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Extensions of the Maximum Principle: Exponential Preservation by the Heat Equation

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I. INTRODUCTION

It is well known that the Cauchy problem for the heat equation

$$\begin{aligned} &\partial u/\partial t = \tfrac{1}{2}\Delta u,\ u = u(t, x),\ t > 0,\ x = (x_1, ..., x_n) \in \mathbb{R}^n,\ n \geqslant 1, \\ &\lim_{t \downarrow 0} u(t, \cdot) = f, \end{aligned} \tag{1.1}$$

preserves positivity of the initial data f. This follows from the maximum principle [9, Chap. 2], or, more simply, from the fact that the heat transformation

$$\begin{aligned} T_t: f(x_1, ..., x_n) &\to (T_t f)(y_1, ..., y_n) \\ &= (1/2\pi t)^{n/2} \int_{\mathbb{R}^n} \exp[-(1/2t)\,|\,y - x\,|^2] f(x)\,dx \end{aligned}$$

preserves positivity of f for $t > 0$. Less trivial is the fact that the heat equation preserves log-concavity of the initial data [1; 2, Theorem 1.3]: $f > 0$ and $\ln f$ concave in $\mathbb{R}^n$ implies $\ln u(t, \cdot)$ concave in $\mathbb{R}^n$ for $t > 0$. In this paper, we investigate other instances of such phenomena, where global properties of $\ln f$ are preserved by the heat flow (1.2).

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Our research on this problem has been motivated in part by work on Ising spin systems. Indeed, as we point out at the end of Section II, there are close connections between correlation inequalities for Ising spin systems and exponential preservation by the heat equation. Also, the proof of one of our theorems depends on a result on the comparison of measures (Lemma 2.4), which is related to work on phase transitions [14, Sect. VI; 16].

Our methods of proof are twofold: (1) to study the transformation (1.2) directly; (2) to apply the maximum principle in a novel way (to derivatives of $\ln u$). The second method is formal in that it requires certain a priori bounds which we are unable to obtain in general. However, we have included the second method because it confirms in a graphic way most of the results found by the first method and because it yields some new results of its own. Both methods extend to parabolic equations of the form $\partial u/\partial t = \Delta u - Vu$ for certain potential functions V.

Section II of this paper states the main results. They are proved in Sections III and IV by means of the first and second methods, respectively.

II. Main Results

Given $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we write $r = |x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. Let $\mathscr{W}$ be a class of smooth functions W which map $\mathbb{R}^n \to \mathbb{R}^1$, some $n = 1, 2, ...,$ and which satisfy

$$\liminf_{r\to\infty} W(x)/r^2 > -\infty. \tag{2.1}$$

We denote by $\exp(-\mathscr{W})$ the class of all functions f of the form $f = \exp(-W)$ for some $W \in \mathscr{W}$. We retain the minus sign in the notation in order to stay consistent with [6, 7].

Definition 2.1. $\mathscr{W}$ is exponentially preserved by the heat equation (1.1) if whenever $f \in \exp(-\mathscr{W})$, then $u(t, \cdot) \in \exp(-\mathscr{W})$ for $t > 0$; i.e., if

$$T_t : \exp(-\mathscr{W}) \to \exp(-\mathscr{W}) \qquad \text{for} \quad t > 0,$$

where T_t denotes the heat flow (1.2).

We need a hypothesis on W like (2.1) so that the integral in (1.2) converges for $f = \exp(-W)$ and for suitably small t. This convergence, and hence the existence of $u(t, \cdot)$, for only small t will be the case if the limit infimum in (2.1) is strictly negative. We assume that, whenever necessary, the t-values are suitably restricted.

Our first theorem lists a number of exponentially preserved classes. The exponential preservation of certain of these classes follows from known facts, as we point out in Remark 2.3(a) below. We include them here since they have not previously been considered from such a general viewpoint. Each of the classes is defined in terms of global properties of certain derivatives of W, of some order k. The superscript appearing in the notation for each class denotes the order k of these derivatives. Our second method of proof throws more light on the fact that only $k = 0, 1, 2, 3$ appear. We use $+$ as a subscript in the notation for certain of the classes (Cases (ii), (iv), (v), (vi)) since there are analogous "minus classes" in these cases. For Case (v), we define this "minus class" $\mathscr{W}^2_{-,r}$ in the theorem because we are able to prove its exponential preservation for a wider range of n than that of $\mathscr{W}^2_{+,r}$. The other "minus classes" (Cases (i), (iv), (vi)) are defined in Remark 2.3(b) below. Their exponential preservation holds for the same values of n as in the theorem. In Remark 2.3(c), we comment on other features of and certain gaps in the theorem; e.g., Case (v)(a) for $n = 3$.

For technical convenience, we work now only with C^∞ functions. See Remark 2.3 (d) concerning a wider choice of initial data. We use subscript notation to denote partial derivatives. Given $W \in C(\mathbb{R}^n)$, $n \geqslant 1$, we say that W is even in x_j if it is invariant under the transformation taking x_j to $-x_j$ and totally even if it is even in x_j for each $j = 1,..., n$.

THEOREM 2.2. *The following subclasses of $C^\infty(\mathbb{R}^n)$ are exponentially preserved by the heat equation:*

(i) $\mathscr{W}^0 = \{W \mid K > W > K' \text{ in } \mathbb{R}^n\}$, *for any constants* $\infty \geqslant K \geqslant K' > -\infty$ *and* $n \geqslant 1$;

(ii) $\mathscr{W}_+^1 = \{W \mid W \text{ even in } x_i \text{ on } \mathbb{R}^n;\ W_{x_i} \geqslant 0 \text{ for } x_i \geqslant 0,\ x_j \text{ real } (j \neq i)\}$, *for* $n \geqslant 1$, *and* $i \in \{1,..., n\}$.

