Aspects of the Krook Model of the Boltzmann Equation

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1. Introduction and main results. Let $\sigma(dv)$ be a Borel probability measure on the real line and Π an orthogonal projection in $L^2(\sigma)$ of finite rank d. Let $\mathfrak{N}$ denote the range of Π. In this paper, we consider the integro-differential equation

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} = \frac{1}{\epsilon}(\Pi - I)p, \qquad \lim_{t \downarrow 0} p = f(x, v), \tag{1.1}$$

where

$$p = p^{(\epsilon)}(t, x, v), \quad t > 0, x, v \text{ real}, \quad \epsilon > 0.$$

The right-hand side of (1.1) operates on p as a function of v only; *i.e.*, $(\Pi p)(t, x, v) = (\Pi p(t, x, \cdot))(v)$. The one-dimensional Krook model [9; p. 93], the simplest continuous velocity model of the linearized Boltzmann equation, is obtained by setting

$$(\Pi p)(t, x, v) = \int (1 + vw)p(t, x, w)\sigma(dw), \tag{1.2}$$

$$\sigma(dw) = (2\pi)^{-1/2}e^{-w^2/2}\,dw.$$

In this paper we formulate a limit theorem and a probabilistic representation for solutions of (1.1). We also discuss an eigenvalue problem connected with this equation.

After completing the present work, we began a full-scale study of limit theorems and asymptotics for the linearized Boltzmann equation [2, 3]. This study includes Theorem 1.1 below. Our methods involve an explicit representation for the solution and are much more complicated than the Taylor expansion argument used in the present paper. We therefore feel that this separate proof is worthwhile.

Concerning the organization of this paper, in Section 2 we prove the existence

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and uniqueness of a semigroup solution of (1.1) for a wide class of initial data. In Section 3 we prove the limit theorem which generalizes a result of Ellis–Pinsky [1]. For $f, g \in L^2(\sigma)$, $\langle f, g\rangle \equiv \int_{\mathbf{R}} \bar{f} g \, d\sigma$. For each $t > 0$, the functions in (1.3) converge (to the stated limit) in $L^2(\sigma)$ with respect to v and uniformly in x.

Theorem 1.1. *For initial data f suitably smooth and bounded, we have*

$$\lim_{\epsilon \downarrow 0} p^{(\epsilon)}(t, x, v) = \sum_{j=1}^{d} \langle e_0^{(j)}, f(x + \alpha_1^{(j)} t, \cdot)\rangle \, e_0^{(j)}(v), \tag{1.3}$$

where $\{e_0^{(j)}; j = 1, \cdots, d\}$ is an orthonormal basis of $\mathfrak{N}$ and $\{\alpha_1^{(j)}; j = 1, \cdots, d\}$ are real numbers given by the formulae

$$\alpha_1^{(j)} = -\langle e_0^{(j)}, v e_0^{(j)}\rangle. \tag{1.4}$$

By the methods of [2], we are also able to obtain the following result, which we mention in passing. The Taylor expansion method used to prove Theorem 1 yields a formal proof of (1.5) which we cannot make completely rigorous.

Theorem 1.2. *Assume that the $\{\alpha_1^{(j)}\}$ are distinct. Then for f suitably smooth and bounded, $j = 1, \cdots, d$, we have*

$$\lim_{\epsilon \downarrow 0} p^{(\epsilon)}\left(\frac{t}{\epsilon}, x - \alpha_1^{(j)} \frac{t}{\epsilon}, v\right) = e_0^{(j)}(v)(4\pi \alpha_2^{(j)} t)^{-1/2} \int e^{-(x-y)^2/4\alpha_2^{(j)} t} \langle e_0^{(j)}, f(y, \cdot)\rangle \, dy, \tag{1.5}$$

where $\{\alpha_2^{(j)}; j = 1, \cdots, d\}$ are positive real numbers given by the formulae

$$\alpha_2^{(j)} = \langle (v + \alpha_1^{(j)}) e_0^{(j)}, (v + \alpha_1^{(j)}) e_0^{(j)}\rangle. \tag{1.6}$$

We comment below on the hypothesis that the $\{\alpha_1^{(j)}\}$ are distinct.

