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Asymptotic Equivalence of the Linear Navier–Stokes and Heat Equations in One Dimension

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1. INTRODUCTION

In a previous paper [2], the following system of equations was treated as a model of the Boltzmann equation

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} = \frac{1}{\epsilon}Qp \qquad (t > 0, \epsilon > 0, x \text{ real}, v \in \{v_1, \ldots, v_N\}),$$
$$\lim_{t \downarrow 0} p(t, x, v) = f(x, v). \tag{1.1}$$

In Eq. (1.1), $p = p_\epsilon(t, x, v)$ and

$$Qp(t, x, v_i) = \sum_{1 \leqslant j \leqslant N} q_{ij} p(t, x, v_j);$$

$Q = (q_{ij})$ is a symmetric negative semidefinite $N \times N$ matrix which has a d-dimensional nullspace $\mathcal{N}(1 \leqslant d < N)$ and $q_{ii} \neq 0$; $v = \text{diag}(v_j)$, where the numbers $\{v_j\}$ are real and distinct; f is an N-tuple of nice functions of x. The Navier–Stokes equations are the second in a hierarchy of approximate equations (the first being Euler) for the hydrodynamical moments of solutions of Eq. (1.1) with $\epsilon = 1$. For the special orthonormal basis $\{e_0^{(k)}; 1 \leqslant k \leqslant d\}$ of $\mathcal{N}$ introduced in [2], it was shown that the Navier–Stokes equations for the moments

$$n_k(t, x) = \langle p, e_0^{(k)} \rangle \equiv \sum_{1 \leqslant j \leqslant N} p(t, x, v_j)\, e_0^{(k)}(v_j)$$

are

$$n_k^{\cdot} = \alpha_1^{(k)} n_k{}' + \sum_{1 \leqslant l \leqslant d} \alpha_2^{(kl)} n_l'',$$
$$\lim_{t \downarrow 0} n_k(t, x) = f_k(x) \equiv \langle f, e_0^{(k)} \rangle, \; 1 \leqslant k \leqslant d. \tag{1.2}$$

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In the previous equation $\{\alpha_1^{(k)}\}$ and $\{\alpha_2^{(kl)}\}$ are constants of the problem (defined in Subsection 2A) and $\alpha_2^{(k)} \equiv \alpha_2^{(kk)} > 0$ (proved in Lemma 2.1 below). Also, $\dot{}$ denotes $\partial/\partial t$, and $'$ $\partial/\partial x$.

We define functions $m_k(t, x)$ which are solutions of the uncoupled heat equations

$$m_k^{\cdot} = \alpha_2^{(k)} m_k'', \lim_{t \downarrow 0} m_k(t, x) = f_k(x), \; 1 \leqslant k \leqslant d. \tag{1.3}$$

The purpose of this paper is to investigate the asymptotic behavior as $t \to \infty$ of the functions $n_k(t, x - \alpha t)$, $1 \leqslant k \leqslant d$, for different choices of the constant α. The first part of Theorem 1 states that if $\alpha = \alpha_1^{(k)}$, then the difference between the translated n_k and the corresponding solution m_k of Eq. (1.3) tends to zero like $1/t$ as $t \to \infty$. Since m_k tends to zero as $t \to \infty$ at the exact rate $t^{-1/2}$, uniformly for x in compacta (see Proposition 2.4), the first part of Theorem 1 gives the asymptotic form of $n_k(t, x - \alpha_1^{(k)} t)$ as $t \to \infty$.

We first prove the theorem under the following assumption.

(C) The numbers $\{\alpha_1^{(k)}\}$ are distinct.

In [1; Section 6] it was shown that (C) holds in cases of physical interest; e.g., if $\mathcal{N} = \text{span}\{v^j; 0 \leqslant j \leqslant d-1\}$.

THEOREM 1. *Let* (C) *hold and assume that for each* $1 \leqslant k \leqslant d$, f_k *is a nonnegative function, rapidly decreasing at infinity with summable Fourier transform. Then for each k we have the following statements*:

$$\sup_{-\infty<x<\infty} | n_k(t, x - \alpha_1^{(k)} t) - m_k(t, x)| = O(1/t), \quad \textit{as} \quad t \to \infty; \tag{1.4}$$

$$\sup_{-\infty<x<\infty} | n_k(t, x - \alpha_1^{(j)} t)| = O(1/t), \quad \textit{as} \quad t \to \infty, \tag{1.5}$$

for $\alpha_1^{(j)} \neq \alpha_1^{(k)}$;

$$\sup_{-\infty<x<\infty} | n_k(t, x - \alpha t)| = O(1/t^n), \quad \textit{as} \quad t \to \infty, \tag{1.6}$$

for $\alpha \notin \{\alpha_1^{(1)}, \ldots, \alpha_1^{(d)}\}$ *and any positive integer* n.

The theorem is proved in Section 2. In Section 3, we show how to extend the method in the case when (C) fails.

Our method of proof has close connections with the general method of Sirovich [8] for the asymptotic evaluation of a class of multidimensional integrals. He obtained estimates of the type (1.4)–(1.5), but with a slightly larger remainder term. Estimates of the type (1.6) take advantage of the oscillatory character of the integrand and do not appear to have been considered in Ref. [8].

