.

Now take $n \geq 2$. Again f is real. We write the argument t of f as $t = \alpha u$, where α is real and $u \in S^n$, the unit sphere in \mathbb{R}^n ; thus, $f = f(\alpha, u)$. Condition (1b) implies that for each $u \in S^n$ (here prime denotes $d/d\alpha$)

$$f''(0, u) = -\int (u \cdot x)^2 dF(x) < 0$$

Hence by the joint continuity of $f''(\alpha, u)$ in α and u , there exists, for each $u \in S^n$, $\delta(u) > 0$ and a neighborhood $\mathcal{N}(u)$ of u so that $f''(\alpha, v) < 0$ for all $|\alpha| < \delta(u)$ and all $v \in \mathcal{N}(u)$. By the compactness of S^n , there exists an integer k and k points u_1, \dots, u_k so that

$$S^n = \bigcup_{i=1}^k \mathcal{N}(u_i)$$

We set $\delta = \min_{i=1, \dots, k} \delta(u_i) > 0$; $f''(\alpha, u) < 0$ for all $|\alpha| < \delta$ and all $u \in S^n$. Condition (1a) implies that $f'(0, u) = 0$ for each $u \in S^n$. Thus we can carry over the proof for $n = 1$ and conclude that for $\epsilon > 0$

$$f(\alpha(1 + \epsilon), u) < f(\alpha, u), \quad \text{all } |\alpha| < \delta/(1 + \epsilon), \quad \text{all } u \in S^n$$

This completes the proof of the theorem.

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REFERENCE

1. J. L. Monroe and A. J. F. Siegert, to be published.