In Section 4, we discuss the probabilistic aspects of our limit results. Indeed, if $\mathfrak{N} = \text{span}\,\{1\}$, then replacing v by $-v$ in (1.1) yields the backward equation of a certain joint Markov process. The limit results are respectively the weak law of large numbers and the central limit theorem for one component of this process. We shall obtain a probabilistic representation for the solution of (1.1) which illustrates a connection between our work and the theory of random evolutions [5]. Our representation has recently been generalized by Pinsky [6].

Concerning the hypothesis of Theorem 1.2, we show in Section 5 of this paper that the $\{\alpha_1^{(j)}\}$ are distinct whenever $\mathfrak{N}$ is spanned by d polynomials the orders of which are consecutive integers. This includes, for example, the Krook case (1.2).

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2. Existence of solutions of (1.1). Let H_γ denote the set of complex vector-valued functions $f_\gamma(v)$, γ real, which have the property $f_\gamma(\cdot) \varepsilon L^2(\sigma)$ for each γ. We denote by $||\cdot||$ the norm of $L^2(\sigma)$. We first show that the operator $Q_\gamma^{(\epsilon)} = \Pi - I - i\epsilon\gamma v$ generates a strongly continuous contraction semigroup on H_γ. Consider the integral equation

$$(2.1) \qquad p_\gamma(t, v) = e^{-t(1+i\epsilon\gamma v)} f_\gamma(v) + \int_0^t e^{-(t-s)(1+i\epsilon\gamma v)} \Pi p_\gamma(x, v)\, ds,$$

which is the integrated form of the equation ($\dot{}$ denotes $\partial/\partial t$) $p_\gamma^{\cdot} = Q_\gamma^{(\epsilon)} p_\gamma$, $\lim_{t\downarrow 0} p_\gamma = f_\gamma$. Iteration yields existence of an H_γ-solution for each $t > 0$, $\epsilon > 0$, γ real. From (2.1), we find

$$||p_\gamma(t, \cdot)|| \leqq \exp(-t)||f_\gamma|| + \int_0^t e^{-(t-s)} ||\Pi p_\gamma(s, \cdot)||\, ds,$$

which implies

$$(2.2) \qquad ||p_\gamma(t, \cdot)|| \leqq ||f_\gamma||.$$

The uniqueness and strong continuity follow easily. We note that $D(Q_\gamma) \supseteq \{f_\gamma \varepsilon H_\gamma : |v|\, f_\gamma \varepsilon H_\gamma\}$.

Let H denote the set of complex-valued functions $f(x, v)$ of the form $f(x, v) = \int e^{i\gamma x} f_\gamma(v)\, d\gamma$, where for each γ $f_\gamma(\cdot) \varepsilon H_\gamma$ and $\int ||f_\gamma||\, d\gamma < \infty$. We define a norm on H by $|||f||| = \int ||f_\gamma||\, d\gamma$. Let $H_{k,\ell}$ denote functions $f \varepsilon H$ such that $\int |\gamma|^\ell\, ||v^k f_\gamma||\, d\gamma < \infty$. For $f \varepsilon H_{1,1}$, we obtain a semigroup solution of (1.1) by defining $p^{(\epsilon)}(t, x, v) = \int e^{i\gamma x} p_\gamma^{(\epsilon)}(t, v)\, d\gamma$, where $p_\gamma^{(\epsilon)}(t, v) = p_\gamma(t/\epsilon, v)$.

3. Proof of Theorem 1.1. We first discuss the selection of the vectors $\{e_0^{(i)}\}$. We make the following assumption:

(A) For each $\phi \varepsilon \mathfrak{N}$, ϕ and $v\phi$ are in $\mathfrak{D}(v) = \{g \varepsilon L^2(\sigma) : vg \varepsilon L^2(\sigma)\}$.