(iii) $\mathscr{W}_v^2 = \{W \mid W \text{ convex in } \mathbb{R}^n\}$, *for* $n \geqslant 1$;

(iv) $\mathscr{W}_+^2 = \{W \mid W \text{ totally even in } \mathbb{R}^n;\ \text{for all } 1 \leqslant i \neq j \leqslant n\ W_{x_i x_j} \geqslant 0$ *for* $x_i, x_j \geqslant 0$, x_k *real* $(k \neq i, j)\}$, *for* $n = 2$;

(v) (a) $\mathscr{W}^2_{+,r} = \{W \mid \mathscr{W}_+^2;\ W \text{ a function only of } r^2\}$, *for* $n = 2$ *or* $n \geqslant 4$;

(b) $\mathscr{W}^2_{-,r} = \{W \mid W \in \mathscr{W}_-^2;\ W \text{ a function only of } r^2\}$ *for* $n \geqslant 2$;

(vi) $\mathscr{W}_+^3 = \{W \mid W \text{ even in } \mathbb{R}^1;\ W_{x_1 x_1 x_1} \geqslant 0 \text{ for } x_1 \geqslant 0\}$.

Remark 2.3. (a) $\mathscr{W}^0$ is exponentially preserved because of the standard maximum principle for the heat equation. The exponential preservation of $\mathscr{W}_+^1$ for $n = 1$ follows from a known fact concerning one-dimensional unimodal functions: for $n = 1$, $f \in \exp(-\mathscr{W}_+^1)$ is symmetric and unimodal; thus $T_t f$, as the convolution of two symmetric, unimodal functions, is symmetric and unimolal [12 p. 98]. Since it is smooth, it belongs to $\exp(-\mathscr{W}_+^1)$ for $t > 0$. The exponential preservation of $\mathscr{W}_v^2$ follows from [1; 2, Theorem 1.3] and that of $\mathscr{W}_+^3$ from [6, Theorem 1.1]. The other cases are new.

(b) Define classes $\mathscr{W}_-^k$, $k = 1, 2, 3$, as follows: $\mathscr{W}_-^1$ is obtained from $\mathscr{W}_-^1$ be replacing the derivative inequality in (ii) with $W_{x_i} \leqslant \delta x_i$ (some δ real);

$\mathscr{W}_-^2$ and $\mathscr{W}_-^3$ are obtained from $\mathscr{W}_+^2$ and $\mathscr{W}_+^3$, respectively, by reversing the senses of the derivative inequalities in (iv) and (vi). These are all exponentially preserved: $\mathscr{W}_-^1$ and $\mathscr{W}_-^2$, by modifying the proofs of Section III; $\mathscr{W}_-^3$ by [7, Theorem 1].

(c) We are unable to prove the exponential preservation of $\mathscr{W}_\pm^2$ for $n > 2$ by either of the two methods; method one yields it only with the extra assumption of spherical symmetry (Case (v)). The exponential preservation of $\mathscr{W}_{+,r}^2$ for $n = 3$, not covered by Case (v)(a), is presumably true. This gap in Theorem 2.2 should be viewed as a fluke. In Remark 3.1, we do show that for $n = 3$

$$T_t\colon \mathscr{W}_{+,r}^2 \cap \mathscr{W}_{+,r}^1 \to \mathscr{W}_{+,r}^2, \qquad t > 0, \tag{2.2}$$

where

$$\mathscr{W}_{+,r}^1 = \{W \mid W(x) = Q(r^2) \text{ for some } C^\infty \text{ function } Q \text{ such that } Q' \geqslant 0 \text{ on } (0, \infty)\}.$$

We note an alternate definition of the classes $\mathscr{W}_{\pm,r}^2$ which will be of use in Section III.

$$\begin{Bmatrix} \mathscr{W}_{+,r}^2 \\ \mathscr{W}_{-,r}^2 \end{Bmatrix} = \left\{ W \mid W(x) = Q(r^2) \text{ for some } C^\infty \begin{Bmatrix} \text{convex} \\ \text{concave} \end{Bmatrix} \text{ function } Q \text{ on } [0, \infty) \right\}. \tag{2.3}$$

Indeed, for W of this form, $W_{x_1 x_2} = x_1 x_2 Q''$; this explains why we write our radial functions with argument r^2 instead of r. The exponential preservation of the classes $\mathscr{W}_\pm^3$ does not seem to extend to $n > 1$ via either method.

(d) The assumption that the initial data be C^∞ is unnecessarily restrictive. In fact, it is natural to allow the initial data to be a measure and to ask for which measures will $u(t, \cdot) \in \overline{\exp(-\mathscr{W})}$ for a given class $\mathscr{W}$. This is equivalent to determining the set $\overline{\exp(-\mathscr{W})}$ of weak limits of measures with densities in $\exp(-\mathscr{W})$. In certain cases, this can be done explicitly. For $\mathscr{W} = \mathscr{W}_+^3$ or $\mathscr{W}_-^3$, we refer to [6, Theorem 2.4; 7, Theorem 6]. Here it is shown that the set of all measures $d\rho$ in $\overline{\exp(-\mathscr{W})}$ absolutely continuous with respect to Lebesgue measure with $d\rho/dx > 0$ in $\mathbb{R}^1$, all have the form $d\rho = f dx$ for some $f \in \exp(-\mathscr{W}^*)$; $\mathscr{W}^*$ is defined analogously to $\mathscr{W}$ except that the C^∞ assumption is dropped and the inequality on the derivative of W is replaced by a suitable convexity/concavity condition. The analogous statement can be proved for Cases (i), (ii), (iii), and (v) (the latter for all $n \geqslant 2$).