The significance of Theorem 1 is best grasped in the context of the limit theorems proved in [2]. These results imply that

$$\lim_{\epsilon\downarrow 0} \langle p_\epsilon(t, x, \cdot), e_0^{(k)}(\cdot)\rangle = f_k(x + \alpha_1^{(k)} t), \tag{1.7}$$

$$\lim_{\epsilon\downarrow 0} \langle p_\epsilon((t/\epsilon), x - \alpha_1^{(k)} (t/\epsilon), \cdot), e_0^{(k)}(\cdot)\rangle = m_k(t, x). \tag{1.8}$$

A problem posed in [2, Section 5] was to discover the relationship between these limit results and the equations of hydrodynamics associated with Eq. (1.1). As we showed in Ref. [2], the aggregate of functions $\{f_k(x + \alpha_1^{(k)} t)\}$ solves the Euler equations associated with Eq. (1.1). Therefore, one of the points of the present research is to connect the Navier–Stokes equations for Eq. (1.1) with the second limit theorem (1.8) for p_ϵ. Also, comparing Eqs. (1.4) and (1.8) shows that in a certain sense the Boltzmann equation and the Navier–Stokes equations are asymptotic as $t \to \infty$. All these remarks should be compared to those of Grad [5, pp. 175–180; and 3].

Finally, we point out that the statement and proof of Theorem 1 apply equally well to the so-called "one dimensional linearized Navier–Stokes equations" [7, 9]

$$\begin{aligned} &\rho_t + u_x = 0 \qquad (\eta, \lambda > 0, \beta = (\tfrac{2}{3})^{\frac{1}{2}}),\\ &u_t + \rho_x + \beta T_x = \eta u_{xx},\\ &T_t + \beta u_x = \lambda T_{xx}. \end{aligned} \tag{1.2$'$}$$

This case was already treated by Sirovich [9, Eqs. (3.13) and (3.15)]. By diagonalization (1.2)′ is of the form (1.2) for some choice of the symmetric nonnegative matrix $\alpha_2^{(jk)}$. *But these* $\alpha_2^{(jk)}$ *cannot be computed from a model Boltzmann equation via the formula* (2.5). Indeed, (1.2)′ is usually derived from the three dimensional Navier–Stokes equations by specializing to solutions independent of (y, z).

We end this Introduction by stating several extra results for p_ϵ which emphasize even more the deep duality between the problems (1.1) and (1.2). The three statements in the next theorem correspond respectively to Eqs. (1.4)–(1.6) in Theorem 1.

THEOREM 2. *Let f in Eq.* (1.1) *be an N-tuple of functions f_k, where each f_k is as in the previous theorem. Assume Condition C. Then for each* $1 \leqslant k \leqslant d$ *and fixed* $t > 0$, *we have the following facts*:

$$\sup_{-\infty<x<\infty} | \langle p_\epsilon((t/\epsilon), x - \alpha_1^{(k)} (t/\epsilon), \cdot), e_0^{(k)}(\cdot)\rangle - m_k(t, x)| = O(\epsilon), \quad \text{as } \epsilon \to 0; \tag{1.9}$$

$$\sup_{-\infty<x<\infty} |\langle p_\epsilon((t/\epsilon), x - \alpha_1^{(j)} (t/\epsilon), \cdot)\, e_0^{(k)}(\cdot)\rangle| = O(\epsilon), \qquad \text{as } \epsilon \to 0, \tag{1.10}$$

for $\alpha_1^{(j)} \neq \alpha_1^{(k)}$;

$$\sup_{-\infty<x<\infty} |\langle p_\epsilon((t/\epsilon), x - \alpha(t/\epsilon), \cdot), e_0^{(k)}(\cdot)\rangle| = O(\epsilon^n), \qquad \textit{as} \quad \epsilon \to 0, \tag{1.11}$$

for $\alpha \notin \{\alpha_1^{(1)},..., \alpha_1^{(d)}\}$ *and any positive integer* n.

The first statement (1.9) was noted in Ref. [2], but the other two are new. The proof of Theorem 2 is entirely analogous to that of Theorem 1, as we shall point out at the end of the next section.

2. Proof of Theorem 1

This section is divided into four subsections. In Subsection 2A we state some facts from linear algebra which are needed in the proof. In 2B, we prove Eq. (1.4) in Theorem 1 as well as the statement on the decay of m_k as $t \to \infty$. In 2C, we give the proofs of three key lemmas stated in 2A. In 2D, we prove Eqs. (1.5) and (1.6) in Theorem 1 and remark on the proof of Theorem 2.

2A. *Preliminaries on Linear Algebra*

We first define the constants appearing in Eqs. (1.2) and (1.3). For λ complex, let $\{\alpha_j(\lambda); 1 \leqslant j \leqslant d\}$ be the eigenvalues of $Q - \lambda v$ defined by the problem

$$(Q - \lambda v)\, e_j(\lambda) = \alpha_j(\lambda)\, e_j(\lambda), \alpha_j(0) = 0. \tag{2.1}$$

One shows [2, Corollary 2A.3] that $\alpha_j(\lambda)$ and the corresponding eigenvector $e_j(\lambda)$ are analytic functions of λ for $|\lambda| \leqslant \lambda_0$, some $\lambda_0 > 0$. Thus, we may write

$$\alpha_j(\lambda) = \sum_{k\leqslant 1} \alpha_k^{(j)} \lambda^k, \tag{2.2}$$

$$e_j(\lambda) = \sum_{k\geqslant 0} e_k^{(j)} \lambda^k,\ 1 \leqslant j \leqslant d, \qquad \text{for} \quad |\lambda| \leqslant \lambda_0. \tag{2.3}$$