By (A), the operator $\Pi v \Pi$ is a bounded operator of rank r, where $0 \leqq r \leqq d$. Hence, we can find r orthonormal vectors $\{e_0^{(j)}; j = 1, \cdots, d\}$ in $\mathfrak{N}$ and r non-zero numbers $\{\alpha_1^{(j)}; j = 1, \cdots, r\}$ such that $\Pi v \Pi = -\sum_j \alpha_1^{(j)} e_0^{(j)} \otimes e_0^{(j)}$. If $r < d$, then let $\{e_0^{(j)}; j = r + 1, \cdots, d\}$ be any $d - r$ orthonormal vectors in $\mathfrak{N}$ such that $\{e_0^{(j)}; j = 1, \cdots, d\}$ is an orthonormal basis of $\mathfrak{N}$, and set $\alpha_1^{(j)}$ equal to 0 for $j = r + 1, \cdots, d$. Thus,

$$(3.1) \qquad \Pi = \sum_i e_0^{(i)} \otimes e_0^{(i)},$$

$$(3.2) \qquad \langle e_0^{(i)}, v e_0^{(j)} \rangle = -\delta_{ij} \alpha_1^{(i)},\ 1, j = 1, \cdots, d.$$

By (A), each $e_0^{(j)}$, $v e_0^{(j)} \varepsilon \mathfrak{D}(v)$. In general, the $\{e_0^{(j)}\}$ are not unique. Indeed, the above specifies the $\{e_0^{(j)}\}$ uniquely provided the $\{\alpha_1^{(j)}; j = 1, \cdots, d\}$ are distinct. This question is treated in Section 5.

We now sketch a proof of Theorem 1.1, justifying the steps later. Given $f \varepsilon H$, let $p_\gamma^{(\epsilon)}$ satisfy

$$(3.3) \qquad p_\gamma^{(\epsilon)}(t, v) = e^{-t(1+i\epsilon\gamma v)/\epsilon} f_\gamma(v) + \frac{1}{\epsilon}\int_0^t e^{-(t-s)(1+i\epsilon\gamma v)/\epsilon}\, \Pi p_\gamma^{(\epsilon)}(s, v)\, ds,$$

and define $p_\gamma \equiv \sum_k \langle e_0^{(k)}, f_\gamma\rangle e_0^{(k)} \exp(i\alpha_1^{(k)}\gamma t)$. We first show that for almost every γ

$$(3.4) \qquad \Pi p_\gamma^{(\epsilon)} \to p_\gamma \quad \text{as} \quad \epsilon \downarrow 0.$$

Later we prove that for almost every γ, $(1 - \Pi)p_\gamma^{(\epsilon)} \to 0$ as $\epsilon \downarrow 0$.

Setting $e = e_0^{(k)} \varepsilon \mathfrak{N}$, we have

$$(3.5) \qquad \langle e, p_\gamma^{(\epsilon)}(t, \cdot)\rangle = e^{-t/\epsilon}\langle e, e^{-it\gamma v} f_\gamma\rangle + \frac{1}{\epsilon}\int_0^t e^{-(t-s)/\epsilon}\, \Sigma_i \langle e, e^{-i\gamma(t-s)v} e_0^{(i)}\rangle\langle e_0^{(i)}, p_\gamma^{(\epsilon)}(s, \cdot)\rangle\, ds.$$

By Taylor's theorem, we can write

$$(3.6) \qquad \langle e, p_\gamma^{(\epsilon)}(s, \cdot)\rangle = \langle e, p_\gamma^{(\epsilon)}(t, \cdot)\rangle + (s - t)\langle e, p_\gamma^{(\epsilon)}(t, \cdot)\rangle^{\cdot} + \frac{(s - t)^2}{2} r_\gamma^{(\epsilon)}(t'),$$

where

$$r_\gamma^{(\epsilon)}(t') = \langle e, p_\gamma^{(\epsilon)}(t', \cdot)\rangle^{\cdot\cdot}, \; s \leqq t' \leqq t.$$