(e) It is known that the classes $\exp(-\mathscr{W})$ for $\mathscr{W}$ in (ii) (with $n = 1$) and for $\mathscr{W}$ in (iii) are closed under convolutions [12, p. 98; 1; 2, Theorem 1.3]. The Gaussian kernel $\exp(-|x|^2/2t)$ belongs to $\exp(-\mathscr{W})$ for $\mathscr{W}$ in (ii) (with $n \geqslant 1$) and for $\mathscr{W}$ in (iv), (v)(a)–(b), and (vi). This leads to the conjecture that some of these $\exp(-\mathscr{W})$ classes are also closed under convolutions, a fact which would generalize Theorem 2.2. It can be shown however, that this is not the case for (vi). (Any $f \in \exp(-\mathscr{W}_+^3)$ can have at most two maxima, but two such f's can be found whose convolution has three maxima.)

The proof of the exponential preservation of the classes $\mathscr{W}_{\pm}^1$, case (ii) of Theorem 2.2, is based on the following result, of some independent interest. We emphasize that for $n = 1$, condition (2.5)(b) is always satisfied. In (2.4) and for the rest of the paper, we omit the region of integration when it is all of $\mathbb{R}^n$ and there is no danger of confusion.

LEMMA 2.4. *We are given $n \geqslant 1$, some $i \in \{1,..., n\}$ and P, $\bar{P} \in C^1(\mathbb{R}^n)$, even in x_i. Sufficient conditions that*

$$\frac{\int x_i^k \exp(x \cdot y - P(x))\, dx}{\int \exp(x \cdot y - P(x))\, dx} \leqslant \frac{\int x_i^k \exp(x \cdot y) - \bar{P}(x))\, dx}{\int \exp(x \cdot y - \bar{P}(x))\, dx} \qquad \textit{for all } k = 1, 2,..., y_i \geqslant 0,\ y_j \textit{ real } (j \neq i), \tag{2.4}$$

are that

$$\begin{aligned} &\text{(a)}\quad \partial P/\partial x_i \geqslant \partial \bar{P}/\partial x_i \textit{ for } x_i \geqslant 0,\ x_j \textit{ real } (j \neq i),\\ &\text{(b)}\quad \partial \bar{P}/\partial x_i \textit{ be a function of } x_i \textit{ only}. \end{aligned} \tag{2.5}$$

Given a class $\mathscr{W}$ in Theorem 2.2, a natural problem is to determine for which potential functions V is $\mathscr{W}$ exponentially preserved by the equation

$$\partial u/\partial t = (\Delta - V)u, \lim_{t \downarrow 0} u = f. \tag{2.6}$$

An answer is essentially that it suffices for V to be in this class $\mathscr{W}$. This can be shown to follow from Theorem 2.2 via the Trotter product formula; for details in two cases, see [1; 2, Theorem 2.10] and [6, Theorem 1.1]. In order to avoid some technicalities, we choose to include this problem in our maximum principle approach where it involves no extra effort.

We now turn to method two. This determines exponential preservation by the (formal) application of the maximum principle to derivatives of $\ln u$, where u solves (1.1) or (2.6). We define new classes $\tilde{\mathscr{W}}_{\pm}^k$, $k = 1, 2, 3$, by modifying the classes $\mathscr{W}_{\pm}^k$, Cases (ii), (iv), (vi) of Theorem 2.2: we drop the hypothesis of eveness, but require that the respective derivative inequalities hold for all real values of the relevant variables.

THEOREM 2.5. *Assume the a priori bounds* (4.4). *The following subclasses of $C^\infty(\mathbb{R}^n)$ are exponentially preserved by the heat equation:*

(i) $\tilde{\mathscr{W}}_{\pm}^1$, $\delta = 0$, $n \geqslant 1$;

(ii) $\tilde{\mathscr{W}}_{\pm}^2$, $n = 2$;

(iii) $\mathscr{W}_S^2 = \{W \mid \partial^2 W/\partial x_i^2 \leqslant 0 \textit{ in } \mathbb{R}^n\}$, *for some* $i \in \{1,..., n\}$, $n \geqslant 1$;

(iv) $\mathscr{W}_\Gamma^2 = \{W \mid \frac{1}{2}\sum_{i,j=1}^n \gamma_{ij} W_{x_i x_j} \leqslant 0 \textit{ in } \mathscr{R}^n\}$, *for some symmetric positive definite matrix* $\Gamma = (\gamma_{ij})$, $n \geqslant 1$;

(v) $\tilde{\mathscr{W}}_{\pm}^3$.

Also assume that $V \in \mathscr{W}$, *for some* $\mathscr{W}$ *in Cases* (i)–(v) *above. Then this* $\mathscr{W}$ *is exponentially preserved by Eq.* (2.6).

Remark 2.6. (a) It suffices to prove the last part of the theorem; the first part follows by taking $V \equiv 0$.

(b) The second method together with a symmetry argument yields the exponential preservation of the classes $\mathscr{W}_\pm^k$, $k = 1, 2, 3$, of Theorem 2.2 (with $\delta = 0$ for $\mathscr{W}_-^1$); see Remark 4.1(a) for details. The second method also yields the exponential preservation of the class $\mathscr{W}_v^2$, of Theorem 2.2, for $n = 1$; see Eq. (4.6)(b).

(c) We define

$$Lu = \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i}, \tag{2.7}$$

where a_{ij}'s and b_i's are real constants and the matrix $A = (a_{ij})$ is symmetric and positive definite. Method two shows that classes (i), (iii), (iv), (v) of Theorem 2.5 as well as the classes $\mathscr{W}_\pm^k$, $k = 1, 2, 3$, of Theorem 2.2 (with $\delta = 0$ for $\mathscr{W}_-^1$ and the first-order terms in L zero) are exponentially preserved by the equation $\partial u/\partial t = Lu$; for class (ii) of Theorem 2.5 we require $a_{12} = 0$. A similar statement as in the theorem holds for the equation $\partial u/\partial t = Lu - Vu$.