The coefficients of λ and λ^2 in (2.2) give respectively the constants $\{\alpha_1^{(j)}\}$, $\{\alpha_2^{(j)}\}$ which appear in Eqs. (1.2) and (1.3). The special orthonormal basis $\{e_0^{(j)}\}$ of $\mathscr{N}$ is given by the coefficients of λ^0 in (2.3). One may show that

$$\alpha_1^{(j)} \delta_{kj} = -\langle e_0^{(k)}, v e_0^{(k)} \rangle. \tag{2.4}$$

We define the constants $\alpha_2^{(kl)}$ in Eq. (1.2) by

$$\alpha_2^{(kl)} = -\langle (v + \alpha_1^{(k)}) e_0^{(k)}, Q^{-1}(v + \alpha_1^{(l)}) e_0^{(l)} \rangle, \tag{2.5}$$

where Q^{-1} denotes the inverse of Q as a symmetric negative definite operator from $\mathscr{N}^{\perp}$ to $\mathscr{N}^{\perp}$. The $\alpha_2^{(kl)}$ are well-defined since by Eq. (2.4)

$$(v + \alpha_1^{(l)})\, e_0^{(l)} \in \mathscr{N}^{\perp}.$$

One can show that $\alpha_2^{(k)} = \alpha_2^{(kk)}$. The fact that $\alpha_2^{(k)} > 0$ is stated in the following lemma, the proof of which we save for Subsection 2C.

LEMMA 2.1. *The numbers*

$$\alpha_2^{(k)} = -\langle y, Q^{-1}y\rangle, \; y = (v + \alpha_1^{(k)})e_0^{(k)}, \; 1 \leqslant k \leqslant d,$$

are all positive.

In order to prove Theorem 1, we consider the eigenvalue problem for the matrix $\lambda A + \lambda^2 B$, λ complex, where A and B are the $d \times d$ matrices defined by

$$A = \operatorname{diag}(\alpha_1^{(k)}), \qquad B = (\alpha_2^{(kl)}).$$

We define $\beta_j(\lambda)$ and $g_j(\lambda)$, $1 \leqslant j \leqslant d$, to be the eigenvalues and corresponding eigenfunctions of $\lambda A + \lambda^2 B$:

$$(\lambda A + \lambda^2 B)\, g_j(\lambda) = \beta_j(\lambda)\, g_j(\lambda), \tag{2.6}$$

where the $g_j(\lambda)$ are normalized so that $\langle g_j(\lambda), g_j(\lambda)\rangle = 1$. In general, $\langle \cdot, \cdot \rangle$ denotes the complex inner product on $\mathscr{C}^d$. We also define $Y_{j,k}(\lambda)$, $1 \leqslant j \leqslant d$, $1 \leqslant k \leqslant d-1$, to be the component matrices of $\lambda A + \lambda^2 B$[6, pp. 173–174]. These are the matrices which appear in the decomposition of the matrix exponential

$$\exp[t(\lambda A + \lambda^2 B)] = \sum_{1 \leqslant j \leqslant d} \{Y_{j,1}(\lambda) + tY_{j,2}(\lambda) + \cdots + t^{d-1}Y_{j,d-1}\} \exp(t\beta_j(\lambda)). \tag{2.7}$$

Later on we shall need the facts stated in the next two lemmas concerning the behavior of $\beta_j(\lambda)$ and $Y_{j,k}(\lambda)$ for $\lambda \to 0$ and for $\lambda \to \infty$. The proofs are given in Subsection 2C. We point out that (2.10) and Lemma 2.3 will be proven without the use of condition (C).

LEMMA 2.2. *Assume* (C) *holds. Then for each* $1 \leqslant j \leqslant d$, *we can find a* $k = k(j)$ *such that*

$$\beta_j(\lambda) = \lambda\alpha_1^{(k)} + \lambda^2\alpha_2^{(k)} + O(\lambda^3), \qquad as \quad |\lambda| \to 0, \tag{2.8}$$

$$(Y_{j,1}(\lambda))_{lm} = \delta_{kl}\,\delta_{km} + O(\lambda),\ as\ |\lambda| \to 0, \tag{2.9}$$

$$Y_{j,l}(\lambda) \equiv 0,\ 2 \leqslant l \leqslant d-1,\ |\lambda| \leqslant \lambda_0,\ some\ \lambda_0 > 0. \tag{2.10}$$

LEMMA 2.3. *For any* $\theta > 0$ *and each* $1 \leqslant j \leqslant d$, *we can find a constant* $M > 0$ *such that*

$$\sup_{\substack{\gamma \text{ real} \\ |\gamma| \geqslant \theta}} \operatorname{Re} \beta_j(i\gamma) \leqslant -M \qquad (i = (-1)^{\frac{1}{2}}). \tag{2.11}$$

We also have Eq. (2.10) *for* $|\lambda| \to \infty$ *and*

$$Y_{j,1}(\lambda) = O(1), \text{ as } |\lambda| \to \infty. \tag{2.12}$$

Remark. We rename the $\beta_j(\lambda)$ and $Y_{j,1}(\lambda)$, if necessary, so that in Lemma 2.2 $k(j) = j$ for each $1 \leqslant j \leqslant d$.