We now substitute (3.6) into (3.5) and apply $\exp(-i\gamma(t - s)v) = 1 - i\gamma(t - s)v + O(\gamma^2(t - s)^2 v^2)$ to the term on the right-hand side of (3.5) containing $\langle e, p_\gamma^{(\epsilon)}\rangle$, $\exp(-i\gamma(t - s)v) = 1 + O(\gamma v(t - s))$ to the term containing $\langle e, p_\gamma^{(\epsilon)}\rangle^{\cdot}$. After some manipulation (using (3.2)) we find

$$(3.7) \qquad \langle e_0^{(k)}, p_\gamma^{(\epsilon)}(t, \cdot)\rangle^{\cdot} = i\gamma\alpha_1^{(k)}\langle e_0^{(k)}, p_\gamma^{(\epsilon)}(t, \cdot)\rangle + o(1), \; \epsilon \downarrow 0.$$

The $o(1)$ term involves $|\gamma|$, $|\gamma|^2$, and $||f_\gamma||$. We claim that there exist smooth functions $p_{\gamma,k}(t)$ such that

$$(3.8) \qquad \begin{aligned} \lim_{\epsilon \downarrow 0} \langle e_0^{(k)}, p_\gamma^{(\epsilon)}(t, \cdot)\rangle &= p_{\gamma,k}(t), \\ \lim_{\epsilon \downarrow 0} \langle e_0^{(k)}, p_\gamma^{(\epsilon)}(t, \cdot)\rangle^{\cdot} &= p_{\gamma,k}^{\cdot}(t). \end{aligned}$$

Since $f \varepsilon H$ implies $||f_\gamma|| < \infty$ for almost every γ, we see from (3.7) and (3.8) that $p_{\gamma,k}(t) = \langle e_0^{(k)}, f_\gamma\rangle \exp(i\gamma\alpha_1^{(k)} t)$ *a.e.* and (3.4) follows. To show $(1 - \Pi)p_\gamma^{(\epsilon)} \to 0$ as $\epsilon \downarrow 0$, we prove that for any $\delta > 0$

$$(3.9) \qquad \sup_{t > \delta} ||(1 - \Pi)p_\gamma^{(\epsilon)}(t, \cdot)|| \leqq K\epsilon|\gamma|\; ||f_\gamma||, \; K \text{ constant}.$$

Indeed, $||(1 - \Pi)p_\gamma^{(\epsilon)}||^2 = \sum_{k \geq 0} |\langle \psi_k, p_\gamma^{(\epsilon)}\rangle|^2$, where $\{\psi_k\}$ is any orthonormal basis of $\mathfrak{N}^{\perp}$. From (3.3), using $\langle \psi_k , e_0^{(i)} \rangle = 0$, $\exp(-i(t-s)\gamma v) = 1 + O(\gamma v(t-s))$, and (2.2), we find

$$\|(1-\Pi)p_\gamma^{(\epsilon)}\|^2 \leqq 2e^{-2t/\epsilon} \|f_\gamma\| + K\epsilon^2|\gamma|^2 \sup_{0\leqq s\leqq t} \|p_\gamma^{(\epsilon)}(s, \cdot)\|^2$$
$$\leqq K(e^{-2t/\epsilon} + \epsilon^2|\gamma|^2) \|f_\gamma\|^2,$$

which implies (3.9).

If we define $p = \int \exp(i\gamma x)p_\gamma \, d\gamma$ and take $f \in H_{1,1}$, then Theorem 1.1 follows from (3.4) and (3.9).