(d) In Remark 4.1(b) we sketch the formal argument which shows that the classes ($W^{(j)}$ denotes $\partial^j W/\partial x_1^j$)

$$\begin{aligned} \tilde{\mathscr{W}}_+^k &= \{W \mid W^{(3)} \geqslant 0, W^{(4)} \leqslant 0, \ldots, (-1)^{k-1} W^{(k)} \geqslant 0 \text{ in } \mathbb{R}^1\}, \\ \tilde{\mathscr{W}}_-^k &= \{W \mid W^{(3)} \leqslant 0, W^{(4)} \leqslant 0, \ldots, W^{(k)} \leqslant 0 \text{ in } \mathbb{R}^1\}, \end{aligned} \tag{2.8}$$

are exponentially preserved by (2.6) for each $k = 4, 5, \ldots$, provided $V \in \tilde{\mathscr{W}}_\pm^k$. A difficulty with this is that for any *smooth* function W in either of the classes $\tilde{\mathscr{W}}_\pm^k$ with $W^{(3)} \not\equiv 0$, (2.1) cannot be satisfied. We discuss how one might get around this by working with $W \in C^\infty(\mathbb{R}^1 \backslash \{0\})$. We mention the classes $\tilde{\mathscr{W}}_\pm^k$ because, surprisingly, they also arise in the apparently unrelated context of [11, Theorem 3.1]. Here, among other things, monotonicity properties of the zeros of solutions of certain Sturm–Lionville systems on $(0, \infty)$ are studied. Analogous, but far less detailed, information on the eigenvalues of the operator $-d^2/dx^2 + V$, for certain potentials V, is known [5, p. 169].

We end this section by briefly commenting on the connection between exponential preservation by the heat equation and correlation inequalities for Ising spin systems. The fact that $\mathscr{W}_+^3$ is exponentially preserved is related to necessary and sufficient conditions, on single-site spin measures, for the Griffiths–Hurst–Sherman (GHS) inequality to hold [6]. $\mathscr{W}_-^3$ is similarly related to reverse GHS inequalities [7]. That for $n = 2$ the classes $\mathscr{W}_\pm^2$ are exponentially preserved is related to necessary and sufficient conditions, on single site two-dimensional vector-spin measures, for certain forms of the Griffiths–Kelly–Sherman (GKS II) inequalities to hold (unpublished). This latter result generalizes inequalities found first in [13] and extended in [3, 4, 10].

III. Proof of Theorem 2.2

We define $W_0 \equiv -\ln f$, where f is given in (1.1), and set $P(x) = W_0 + |x|^2/2t$. We write

$$u(t, \cdot) = e^{-W_t(\cdot)}, \tag{3.1}$$

where

$$W_t(y) = -\ln T_t(\exp(-W_0)) = \frac{|y|^2}{2t} - \ln \int \exp\left(\frac{x \cdot y}{t} - \frac{|x|^2}{2t} - W_0\right) dx + \frac{n}{2}\ln(2\pi t). \tag{3.2}$$

For G a function of x, we define

$$\langle G\rangle_P(y) \equiv \frac{\int G(x)\exp((x \cdot y/t) - P(x))\, dx}{\int \exp((x \cdot y/t) - P(x))\, dx}. \tag{3.3}$$

When $n = 1$, we write x for x_1, y for y_1.

Case (ii). We prove that if W_0 is even in x_i and there exist constants $\delta_2 < \delta_1$ so that

$$\delta_2 x_i \leqslant (W_0)_{x_i} \leqslant \delta_1 x_i \qquad \text{for all } x_i \geqslant 0, \qquad x_j \text{ real } (j \neq i), \tag{3.4}$$

then

$$\delta_2 y_i/(1 + \delta_2 t) \leqslant (W_t)_{y_i} \leqslant \delta_1 y_i/(1 + \delta_1 t), \text{ for all } y_i \geqslant 0, y_j \text{ real } (j \neq i), t > 0, \tag{3.5}$$

as long as the denominators stay finite. The exponential preservation of $\mathscr{W}_+^1$ (resp., $\mathscr{W}_-^1$) follows from the left-hand (resp., right-hand) inequality by setting $\delta_2 = 0$ (resp., $\delta_1 = \delta$). Our proof that (3.4) implies (3.5) depends on Lemma 2.4, which is proved at the end of this section. From (3.2) and (3.3), we have

$$(-W_t)_{y_i} = -y_i/t + \langle x_i/t\rangle_P.$$

By the hypothesis (3.4) on W_0 and by Lemma 2.4, it follows that

$$-y_i/t + \langle x_i/t\rangle_{(\delta_1 + t^{-1})x_i^2/2} \leqslant (-W_t)_{y_i} \leqslant -y_i/t + \langle x_i/t\rangle_{(\delta_2 + t^{-1})x_i^2/2}$$

It is easily calculated that

$$\langle x_i/t\rangle_{(\delta+t^{-1})x_i^2/2} = y_i/t(1+\delta t),$$

and so we obtain (3.5). ∎

Case (*iv*). We define $\mathbb{R}_+^n = \{x \mid x_1 > 0,..., x_n > 0\}$, $n \geqslant 1$. We prove that

$$(W_0)_{x_1x_2} \geqslant 0 \text{ on } \mathbb{R}_+^2 \Rightarrow (W_t)_{x_1x_2} \geqslant 0 \text{ on } \mathbb{R}_+^2 \text{ for } t > 0. \tag{3.6}$$

Our method parallel that of [5, Sect. 4], which was used to prove Case (vi) of Theorem 2.2. Using the notation (3.3), we see that

$$(-W_t)_{y_1y_2} = \left\langle \frac{x_1}{t}\frac{x_2}{t}\right\rangle_P (y) - \left\langle \frac{x_1}{t}\right\rangle_P (y)\cdot\left\langle \frac{x_2}{t}\right\rangle_P (y),$$

which can be rewritten in terms of new variables $z = (z_1, z_2)$ as

$$(-W_t)_{y_1y_2} = \frac{1}{2t^2}\frac{\int (x_1-z_1)(x_2-z_2)\, e^{y\cdot(x+z)/t}e^{-P(x)-P(z)}\, dx\, dz}{[\int e^{x\cdot y/t}e^{-P(x)}\, dx]^2}.$$