2B. *Proof of Eq.* (1.4) *in Theorem* 1

We use Fourier transforms. Let

$$n(t, x) = \begin{pmatrix} n_1(t, x) \\ \vdots \\ n_d(t, x) \end{pmatrix}, \qquad f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_d(x) \end{pmatrix}.$$

Then

$$n(t, x) = \int e^{i\gamma x}\, \hat{n}(t, \gamma)\, d\gamma,$$

where $\hat{n}(t, \gamma) = (2\pi)^{-1} \int e^{-i\gamma x} n(t, x)\, dx$ satisfies

$$\frac{\partial \hat{n}}{\partial t} = (i\gamma A - \gamma^2 B)\hat{n}, \ \hat{n}\,|_{t=0^+} = \hat{f}.$$

Thus

$$n = \int e^{i\gamma x} \exp[t(i\gamma A - \gamma^2 B)]\, \hat{f}\, d\gamma. \tag{2.13}$$

Here and in the sequel, integration extends over the whole real line when no limits are indicated.

By Eq. (2.10), the higher component matrices $Y_{j,k}$, $1 \leqslant j \leqslant d$, $2 \leqslant k \leqslant d - 1$, are zero for $|\gamma| \leqslant \theta$, some $\theta > 0$. By Lemma 2.3, (2.7), and the summability of $\hat{f}$, the contribution to the integral in (2.13) for $|\gamma| > \theta$ can be bounded by Me^{-Mt} (here and below M denotes a generic positive constant). Thus by Eq. (2.7) we may write

$$n_k(t, x - \alpha_1{}^k t) = \sum_{1 \leqslant j, l \leqslant d} \int_{|\gamma| \leqslant \theta} e^{i\gamma x} (Y_{j,1}(i\gamma))_{kl} \exp[t(\beta_j(i\gamma) - i\alpha_1^{(k)}\gamma)]\, \hat{f}_l(\gamma)\, d\gamma$$
$$+ O(e^{-Mt}). \tag{2.14}$$

On the other hand,

$$m_k(t, x) = \int e^{i\gamma x} e^{-t\alpha_2^{(k)}\gamma^2} \hat{f}_k(\gamma)\, d\gamma$$
$$= \int_{|\gamma| \leqslant \theta} e^{i\gamma x} e^{-t\alpha_2^{(k)}\gamma^2} \hat{f}_k(\gamma)\, d\gamma + O(e^{-Mt}). \tag{2.15}$$

From Lemma 2.2 we have the following estimates

$$|(Y_{j,1}(i\gamma))_{kl} - \delta_{jk}\delta_{jl}| \leqslant M |\gamma|, \qquad \text{for} \quad |\gamma| \leqslant \theta, \tag{2.16}$$

$$|\exp[t(\beta_k(i\gamma) - i\alpha_1^{(k)}\gamma)] - e^{t\alpha_2^{(k)}\gamma^2}| \leqslant Mt |\gamma|^3 e^{-Mt\gamma^2} \quad \text{for} \ |\gamma| \leqslant \theta. \tag{2.17}$$

Hence from Eqs. (2.14) and (2.15)

$$n_k(t, x - \alpha_1^{(k)}t) - m_k(t, x)$$
$$= \sum_{j=1}^{d} \int_{|\gamma| \leqslant \theta} e^{i\gamma x}((Y_{j,1}(i\gamma))_{kl} - \delta_{jk}\delta_{jl}) \exp[t(\beta_j(i\gamma) - i\alpha_1^{(k)}\gamma)] \hat{f}_l\, d\gamma$$
$$+ \int_{|\gamma| \leqslant \theta} e^{i\gamma x}(\exp[t(\beta_k(i\gamma) - i\alpha_1^{(k)}\gamma)] - e^{-t\alpha_2^{(k)}\gamma^2}) \hat{f}_k(\gamma)\, d\gamma$$
$$+ O(e^{-Mt}). \tag{2.18}$$

For the first term on the right-hand side of Eq. (2.18) we use Eq. (2.16); on the second term we use Eq. (2.17). Thus

$$|n_k(t, x - \alpha_1^{(k)}t) - m_k(t, x)| \leqslant M \int_{|\gamma| \leqslant \theta} |\gamma|\, e^{-Mt\gamma^2}\, d\gamma$$
$$+ M \int_{|\gamma| \leqslant \theta} t\, |\gamma|^3\, e^{-Mt\gamma^2} + O(e^{-Mt}) \leqslant \frac{M}{t}.$$

This completes the proof of Eq. (1.4). ▼

We now prove the fact stated in the Introduction concerning the decay of $m_k(t, x)$ as $t \to \infty$.

PROPOSITION 2.4. *Take f_k as in Theorem 1 not identically zero. Then for any compact set K there are constants $M_\pm$ such that*

$$\frac{M_-}{t^{1/2}} \leqslant \inf_{x \in K} m_k(t, x) \leqslant \sup_{-\infty < x < \infty} m_k(t, x) \leqslant \frac{M_+}{t^{1/2}}, \qquad \textit{as} \quad t \to \infty.$$

Proof. This is a consequence of Eq. (2.15) and the fact that

$$\hat{f}_k(\gamma) = \hat{f}_k(0) + O(\gamma), \qquad \text{for} \quad |\gamma| \leqslant \theta,$$
$$\hat{f}_k(0) = (2\pi)^{-1} \int f_k dx > 0. \quad ▼$$