To justify the first part of the proof, we claim that the $r_\gamma^{(\epsilon)}$ term in (3.6) gives rise to a $O(\epsilon)$ error when substituted into (3.5). Indeed, using the fact that

$$p_\gamma^{(\epsilon)\cdot} = \frac{1}{\epsilon}(\Pi - 1)p_\gamma^{(\epsilon)} - i\gamma v p_\gamma^{(\epsilon)},$$

we see that

$$r_\gamma^{(\epsilon)}(t') = -\gamma^2\langle v^2 e, p_\gamma^{(\epsilon)}(t', \cdot)\rangle - i\gamma\langle ve, \epsilon^{-1}(1-\Pi)p_\gamma^{(\epsilon)}(t', \cdot)\rangle.$$

By (2.2), (A), and (3.9), this is bounded, for almost all γ, uniformly for $t' \geqq \delta > 0$, and thus the claim follows. This term ultimately contributes to the $o(1)$ term in (3.7). The other contributions to this $o(1)$ term arise from the equiboundedness, on compact subsets of $\{t: t > 0\}$, of the classes of functions $\{\langle e_0^{(k)}, p_\gamma^{(\epsilon)}(t, \cdot)\rangle; \epsilon > 0\}$ and $\{\langle e_0^{(k)}, p_\gamma^{(\epsilon)}(t, \cdot)\rangle^{\cdot}; \epsilon > 0\}$. The equiboundedness and equicontinuity of these classes imply the existence and properties (3.8) of the function $p_{\gamma,k}$. The proofs of the equiboundedness and equicontinuity are elementary and will be omitted. This completes the proof of Theorem 1.1.

Remark. A more general version of (1.1) can be handled by the same methods. We replace π by $a^{-1}G$, where G is a symmetric compact operator on $L^2(\sigma)$ of finite rank, norm a. Denoting by $\{\varphi_i\}$ the eigenfunctions of G, we assume that φ_i, $v\varphi_i \in \mathfrak{D}(v)$ if $\varphi_i \in \mathfrak{N}(a^{-1}G - I)$, $\varphi_i \in \mathfrak{D}(v)$ if $\varphi_i \notin \mathfrak{N}(a^{-1}G - I)$. Then everything in Section 2 and 3 goes over to this case.

4. Probabilistic representation. We derive a probabilistic representation for solutions of the equation

$$\text{(4.1)} \qquad p^{\cdot} = vp' + (a^{-1}G - I)p, \qquad \lim_{t\downarrow 0} p = f \in H_{1,1},$$

where G is a symmetric compact operator of norm a on $L^2(\sigma)$ satisfying condition (B) below. Since $\|a^{-1}G\| = 1$, we may assume without loss of generality that $a = 1$. We define a regular step process $\{v(t, \omega); t \geqq 0, \omega \in \Omega\}$ (the velocity process) and let $x(t, \omega) = \int_0^t v(s, \omega)\, ds$. We then prove that the semigroup solution p of (4.1) obtained as in Section 2 can be represented as

$$\text{(4.2)} \qquad p(t, x, v) = E_v\{M(t, \omega)f(x + x(t, \omega), v(t, \omega))\}.$$

E_v denotes the integral with respect to the measure $P_v(d\omega)$ on paths $v(t, \omega)$

satisfying $v(0, \omega) = v$ *a.e.*, and $M(t, \omega)$ denotes a certain random function to be defined. From now on, we will usually suppress the ω-dependence. In a remark at the end of this section, we restate the limit result (1.3) for the expectation in (4.2).

Given G as above, we let $\{\mu_j ; j \geqq 1\}$ and $\{\varphi_j ; j \geqq 1\}$ be its eigenvalues and corresponding eigenfunctions. We make the following assumptions:

(B) (i) G is nuclear ($\sum |\mu_j| < \infty$) and (ii) $\sup_j \int |\varphi_j|^4 \sigma(dv) < \infty$. For any $f \in L^2(\sigma)$, we have [7; p. 234] $Gf(v) = \int K(v, z)f(z)\sigma(dz)$, where $K(v, z) = \sum \mu_j \varphi_j(v)\bar{\varphi}_j(z)$.