Since $y_1, y_2 > 0$, (3.6) will follow if we can show

$$\begin{aligned} 0 &\geqslant \int_{\mathbb{R}^2\times\mathbb{R}^2} (x_1+z_1)^{k_1}(x_1-z_1)(x_2+z_2)^{k_2}(x_2-z_2)\, e^{-P(x)-P(z)}\, dx\, dz \\ &= 2^{k_1+k_2+4}\int_{\mathbb{R}^4} s^{k_1}\bar{s}u^{k_2}\bar{u}e^{-P(s+\bar{s},u+\bar{u})-P(s-\bar{s},u-\bar{u})}\, ds\, d\bar{s}\, du\, d\bar{u}, \\ &\qquad\qquad \text{for all } k_1, k_2 \geqslant 0. \end{aligned} \tag{3.7}$$

Because P is totally even, this integral vanishes unless both k_1 and k_2 are odd. Now take k_1 and k_2 both odd. Via a change of variables in each of the fifteen orthants of $\mathbb{R}^u$ where at least one of the variables s, $\bar{s}$, u or $\bar{u}$ is negative, we rewrite the integral as

$$8\int_{\mathbb{R}_+^4} s^{k_1}\bar{s}\bar{u}^{k_2}\bar{u}\, dp(s,\bar{s},u,\bar{u}),$$

where

$$\begin{aligned} dp(s,\bar{s},u,\bar{u}) = \{&\exp[-P(s+\bar{s},u+\bar{u})-P(s-\bar{s},u-\bar{u})] \\ &-\exp[-P(s+\bar{s},u-\bar{u})-P(s-\bar{s},u+\bar{u})]\}\, ds\, d\bar{s}\, du\, d\bar{u}. \end{aligned}$$

Thus, we obtain (3.6) if $dp \leqslant 0$ in $\mathbb{R}_+^4$, which occurs if and only if

$$0 \leqslant P(s+\bar{s},u+\bar{u}) - P(s+\bar{s},u-\bar{u}) - P(s-\bar{s},u+\bar{u}) + P(s-\bar{s},u-\bar{u}),$$
$$\text{for all}\quad s, \bar{s}, u, \bar{u} > 0.$$

We claim that (3.8) holds if and only if $P_{x_1x_2} \equiv (W_0)_{x_1x_2} \geqslant 0$ on $\mathbb{R}_+{}^2$. Assuming that $s \geqslant \bar{s}$, $u \geqslant \bar{u}$ (by symmetry), we prove this claim by rewriting (3.8) in the form

$$0 \leqslant \int_{s-\bar{s}}^{s+\bar{s}} \int_{u-\bar{u}}^{u+\bar{u}} \frac{\partial^2}{\partial x_1\, \partial x_2} P(x_1, x_2)\, dx_1\, dx_2 . \quad \blacksquare$$

Case (v). For $n = 2$, Cases (v) (a) and (b) are clear because $\mathscr{W}^2_{*,r} \subset \mathscr{W}^2_*$, $* = +$ or $-$, $n = 2$, and because $T_t \exp(-W_0)$ is a function of r^2 if W_0 is. The proof for $n > 2$ proceeds via a reduction to $n = 2$, but we need different methods for (a) and (b).

Proof of (v) (a) *for* $n \geqslant 4$. Since $W_t = W_t(|\, y\,|^2)$, we may choose $y = (y_1, y_2, 0, 0,\ldots, 0)$. Because of symmetry, it suffices to prove that

$$(W_t)_{y_1y_2} \geqslant 0 \text{ for } t > 0, y_1, y_2 > 0, \text{ given } W_0 = Q_0(|\, x\,|^2), Q_0'' \geqslant 0. \tag{3.9}$$

We define $\bar{x} = (x_1, x_2)$, $\bar{r} = |\, \bar{x}\,|$, $\tilde{x} = (x_3,\ldots, x_n)$, $\tilde{r} = |\, \tilde{x}\,|$, and

$$U_0(\bar{r}^2) = -\ln \int \exp[-\tilde{r}^2/2t - Q_0(\bar{r}^2 + \tilde{r}^2)]\, d\tilde{x}. \tag{3.10}$$

Then

$$W_t(y) = |\, y\,|^2/2t - \ln \int \exp[x \cdot y/t - \bar{r}^2/2t - U_0(\bar{r}^2)]\, d\bar{x} + (n/2) \ln(2\pi t). \tag{3.11}$$

By Case (v) (a) for $n = 2$ and by (2.3), it suffices to prove that $U_0'' \geqslant 0$. But

$$\begin{aligned} e^{-U_0(\bar{r}^2)} &= \text{const} \int_0^\infty \exp[-\tilde{r}^2/2t - Q_0(\bar{r}^2 + \tilde{r}^2)]\, \tilde{r}^{n-3}\, d\tilde{r} \\ &= \text{const} \int_0^\infty \exp(-[s/2t + Q_0(\bar{r}^2 + s) + [(4-n)/2] \ln s])\, ds, \quad s = \tilde{r}^2. \end{aligned}$$

Thus $U_0'' \geqslant 0$ if the function on $(0, \infty) \times (0, \infty)$,

$$(\bar{r}^2, s) \to s/2t + Q_0(\bar{r}^2 + s) + [(4-n)/2] \ln s,$$

is jointly convex for $t \geqslant 0$ [1; 2, Theorem 1.1]. Since $Q_0'' \geqslant 0$, this will be the case provided $n \geqslant 4$. $\blacksquare$

Remark 3.1. Concerning $n = 3$, for which the above argument breaks down, we prove (2.2). For $n = 3$, $U_0'' \geqslant 0$ provided the function

$$(\bar{r}^2, \tilde{r}) \to \tilde{r}^2/2t + Q_0(\bar{r}^2 + \tilde{r}^2)$$

is jointly convex for $t > 0$. This is the case if $Q_0' \geqslant 0$, $Q_0'' \geqslant 0$; thus (2.2) follows. (For any $n \geqslant 2$, $W_0 \in \mathscr{W}^2_{+,r}$ implies that U_0 in (3.10) is jointly convex in x_1, x_2, but this is not the same as showing the convexity of U_0 in $\bar{r}^2$; we need the latter for Case (v) (a).)