2C. *Proofs of Lemmas* 2.1–2.3

Proof of Lemma 2.1. The formula for $\alpha_2^{(k)} = \alpha_2^{(kk)}$ follows from Eq. (2.5). Since Q is negative semidefinite, the lemma will be shown once we prove $y \neq 0$. But if $y = 0$, then $v_i = -\alpha_1^{(k)}$ whenever $e_0^{(k)}(v_i) \neq 0$. Let $Z = \{i : e_0^{(k)}(v_i) = 0\}$. Since $e_0^{(k)} \neq 0$, we know that $|Z| \leqslant N - 1$($|Z|$ denotes the number of elements of Z). If $|Z| \leqslant N - 2$, then $v_i = -\alpha_1^{(k)}$ for at least two different values of i, which contradicts the fact that the $\{v_i\}$ are distinct. If $|Z| = N - 1$, then for some $1 \leqslant i_0 \leqslant N$ $e_0^{(k)}(v_{i_0}) \neq 0$ while $e_0^{(k)}(v_i) = 0$ for $i \neq i_0$. This implies that for each $1 \leqslant i \leqslant N$

$$0 = \sum_{1 \leqslant j \leqslant N} q_{ij} e_0^{(k)}(v_j) = q_{ii_0}.$$

But as $q_{i_0 i_0} \neq 0$ by assumption, we have a contradiction. ▼

Proof of Lemma 2.2. We first prove Eq. (2.8). Since A and B are symmetric matrices, $\beta_j(\lambda)$ is analytic in λ for $|\lambda|$ sufficiently small [6, p. 252]. Take $j = 1$ and write

$$\beta_1(\lambda) = c_1\lambda + c_2\lambda^2 + O(\lambda^3), \ |\lambda| \text{ small.} \tag{2.19}$$

Then

$$\det(\lambda A + \lambda^2 B - \beta_1(\lambda) I) = 0. \tag{2.20}$$

But the lowest order term in λ in the expansion of this determinant is

$$\lambda^d \prod_{1 \leqslant k \leqslant d} (\alpha_1^{(k)} - c_1).$$

Since this equals zero and (C) holds, there is exactly one k such that $c_1 = \alpha_1^{(k)}$. Setting $c_1 = \alpha_1^{(k)}$ in Eq. (2.19), one finds that now the lowest order term in λ in the expansion of the determinant in Eq. (2.20) is

$$\lambda^{d+1} \prod_{j \neq k} (\alpha_1^{(j)} - c_1) \cdot (\alpha_2^{(k)} - c_2).$$

Since this equals 0, we conclude that $c_2 = \alpha_2^{(k)}$, which gives Eq. (2.8).

Statement (2.10) is a consequence of the fact that the matrix $\lambda A + \lambda^2 B$ is diagonable for $|\lambda|$ sufficiently small. For such λ, $Y_{j,1}$ is the projection onto the eigenspace of $\beta_j(\lambda)$. Since $g_j(\lambda)^T$ is a left eigenvector of $\lambda A + \lambda^2 B$ with eigenvalue $\beta_j(\lambda)$, we see that for $|\lambda|$ suitably small

$$Y_{j,1}(\lambda) = g_j(\lambda) \otimes g_j(\lambda)^T. \tag{2.21}$$

As an eigenfunction of the matrix $\lambda A + \lambda^2 B$, which is symmetric for λ real, $g_j(\lambda)$ is analytic for $|\lambda|$ small. Hence Eq. (2.9) is shown once we have proven that

$$(g_j(0))_l = \delta_{k(j),l}. \tag{2.22}$$

Since $g_j(0)$ is an eigenfunction of the diagonal matrix A with eigenvalue $\alpha_1^{k(j)}$, this is a consequence of (C) and the fact that the $\{g_j(0)\}$ form an orthonormal basis of E^d. ▼

Proof of Lemma 2.3. Fix $1 \leqslant j \leqslant d$. We prove Eq. (2.11) by showing for $\gamma \neq 0$ and real that

$$\operatorname{Re} \beta_j(i\gamma) < 0, \tag{2.23}$$

and that the following development holds

$$\beta_j(i\gamma) = -\kappa\gamma^2 + i\gamma\mu + \delta + O(1/\gamma), \text{ as } |\gamma| \to \infty, \tag{2.24}$$

where $\kappa \geqslant 0$, μ real, and $\delta < 0$ if $\kappa = 0$.

To show Eq. (2.23), we claim first that B is positive semidefinite. Indeed, let $\{z_k ; 1 \leqslant k \leqslant d\}$ be complex numbers. Then

$$\sum_{kl} z_k^*(B)_{kl} z_l = -\langle y, Q^{-1}y\rangle,$$

where $y = \sum_k z_k(v + \alpha_1^{(k)})\, e_0^{(k)}$. Since $y \in \mathcal{N}^\perp$, this yields the claim. Notice that in general B is not positive definite since y may equal 0. Since $(B)_{kk} \neq 0$, inequality (2.23) now follows from Lemma 2B.1 in [2, p. 93].