We now define $v(t)$ and $M(t)$. Let $\{v_j ; j \geqq 1\}$ be a collection of independent random variables, each distributed by σ. Let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ be a collection of random variables whose increments $\tau_j - \tau_{j-1}$, $j \geqq 1$, are independent, identically distributed exponential random variables with mean one. We assume that the $\{v_j\}$ and $\{\tau_j\}$ are defined on the same probability space $(\Omega, \mathfrak{F}, P)$. Fixing v, we define

$$v(t, \omega) = \begin{cases} v, & \text{if} \quad \tau_0(\omega) \leqq t < \tau_1(\omega), \\ v_j(\omega), & \text{if} \quad \tau_j(\omega) \leqq t < \tau_{j+1}(\omega), \qquad j \geqq 1. \end{cases}$$

Thus, the $\{\tau_j\}$ are the jump times of the velocity process. Let $\{N(t); t \geqq 0\}$ denote the integer-valued Poisson process constructed from the $\{\tau_j\}$ and define

$$M(t, \omega) = \begin{cases} 1, & \text{if} \quad N(t, \omega) = 0, \\ \prod_{j=1}^{N(t,\omega)} K(v_{j-1}(\omega), v_j(\omega)), & \text{if} \quad N(t, \omega) > 1. \end{cases}$$

Proposition 4.1. *Assume* (B) (i) *and let* $v(0)$ *be distributed by* σ. *Define* $\mathfrak{F}_t$ ' $t \geqq 0$, *to be the smallest* σ-*subfield of* $\mathfrak{F}$ *which measures* $\{v(s); 0 \leqq s \leqq t\}$, $\{N(s); 0 \leqq s \leqq t\}$. *For each* $t \geqq 0$, $M(t)$ *is* $\mathfrak{F}_t$-*measurable and* $E\,|M(t)| < \infty$.

Proof. The first statement is clear. For the second, we define $S_m(t) = \{\omega : N(t, \omega) = m\}$, $m \geqq 0$, and using the independence of the $\{v_j\}$ and $N(t)$, we calculate

$$E\{|M(t)|; S_m(t)\} \leqq \rho_m(t)(\sum |\mu_j|)^m,$$

where $\rho_m(t) = P(S_m(t)) = \exp(-t)t^m/m!$ Thus, $E\,|M(t)| < \infty$. ■

In order to state the main result of this section, we define $q_\gamma = E_v\{M(t) \exp(i\gamma x(t))f_\gamma(v(t))\}$, $q = \int e^{i\gamma x} q_\gamma \, d\gamma$.

Theorem 4.2. *Assume* (B) (i), (ii), *and take* $f \in H_{1,1}$. *Then* $q(t, \cdot, \cdot) \in H_{1,1}$ *for each* $t \geqq 0$ *and is the unique semigroup solution of* (4.1).

Proof. We first prove that $q(t, \cdot, \cdot) \in H_{1,1}$. We have for each γ

$$\begin{aligned}
||q_\gamma||^2 &\leqq 2 \int \sigma(dv)[|E_v\{M(t)e^{i\gamma x(t)}f_\gamma(v(t));\ \tau_1 > t\}|^2 \\
&\quad + |E_v\{M(t)e^{i\gamma x(t)}f_\gamma(v(t));\ \tau_1 \leqq t\}|^2] \\
&\leqq 2P(S_0(t))\ ||f_\gamma||^2 + 2 \sum_{m\geqq 1} E\{|M(t)|^2;\ S_m(t)\}E\{|f_\gamma(v(t))|^2;\ \tau_1 < t\} \\
&\leqq 2\ ||f_\gamma||^2 \sum_{m\geqq 0} E\{|M(t)|^2;\ S_m(t)\}.
\end{aligned}$$