Proof of (v) (b) for $n \geqslant 3$. We prove (3.9) with the sense of the derivative inequalities reversed. By Case (v) (b) for $n = 2$ and by (2.3), it suffices to prove $U_0'' \leqslant 0$. But using the notation (3.3) and denoting $\langle\ \rangle_{W_0+\bar{r}^2/2t}$ by $\langle\ \rangle$, we have

$$U_0'' = \langle Q_0''\rangle(0) - \langle(Q_0' - \langle Q_0'\rangle)^2\rangle(0) \leqslant 0. \quad \blacksquare$$

Proof of Lemma 2.4. We first prove the lemma for $n = 1$, then reduce the proof for $n > 1$ to this case. We write x for x_1, y for y_1. Cross-multiplying in (2.4) and expressing the result as a single integral, we see that (2.4) is equivalent to

$$0 \leqslant \int_{\mathbb{R}^2} (x^k - z^k)\, e^{y(x+z)} e^{-\bar{P}(x)-P(z)}\, dx\, dz, \qquad y \geqslant 0, \qquad k = 1, 2,\ldots. \quad (3.12)$$

Since

$$\begin{aligned} x^k - z^k &= \left[\frac{x+z}{2} + \frac{x-z}{2}\right]^k - \left[\frac{x+z}{2} - \frac{x-z}{2}\right]^k \\ &= \Sigma a_{l,m}(x+z)^l(x-z)^m \end{aligned}$$

for certain $a_{l,m} \geqslant 0$ and since $y \geqslant 0$, (3.12) holds provided

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^2} (x+z)^l(x-z)^m e^{-\bar{P}(x)-P(z)}\, dx\, dz, \qquad \text{for all} \quad l, m \geqslant 0, \\ &= 2^{l+m+1} \int_{\mathbb{R}^2} s^l u^m e^{-\bar{P}(s+u)-P(s-u)}\, ds\, du. \end{aligned}$$

Because of the evenness of P, $\bar{P}$, this integral vanishes unless l and m are either both even or both odd. If l and m both are even, then the integral is clearly positive. Now take l and m both odd. Via a change of variables in the second, third, and fourth quadrants of $\mathbb{R}^2$, we rewrite the integral as

$$2 \int_{\mathbb{R}_+{}^2} s^l u^m [\exp(-\bar{P}(s+u) - P(s-u)) - \exp(-\bar{P}(s-u) - P(s+u))]\, ds\, du.$$

As in the proof of case (iv) of Theorem 2.2, the last integral is nonnegative for all l, m odd if

$$P(s+u) - P(s-u) \geqslant \bar{P}(s+u) - \bar{P}(s-u), \quad \text{all} \quad s, u \geqslant 0,$$

which is equivalent to $\partial P/\partial x \geqslant \partial \bar{P}/\partial x$ for $x \geqslant 0$. This completes the proof for $n = 1$. Now assume that $n > 1$ and, without loss of generality, that $i = 1$ in (2.4). We define $\tilde{x} = (x_2, \ldots, x_n)$, $\tilde{y} = (y_2, \ldots, y_n)$, and

$$\left.\begin{matrix} U(x_1) \\ \bar{U}(x_1) \end{matrix}\right\} = -\ln \int_{\mathbb{R}^{n-1}} d\tilde{x} \exp(\tilde{x}\cdot\tilde{y}) \cdot \left\{\begin{matrix} \exp(-P(x_1, \tilde{x})) \\ \exp(-\bar{P}(x_1, \tilde{x})). \end{matrix}\right.$$

Then (2.4) may be rewritten as

$$\frac{\int x_1{}^k e^{x_1 y_1 - U(x_1)}\, dx_1}{\int e^{x_1 y_1 - U(x_1)}\, dx_1} \leqslant \frac{\int x_1{}^k e^{x_1 y_1 - \bar{U}(x_1)}\, dx_1}{\int e^{x_1 y_1 - \bar{U}(x_1)}\, dx_1}, \qquad \text{all } k = 1, 2,..., y_1 \geqslant 0. \tag{3.13}$$

By the proof for $n = 1$, (3.13) holds provided

$$\frac{\partial U}{\partial x_1} \equiv \frac{\int \partial P/\partial x_1 \exp(\tilde{x} \cdot \tilde{y} - P)\, d\tilde{x}}{\int \exp(\tilde{x} \cdot \tilde{y} - P)\, d\tilde{x}} \geqslant \frac{\int \partial \bar{P}/\partial x_1 \exp(\tilde{x} \cdot \tilde{y} - \bar{P})\, d\tilde{x}}{\int \exp(\tilde{x} \cdot \tilde{y} - \bar{P})\, d\tilde{x}} \equiv \frac{\partial \bar{U}}{\partial x_1}, \qquad \text{for } x_1 \geqslant 0. \tag{3.14}$$