To show Eq. (2.24), we have for $\lambda \neq 0$ that

$$\det\left(\frac{A}{\lambda} + B - \frac{\beta_j(\lambda)}{\lambda^2}\right) = 0,$$

or setting $\lambda = \nu^{-1}$, $\alpha_j = \beta_j/\lambda^2$, that

$$\det(\nu A + B - \alpha_j) = 0.$$

Since A and B are symmetric, we conclude that α_j is an analytic function of ν for $|\nu|$ suitably small. In this eigenvalue problem, we are perturbing a semidefinite matrix B by a diagonal matrix A, so that we are in the situation of [2, Section 2A]. Hence, using the formulae developed there, we see that if

$$\alpha_j(\nu) = \kappa + \mu\nu + \delta\nu^2 + O(\nu^3), \ |\nu| \text{ small},$$

then $\kappa \geqslant 0$ since κ is an eigenvalue of B; μ is real as

$$\mu = \langle e, Ae\rangle, \qquad \text{where} \quad Be = \kappa e;$$

and since

$$\delta = -\langle(\mu - A)e, (B - \kappa)^{-1}(\mu - A)e\rangle,$$

$\delta < 0$ whenever $\kappa = 0$ by exactly the same proof used in Lemma 2.1 to show $\alpha_2^{(k)} > 0$. Since $\beta_j(\lambda) = \lambda^2\alpha_j(\lambda^{-1})$, Eq. (2.24) is proved.

The fact that Eq. (2.10) holds for $|\lambda| \to \infty$ follows from the observation that the component matrices $\tilde{Y}_{j,l}(\nu)$ of $B + \nu A$ (which is diagonable for $|\nu|$ sufficiently small) are

$$\tilde{Y}_{j,l}(\nu) = \nu^{2(l-1)}\, Y_{j,l}(\nu^{-1}).$$

Concerning Eq. (2.12), we note that $g_j(\nu^{-1})$ is an eigenfunction of $B + \nu A$ with eigenvalue $\alpha_j(\nu)$ and so is bounded as $\nu \to 0$. Since for such ν

$$\tilde{Y}_{j,1}(\nu) = g_j(\nu) \otimes g_j(\nu)^T,$$

Eq. (2.12) follows. This concludes the proof of Lemma 2.3. ▼

2D. *Proof of Eqs.* (1.5) *and* (1.6) *in Theorem* 1; *Remarks on Theorem* 2.

By the same reasoning which led to Eq. (2.14), we have

$$n_k(t, x - \alpha t) = \sum_{1 \leqslant j,l \leqslant d} \int_{|\gamma| \leqslant \theta} e^{i\gamma x} (Y_{j,1}(i\gamma))_{kl} \cdot \exp[t(\beta_j(i\gamma) - i\alpha\gamma)]\, \hat{f}_l(\gamma)\, d\gamma + O(e^{-Mt}), \qquad \text{some} \quad \theta > 0. \tag{2.25}$$

Let us first assume that $\alpha = \alpha_1^{(m)}$, where $\alpha_1^{(m)} \neq \alpha_1^{(k)}$. Then the term on the right-hand side of Eq. (2.25) corresponding to $j = m$ is

$$\sum_{1 \leqslant l \leqslant d} \int_{|\gamma| \leqslant \theta} e^{i\gamma x} (Y_{m,1}(i\gamma))_{kl} \exp[t(-\alpha_2^{(j)}\gamma^2 + O(\gamma^3)]\, \hat{f}_l(\gamma)\, d\gamma.$$

By Eq. (2.16), this can be bounded by

$$M \int_{|\gamma| \leqslant \theta} |\gamma|\, e^{-Mt\gamma^2}\, d\gamma \leqslant \frac{M}{t}.$$

The other terms in the summation in Eq. (2.25) corresponding to $j \neq m$ all have the same form as

$$r(t, x) = \int_{|\gamma| \leqslant \theta} e^{i\gamma x} \exp[t(i\gamma\alpha_1 - \gamma^2\alpha_2)] \exp(tg(\gamma))\, h(\gamma)\, d\gamma, \tag{2.26}$$

where

$$\alpha_1 \neq 0, \alpha_2 < 0, g \text{ and } h \in C^\infty, \quad g(\gamma) = O(\gamma^3), g'(\gamma) = O(\gamma^2). \tag{2.27}$$

The fact that $h \in C^\infty$ follows from the hypothesis that each f_l is rapidly decreasing at infinity. On the other hand, if in Eq. (2.25) $\alpha \notin \{\alpha_1^{(1)}, \ldots, \alpha_1^{(d)}\}$, then *all* the terms in the summation in Eq. (2.25) have the same form as r. Thus, Eqs. (1.5) and (1.6) will be proved once we have shown the following result.

PROPOSITION 2.4. *Let $r(t, x)$ be defined by Eq.* (2.26), *where α_1, α_2, $g(\gamma)$, and $h(\gamma)$ satisfy Eq.* (2.27). *Then for any integer $n \geqslant 0$*

$$\sup_{-\infty<x<\infty} |r(t, x)| = O(1/t^n), \qquad \textit{as} \quad t \to \infty. \tag{2.28}$$

Proof. For ease of notation, we set $\alpha_1 = \alpha_2 = 1$ in Eq. (2.26). We first modify g and h outside $\{|\gamma| \leqslant \theta/2\}$ by introducing a C^∞ function $\varphi(\gamma)$ such that [4, p. 9]

$$\varphi = \begin{cases} 1, & \text{for} \quad |\gamma| \leqslant \theta/2, \\ 0, & \text{for} \quad |\gamma| > \theta. \end{cases}$$

Now define

$$\tilde{g}(\gamma) = \varphi(\gamma) g(\gamma), \qquad \tilde{h}(\gamma) = \varphi(\gamma) h(\gamma),$$

and let

$$\tilde{r}(t, x) = \int e^{i\gamma x} \exp[t(i\gamma - \gamma^2 + \tilde{g}(\gamma))] \tilde{h}(\gamma) \, d\gamma.$$