One can prove that $E\{|M(t)|^2;\ S_m(t)\} \leqq \rho_m(t)(L \sum |\mu_j|)^m$, where for all j, $\int |\varphi_j|^4\ \rho(dv) \leqq L$. Thus $q(t, \cdot\ , \cdot) \in H_{1,1}$. We now show that q_γ solves the integral equation

$$p_\gamma(t, v) = e^{-t(1-i\gamma v)}f_\gamma + \int_0^t e^{-(t-s)(1-i\gamma v)}Gp_\gamma(s, v)\ ds, \tag{4.3}$$

which bears the same relation to (4.1) as (2.1) bears to (1.1). We prove this by using the facts that $v(t)$ is a strong Markov process and that $M(t, \omega) = M(s, \omega)M(t - s, \theta_s\omega)$, $x(t, \omega) = x(s, \omega) + x(t - s, \theta_s\omega)$ *a.e.*, $0 \leqq s \leqq t$, where $\theta_s\omega$ denotes the shifted path. Indeed,

$$\begin{aligned}
q_\gamma(t, v) &= E_v\{M(t)e^{i\gamma x(t)}f_\gamma(v(t));\ \tau_1 > t\} \\
&\quad + E_v\{M(t)e^{i\gamma x(t)}f_\gamma(v(t));\ \tau_1 \leqq t\} = e^{-t}e^{i\gamma vt}f_\gamma(v) \\
&\quad + \int_{\{\tau_1\leqq t\}} M(\tau_1)e^{i\gamma x(\tau_1)}E_{v(\tau_1)}\{M(t - \tau_1)e^{i\gamma x(t-\tau_1)}f_\gamma(v(t - \tau_1))P_v(d\omega) \\
&= e^{-t}e^{i\gamma vt}f_\gamma(v) + \int_0^t e^{-s}e^{i\gamma vs}\left[\int K(v, v_1)q_\gamma(t - s, v_1)\rho(dv_1)\right] ds.
\end{aligned}$$

Changing s to $(t - s)$ shows that q_γ satisfies (5.3). As in Section 2, one can show that (4.3) has a unique solution $p_\gamma \in L^2(\sigma)$, and thus $q_\gamma(t, \cdot) = p_\gamma(t, \cdot)$, $t \geqq 0$. Defining

$$p = \int e^{i\gamma x}p_\gamma\ d\gamma, \qquad f = \int e^{i\gamma x}f_\gamma\ d\gamma,$$

we have that if $f \in H_{1,1}$, then p is the semigroup solution of (4.1). By Fubini's theorem, (4.2) holds for all t, x, and almost everywhere (with respect to σ) in v. ■

Remark. Let $p^{(\epsilon)}$ solve $p^{(\epsilon)\cdot} = vp^{(\epsilon)\prime} + \epsilon^{-1}(G - I)p^{(\epsilon)}$, $\lim_{t\downarrow 0} p^{(\epsilon)} = f \in H_{1,1}$. We assume that G satisfies (B) (ii) as well as the assumptions in the remark at the end of Section 3 (thus G is of finite rank, and so (B) (i) is automatic). One can show the representation

$$p^{(\epsilon)} = E_v\{M(t/\epsilon)f(x + \epsilon x(t/\epsilon), v(t/\epsilon))\}. \tag{4.4}$$

By the remark at the end of Section 3, we thus have a limit theorem as $\epsilon \downarrow 0$

for the expectation in (4.4). Related limit theorems have been obtained [5; §3.4] when the state space of $v(t)$ is finite.

6. Special case of the eigenvalue problem for $\Pi v \Pi$. We consider an important special case of (1.1). Assume $\int |v|^k \, \sigma(dv) < \infty$, $k = 1, \cdots, M$, for some integer M. We denote by $\{S_k ; k = 0, \cdots, M\}$ the polynomials formed by an orthonormalization in $L^2(\sigma)$ of the powers $\{v^k; k = 0, \cdots, M\}$. We make the assumption that $\mathfrak{N} = \text{span}\,\{S_{n_j} ; 1 \leqq j \leqq d\}, 0 \leqq n_1 < n_2 < \cdots < n_d < M$.

Theorem 5.1. *If $\{n_j ; 1 \leqq j \leqq d\}$ are consecutive integers, then the $\{\alpha_1^{(j)}; 1 \leqq j \leqq d\}$ are distinct real numbers. Assume that the measure σ is symmetric. If d is even, then the $\{\alpha_1^{(j)}\}$ come in $d/2$ pairs, the eigenvalues in a pair being negatives of each other. If d is odd, then one $\alpha_1^{(j)}$ is zero and the other $d - 1$ pair up as in the even case.*

Remark. The hypothesis of the theorem is satisfied in the Krook case (1.2), where $\mathfrak{N} = \text{span}\,(S_0, S_1)$. Numerical calculations for the Krook and other cases are given at the end of this section.