But because of conditions (2.5),

$$\frac{\partial U}{\partial x_1} \geqslant \frac{\int \partial \bar{P}/\partial x_1 \exp(\tilde{x} \cdot \tilde{y} - P)\, d\tilde{x}}{\int \exp(\tilde{x} \cdot \tilde{y} - P)\, d\tilde{x}} = \frac{\partial \bar{P}}{\partial x_1} = \frac{\partial \bar{U}}{\partial x_1}, \qquad \text{for} \quad x_1 \geqslant 0,$$

and so (3.14) follows. ∎

IV. Proof of Theorem 2.5

Let u solve (2.6) with $\lim_{t\downarrow 0} u(t, \cdot) = \exp(-W_0)$ and define $W_t(x) = -\ln u$ and $\Omega_0 = (0, T] \times \mathbb{R}^n$, $0 < T < \infty$. Our strategy is to apply the maximum principle of [9, p. 43, Theorem 9] to the (quasilinear) equations satisfied by $Y \equiv DW_t$ for each of the five cases of Theorem 2.5. Here, D is the first-, second-, or third-order differential operator appearing in the definitions of the classes. For each case, we show that Y satisfies an equation of the form

$$\partial Y/\partial t = MY + \varphi(t, x) + DV \text{ in } \Omega_0,$$

$$MY = \tfrac{1}{2}\Delta Y + \sum_{j=1}^{n} \beta_j(t, x)\, Y_{x_j} + \gamma(t, x) Y,$$

$$\lim_{t\downarrow 0} Y = W_0 .$$

The functions β_j, γ, and φ depend on t, x through W and its derivatives, $\varphi \equiv 0$ in Cases (i), (ii) and (v) and $\varphi \leqslant 0$ in Ω_0 in Cases (iii) and (iv). We now assume that W_0 and V belong to $\mathscr{W}$, for a given class $\mathscr{W}$ in the theorem. For the plus classes in (i), (ii) and (v), we have that

$$(M - \partial/\partial t) Y \leqslant 0 \text{ in } \Omega_0, \; Y(0, x) \geqslant 0 \text{ in } \mathbb{R}^n, \tag{4.2}$$

as $\varphi \equiv 0$, DV, $DW_0 \geqslant 0$. For the minus classes in (i), (ii), (v) and in Cases (iii) and (iv), we have that

$$(M - \partial/\partial t)Y \geqslant 0 \text{ in } \Omega_0\,, \; Y(0, x) \leqslant 0 \text{ in } \mathbb{R}^n, \tag{4.3}$$

as φ, DV, $DW_0 \leqslant 0$. Hence, modulo the bounds (4.4), we may conclude that $Y \geqslant 0$ in Ω_0 in (4.2), $Y \leqslant 0$ in Ω_0 in (4.3). In all cases, we see that $W \in \mathscr{W}$.

In order to make this argument rigorous, one must show that the bounds [9, p. 43, Eqs. (4.1)–(4.2)] on Y and on the coefficients in M in (4.2) and in (4.3) above are valid. Comparison with the equations to be derived below shows that we need the following a priori bounds on W and its derivatives for $(t, x) \in \Omega_0$:

$$\begin{aligned}
&\text{Case (i):} \quad |\nabla W| \leqslant K(|x| + 1);\\
&\text{Case (ii):} \quad \pm W_{x_1 x_2} \geqslant -B\exp(C\,|x|^2), \qquad |\nabla W| \leqslant K(|x| + 1),\\
&\qquad\qquad W_{x_i x_i} \geqslant -K(|x|^2 + 1), \qquad i = 1, 2;\\
&\text{Case (iii)–(iv):} \quad W_{x_i x_i} \leqslant B\exp(C\,|x|^2), \qquad i = 1,\ldots, n;\\
&\qquad\qquad |\nabla W| \leqslant K(|x| + 1);\\
&\text{Case (v):} \quad \pm W_{x_1 x_1 x_1} \geqslant -B\exp(C\,|x|^2),\ |W_{x_1}| \leqslant K(|x| + 1),\\
&\qquad\qquad W_{x_1 x_1} \geqslant -K(|x|^2 + 1).
\end{aligned} \tag{4.4}$$

Here, $|\nabla W| \equiv \sup_{1 \leqslant i \leqslant n} |W_{x_i}|$; $0 < K, B, C < \infty$; one chooses the signs in Cases (ii) and (v) of (4.4) according to whether one is working with the plus or minus class in these cases. We are unable to obtain these bounds in any generality; see Remark 4.1(c) for special cases.

We proceed to derive Eqs. (4.1) for the five cases. The following are easily checked ($i, j, l \in \{1,\ldots, n\}$):

$$\begin{aligned}
&\text{(a)} \quad (W_{x_i})_t = \tfrac{1}{2}\Delta(W_{x_i}) + \beta \cdot \nabla(W_{x_i}) + V_{x_i}\,, \qquad \beta = -\nabla W;\\
&\text{(b)} \quad (W_{x_i x_j})_t = \tfrac{1}{2}\Delta(W_{x_i x_j}) + \beta \cdot \nabla(W_{x_i x_j}) + \theta + V_{x_i x_j}\,,\\
&\qquad\qquad \beta = -\nabla W, \qquad \theta = -\nabla(W_{x_i}) \cdot \nabla(W_{x_j});\\
&\text{(c)} \quad (W_{x_i x_j x_l})_t = \tfrac{1}{2}\Delta(W_{x_i x_j x_l}) + \beta \cdot \nabla(W_{x_i x_j x_l}) + \theta + V_{x_i x_j x_l}\,,\\
&\qquad\qquad \beta = -\nabla W,\ \theta = -\nabla(W_{x_i}) \cdot \nabla(W_{x_j x_l}) - \nabla(W_{x_j})\\
&\qquad\qquad\quad \cdot \nabla(W_{x_i x_l}) - \nabla(W_{x_l}) \cdot \nabla(W_{x_i x_j}).
\end{aligned} \tag{4.5}$$