Noting that $0 = \tilde{g}'(0) \neq -i$, we choose θ so small that $\tilde{g}' \neq 2\gamma - i$ for $|\gamma| \leqslant \theta$. Since

$$\sup_{-\infty<x<\infty} |r(t, x) - \tilde{r}(t, x)| \leqslant Me^{-Mt}, \qquad \text{as} \quad t \to \infty,$$

it suffices to prove that for each $n \geqslant 0$

$$\sup_{-\infty<x<\infty} t^n |\tilde{r}(t, x)| \leqslant M_n < \infty. \tag{2.31}$$

First of all, it is clear that $\tilde{r}(t, x)$ is bounded uniformly in x and t. For $n = 1$, we have, after integrating by parts, that

$$\begin{aligned} t\tilde{r}(t, x) &= \int e^{i\gamma x} \left(\frac{d}{d\gamma} \exp[t(i\gamma - \gamma^2 + \tilde{g}(\gamma))]\right) \frac{\tilde{h}(\gamma)}{i - 2\gamma + \tilde{g}'(\gamma)} \, d\gamma \\ &= -\int e^{i\gamma x} \exp[t(i\gamma - \gamma^2 + \tilde{g}(\gamma))] \frac{d}{d\gamma} \left(\frac{\tilde{h}(\gamma)}{i - 2\gamma + \tilde{g}'(\gamma)}\right) d\gamma \\ &= \int e^{i\gamma x} \exp[t(i\gamma - \gamma^2 + \tilde{g}(\gamma))] \bar{h}(\gamma) \, d\gamma, \end{aligned} \tag{2.32}$$

where we have set $\bar{h} = (-\tilde{h}/(i - 2\gamma + \tilde{g}'(\gamma)))'$. As $\bar{h}$ is a C^∞ function with support in $\{|\gamma| \leqslant \theta\}$, the last line in Eq. (2.32) is of the same form as $\tilde{r}(t, x)$ and so is bounded uniformly in x and t. As Eq. (2.31) for $n \geqslant 2$ now follows by induction, we are finished. ▼

We end Section 2 with remarks on the proof of Theorem 2. Statement (1.9) was pointed out at the end of Section 3 in Ref. [2]. Its proof is entirely analogous to that of its counterpart Eq. (1.4). Statements (1.10) and (1.11) can be shown starting from the representation of $\langle p_\epsilon(t/\epsilon, x - \alpha t/\epsilon, \cdot), e_0^{(k)}(\cdot)\rangle$ as a Fourier integral (see Ref. [2, Eq. (3.7)]). This representation has exactly the same form as Eq. (2.14) above after writing t/ϵ^2 for t and $\epsilon\gamma$ for γ in the argument of $Y_{j,1}$ and of the exponential in the latter. Hence letting $\epsilon \to 0$ for p_ϵ is the same as letting $t \to \infty$ for n_k, and the proofs of Eqs. (1.10) and (1.11) follow from the work of this subsection.

3. Reformulation of Theorem 1 When Condition (C) Fails

We first recall a result which is needed later (this was Proposition 2A.2 in Ref. [2, p. 291]).

Lemma 3.1. *If* $\alpha_1^{(k)} = \alpha_1^{(l)}$ *for* $k \neq l$, *then* $\alpha_2^{(kl)} = 0$.

Before we reformulate the general result, we note a degenerate case for which we have exponential decay on compacta as $t \to \infty$. Say all the $\alpha_1^{(k)}$ are equal. Then by Lemma (3.1), all the $\alpha_2^{(kl)} = 0$, and it follows from Eq. (1.2) that

$$n_k(t, x - \alpha_1^{(k)} t) = m_k(t, x).$$

In this case, Eq. (1.4) is an equality; the hypothesis of Eq. (1.5) is never satisfied; and provided the initial data f_k have compact support, then for any compact set K, any $\delta > 0$, and $\alpha \neq \alpha_1^{(k)}$,

$$\sup_{x\in K} | n_k(t, x - \alpha t)| = O(e^{-t^{1-\delta}}).$$

This last statement follows from writing $m_k(t, x)$ as a convolution in x-space and estimating this integral. We omit the details.

We now turn to the general case. Relabel the distinct eigenvalues of A: $a_1 < \cdots < a_k$, where $1 \leqslant k < d$. Let $\{P_j\}_{1\leqslant j\leqslant k}$ be the orthogonal projections onto the eigenspaces of A: $P_jP_{j'} = 0$ for $j \neq j'$,

$$\sum_1^k P_j = I, \; AP_j = P_jA = a_jP_j .$$

In the subspace $M_j \equiv P_j(\mathscr{C}^d)$ we consider the matrix $B_j = P_jBP_j$. By Lemma (3.1) B_j is a diagonal matrix. Relabel the distinct diagonal terms $\{b_j^l\}_{1\leqslant j\leqslant k, 1\leqslant l\leqslant L_k}$. Let $\{P_j^l\}_{1\leqslant l\leqslant k}$ be the orthogonal projections onto the eigenspaces of B_j : $P_j^lP_{j'}^{l'} = 0$ for $j \neq j'$ or $l \neq l'$,

$$\sum P_j^l = I, \; B_jP_j^l = P_j^lB_j = b_j^lP_j^l.$$

Finally let $M_j^l = P_j^l(\mathscr{C}^d)$.