Proof. We note [8; p. 41] that each S_k is a polynomial of order k with leading term $\beta_k v^k$, $\beta_k > 0$, and that the following three-step recursion formula is satisfied:

$$vS_k(v) = r_k S_{k+1} + q_k S_k + t_k S_{k-1}, \qquad 0 \leqq k \leqq N - 1, \tag{5.1}$$

where

$$r_k = \frac{\beta_k}{\beta_{k+1}}, \; q_k = \langle vS_k, S_k\rangle, \; t_k = \frac{\beta_{k-1}}{\beta_k}, \; \beta_{-1} = 0. \tag{5.2}$$

Let L denote the matrix of $\Pi v \Pi$ expressed in the basis $\{S_{n_j}\}$. By (5.1), (5.2), we find

$$(L)_{ij} = \begin{cases} t_{n_i+1} & \text{if } n_j = n_i + 1, \\ q_{n_i} & \text{if } n_j = n_i, \\ t_{n_i} & \text{if } n_j = n_i - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Thus, L is symmetric, $(L)_{ij} = 0$ whenever $|i - j| > 1$, and its eigenvalues are the $\{\alpha_1^{(j)}\}$. L is irreducible if and only if the $\{n_j\}$ are consecutive integers. Thus, the first part of the theorem follows from a theorem on Jacobi matrices [4; p. 80]. Now assume that σ is symmetric. If d is even, then $D(\lambda)$, the characteristic polynomial of L, is a polynomial in λ^2 with positive constant term. If d is odd, the $D(\lambda) = \lambda \tilde{D}(\lambda)$, where $\tilde{D}(\lambda)$ is a polynomial in λ^2 with positive constant term. ■

For σ symmetric, we now calculate the $\{\alpha_1^{(j)}\}$ and $\{e_0^{(j)}\}$ in special cases.

We write $m_k = \int v^k \, d\sigma$.

(a) $\mathfrak{N} = \text{span}\ \{1\}$. Then $\alpha_1^{(1)} = 1$, $e_0^{(1)} = 1$.

(b) $\mathfrak{N} = \text{span}\ \{1, v\}$. Since $S_1(v) = v/(m_2)^{1/2}$,

$$L = \begin{bmatrix} 0 & (m_2)^{1/2} \\ (m_2)^{1/2} & 0 \end{bmatrix},$$

so that $\alpha_1^{(j)} = (-1)^j (m_2)^{1/2}$, $j = 1, 2$. One can show

$$e_0^{(j)} = (v - (-1)^j (m_2)^{1/2})/(2m_2)^{1/2}.$$

(c) $\mathfrak{N} = \text{span}\ \{1, v, v^2\}$. Since $S_2(v) = (v^2 - m_2)/(m_4 - m_2^2)$

$$L = \begin{bmatrix} 0 & q_{12} & 0 \\ q_{12} & 0 & q_{23} \\ 0 & q_{23} & 0 \end{bmatrix},$$

where $q_{12} = (m_2)^{1/2}$, $q_{23} = (m_4 - m_2^2)/(m_2)^{1/2}$. The eigenvalues of L are $\alpha_1^{(j)} = (-1)^j((q_{12}^2 + q_{23}^2))^{1/2} = (-1)^j(m_4)^{1/2}/(m_2)^{1/2}$, $j = 1, 2$, and $\alpha_1^{(3)} = 0$. One can show that $e_0^{(j)} = (v^2 - (-1)^j v(m_4)^{1/2}/(m_2)^{1/2})/(2m_4)$, $j = 1, 2$, and $e_0^{(3)} = (m_2 v^2 - m_4)/(m_4(m_4 - m_2^2))^{1/2}$.

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