Case (i) is covered by Eq. (4.5) (a). For Cases (ii)–(iv), we derive the following:

$$
\begin{aligned}
&\text{(a)}\quad (W_{x_1x_2})_t = \tfrac{1}{2}\Delta(W_{x_1x_2}) + \beta\cdot\nabla(W_{x_1x_2}) + \gamma W_{x_1x_2} + V_{x_1x_2}\,,\\
&\qquad\qquad \beta = -\nabla W, \qquad \gamma = -\Delta W;\\
&\text{(b)}\quad (W_{x_ix_i})_t = \tfrac{1}{2}\Delta(W_{x_ix_i}) + \beta\cdot\nabla(W_{x_ix_i}) + \varphi + V_{x_ix_i}\,,\\
&\qquad\qquad \beta = -\nabla W, \varphi = -\nabla(W_{x_i})\cdot\nabla(W_{x_i});\\
&\text{(c)}\quad (DW)_t = \tfrac{1}{2}\Delta(DW) + \beta\cdot\nabla(DW) + \varphi + DV,\\
&\qquad D = \Sigma\gamma_{ij}\,\partial^2/\partial x_i\partial x_j\,, \qquad \beta = -\nabla W,\\
&\qquad \varphi = -\Sigma\gamma_{ij}\nabla(W_{x_i})\cdot\nabla(W_{x_j}).
\end{aligned}
\tag{4.6}
$$

Case (ii) is covered by (4.6) (a); $\varphi \leqslant 0$ in (4.6) (b), which covers Case (iii); by the positive definiteness of Γ, $\varphi \leqslant 0$ in (4.6) (c), which covers Case (iv). Case (v) is covered by (4.5) (c) with $i = j = l = n = 1$ ($x = x_1$):

$$
\begin{aligned}
(W_{xxx})_t &= \tfrac{1}{2}(W_{xxx})_{xx} + \beta(W_{xxx}) + \gamma W_{xxx} + V_{xxx}\,,\\
\beta &= -W_x\,, \qquad \gamma = -3W_{xx}\,.
\end{aligned}
\tag{4.7}
$$

Remark 4.1. (a) We explain how method two yields the exponential preservation of the classes $\mathscr{W}_\pm^k$, $k = 1, 2, 3$, of Theorem 2.2 (with $\delta = 0$ for $\mathscr{W}_-^1$). We define $\Omega_0 = (0, T] \times G$, where

$$
\begin{aligned}
G &= \{x \in \mathbb{R}^n \mid x_i \geqslant 0\} \qquad \text{for} \quad \mathscr{W}_\pm^1,\\
&= \mathbb{R}_+^2 \qquad \text{for} \quad \mathscr{W}_\pm^2,\\
&= \mathbb{R}_+^1 \qquad \text{for} \quad \mathscr{W}_\pm^3.
\end{aligned}
$$

In each case, the maximum principle yields the exponential preservation provided $DW \geqslant 0$ in $(0, T] \times \partial G$ in the plus cases, $DW \leqslant 0$ in $(0, T] \times \partial G$ in the minus cases (see [15, p. 183, Theorem 10]. But by the evenness of W, $DW \equiv 0$ in $(0, T] \times \partial G$ in all cases.

b) Concerning the exponential preservation of the classes $\mathscr{W}_\pm^k$, $k \geqslant 4$, defined by (2.8), we derive from (4.7) the following equations satisfied by $W^{(k)}$:

$$
\begin{aligned}
(W^{(k)})_t &= \tfrac{1}{2}(W^{(k)})_{xx} + \beta(W^{(k)})_x + \gamma_k W^{(k)} + \varphi_k + V^{(k)},\\
\beta &= -W^{(1)}, \qquad \gamma_k = -[3 + (k-3)]\, W^{(2)},\\
\varphi_k &= -\sum_{m=3}^{k-1} \lambda_m W^{(m)} W^{(k+2-m)}, \qquad \lambda_m > 0,\\
k &\geqslant 4.
\end{aligned}
\tag{4.8}
$$

An induction argument in k shows that in the case of $\tilde{\mathscr{W}}_+^k$, $(-1)^{k-1}\varphi_k \geqslant 0$ while in the case of $\tilde{\mathscr{W}}_-^k$, $\varphi_k \leqslant 0$; hence in both cases the maximum principle may be applied. It is not difficult to produce $W_0 \in \bigcap_{k\geqslant 4} \mathscr{W}_*^k(\mathbb{R}^1\backslash\{0\})$, $* = +$ or $-$; i.e., W_0 which satisfy the derivative inequalities in (2.8) in $\mathbb{R}^1\backslash\{0\}$ (e.g., set $W_0 \equiv 0$ on $(-\infty, 0)$ and use [8, p. 415]). For the status of the maximum principle in such a situation, we refer to [15, pp. 99, 186].

(c) We consider special cases which at least show that the a priori bounds (4.4) are reasonable. If W_0 is totally even and satisfies the bounds

$$|(W_0)_{x_i}| \leqslant K \mid x_i \mid \text{ in } \mathbb{R}^n, \qquad \text{for each} \quad i = 1,\ldots, n, \tag{4.9}$$

(3.5) shows that $|\nabla W| \leqslant M \mid x \mid$ in Ω_0. Under assumption (4.9) and the total evenness of W_0, Lemma 2.4 can be applied to higher derivatives of W to conclude that for $i = 1,\ldots, n$,

$$| W_{x_ix_i} | \leqslant M(| x |^2/t + 1), \qquad | \partial^j/\partial x_i^j W | \leqslant M(| x |^j/t^j + 1), \qquad j \geqslant 3,$$

but these bounds are bad for $t \downarrow 0$. If W_0 is convex, then by [2, Theorem 1.3], we may conclude that $W_{x_ix_i} \geqslant 0$, $i = 1,\ldots, N$, $\Delta W \geqslant 0$ in Ω_0.

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