Now consider the eigenvalue problem

$$(\lambda A + \lambda^2 B)x = \beta x. \tag{3.3}$$

By the theorem of Kato [6, p. 252], we can find d solutions of Eq. (3.3) in the form

$$\beta(\lambda) = \sum_{n\geqslant 0} \beta_n \lambda^{n+1}, \tag{3.4}$$

$$x(\lambda) = \sum_{n\geqslant 0} x_n \lambda^n, \tag{3.4}$$

convergent in an interval $|\lambda| < \lambda_0$. We wish to prove for each β and x that

$$\beta(\lambda) = a_j\lambda + b_j{}^l\lambda^2 + O(\lambda^3), \quad \lambda \to 0, \tag{3.6}$$

for some j and l, and that $x_0 \in M_j{}^l$. Expansion (3.6) is the analogue of Eq. (2.8). To do this, substitute Eqs. (3.4) and (3.5) into Eq. (3.3) and identify coefficients:

$$Ax_0 = \beta_0 x_0, \tag{3.7}$$

$$Ax_1 = (\beta_1 - B)\, x_0 + \beta_0 x_1 . \tag{3.8}$$

Let $(x(\lambda, \beta(\lambda))$ be a solution with $\beta(0) = a_j$. From Eq. (3.7) it follows that $x_0 \in M_j$. Now Eq. (3.8) implies that β_1 is an eigenvector of $B_j = P_j B P_j$. But by Lemma (3.1), B_j is a diagonal matrix with elements diag $(b_j{}^l)$. Therefore $\beta_1 = b_j{}^l$ for some $1 \leqslant l \leqslant L_j$. Now a second application of Eq. (3.8) shows that $x_0 \in M_j{}^l$. The set of all such x_0 can be specified by $\{(x_0)_{j,l,m}\}$, where j and l indicate that $(x_0)_{j,l,m} \in M_j{}^l$ and $1 \leqslant m \leqslant \dim M_j{}^l$. Thus

$$P_j{}^l = \sum_m \langle \cdot, (x_0)_{j,l,m} \rangle\, (x_0)_{j,l,m} .$$

Consider the vector solution $n(t, x)$ of Eq. (1.2). For $\theta > 0$ sufficiently small, this has the form (by Eq. (2.10) and Lemma 2.3)

$$n(t, x) = \sum_{p=1}^{d} \int_{-\theta}^{\theta} e^{i\gamma x} \exp(t\beta_p(i\gamma))\, \langle \hat{f}(\gamma), x^p(\gamma) \rangle\, x^p(\gamma)\, d\gamma + O(e^{-Mt}), \quad M > 0.$$

Hence

$$\begin{aligned} n(t, x) &= \sum_{j,l,m} \int_{-\theta}^{\theta} e^{i\gamma x}\, e^{t(ia_j\gamma - b_j{}^l\gamma^2 + O(\gamma^3))}\, \langle \hat{f}(\gamma), (x_0)_{j,l,m} \rangle\, (x_0)_{j,l,m}\, d\gamma + O(1/t) \\ &= \sum_{j,l,m} \int_{-\theta}^{\theta} \exp(i\gamma x + ita_j\gamma - tb_j{}^l\gamma^2)\, \langle \hat{f}(\gamma), (x_0)_{j,l,m} \rangle\, (x_0)_{j,l,m}\, d\gamma + O(1/t) \\ &= \sum_{j,l} \int_{-\theta}^{\theta} \exp(i\gamma x + ita_j\gamma - tb_j{}^l\gamma^2)\, P_j{}^l \hat{f}(\gamma)\, d\gamma + O(1/t). \end{aligned}$$

Therefore for any $a_j \in \{\alpha_1^{(1)},..., \alpha_1^{(d)}\}$,

$$n(t, x - a_j t) = \sum_{l=1}^{L_j} \int_{-\infty}^{\infty} \exp(i\gamma x - t b_j^l \gamma^2)(P_j^l \hat{f})(\gamma)\, d\gamma + O(1/t). \tag{3.9}$$

The first member on the right-hand side of Eq. (3.9) is a sum

$$\sum_l m_j^l(t, x),$$

where m_j^l is the vector solution of the heat equation

$$(m_j^l)^{\cdot} = b_j^l (m_j)'', \qquad m_j^l(0^+, x) = P_j^l f(x).$$

We now state the analogue of Eqs. (1.4) and (1.5).

THEOREM 3. *For each distinct eigenvalue a_j of A,*

$$\begin{aligned} &\sup_{-\infty<x<\infty} | P_j^l n(t, x - a_j t) - m_j^l(t, x)| = O(1/t), \quad \text{as} \quad t \to \infty; \\ &\sup_{-\infty<x<\infty} | P_{j'}^l n(t, x - a_j t)| = O(1/t), \quad \text{as} \quad t \to \infty, \quad \text{if} \quad j' \neq j. \end{aligned} \tag{3.10}$$

In the special case that each subspace M_j^l has dimension 1, the result (3.10) has exactly the same form as our main result (1.4)–(1.5) (when condition (C) holds). In the case of Eq. (1.6), the methods of the previous section yield the identical conclusion: for any $n = 1, 2,...$, $n(t, x - \alpha t) = O(1/t^n)$, as $t \to \infty$, if $\alpha \notin \{a_1, ..., a_k\}$.

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