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Richard S. Ellis

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INEQUALITIES FOR MULTIVARIATE COMPOUND POISSON DISTRIBUTIONS[1]

BY RICHARD S. ELLIS

University of Massachusetts

This paper proves a converse to a theorem of L. D. Brown and Y. Rinott concerning positive dependence ordering for multivariate compound Poisson distributions.

We say that a one-parameter family of probability distributions $\{Q_t,\ t \geq 0\}$ forms a *compound Poisson family* if for each $t \geq 0$,

$$Q_t = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} F_k, \tag{1}$$

where $\lambda > 0$ is fixed and F_k denotes the k-fold convolution of some distribution F. Q_t represents the distribution of the random sums $U_1 \cdots + U_{N(t)}$, where $\{U_i\}$ are independent with common distribution F and $N(t)$ is a Poisson (λt) variable independent of $\{U_i\}$. In this paper we prove a converse to a theorem of Brown and Rinott (1988) involving compound Poisson families.

Let n be a positive integer, define $\mathscr{A} = \{A\colon A \subseteq \{1, \ldots, n\}\}$, and to each nonempty set A in $\mathscr{A}$ associate a number $t(A) \geq 0$. Let Q_t be a compound Poisson family and let $\{Z_A,\ A \in \mathscr{A}\}$ be independent random variables with Z_A distributed according to $Q_{t(A)}$. Define $\mathbf{X} = (X_1, \ldots, X_n)$ by

$$X_i = \sum_{A\colon i \in A} Z_A, \qquad i = 1, \ldots, n. \tag{2}$$

$\mathbf{X}$ is said to have a *multivariate compound Poisson distribution* with parameter $\mathbf{t} = \{t(A),\ \varnothing \neq A \in \mathscr{A}\}$ based on the family Q_t. Let $\mathbf{Y}$ have a multivariate compound Poisson distribution with parameter $\mathbf{t}^* = \{t^*(A),\ \varnothing \neq A \in \mathscr{A}\}$ based on the same family Q_t. Part (a) of the following theorem proves a converse to Theorem 1.2(i) in Brown and Rinott (1988). Part (b) of the following theorem is proved in Theorem 1.2(ii) in Brown and Rinott (1988) in somewhat greater generality (for multivariate infinitely divisible distributions). We have included part (b) because its proof is similar in spirit to that of part (a).

In a preliminary version of their paper, L. D. Brown and Y. Rinott had a proof of part (a) for the multivariate Poisson case. I am grateful to them for bringing the problem to my attention.

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THEOREM 1. (a) *Assume that* $P\{U_1 > 0\}$ *and that* $E\{\exp(\alpha U_1)\} < \infty$ *for all* $\alpha > 0$. *Then*

$$P\{\mathbf{X} \ge \mathbf{c}\} \le P\{\mathbf{Y} \ge \mathbf{c}\}, \quad \text{for all } \mathbf{c} \in R^n, \tag{3}$$

implies that

$$\sum_{A:\, A \supseteq B} t(A) \le \sum_{A:\, A \supseteq B} t^*(A), \quad \text{for all } B \ne \varnothing,\ B \in \mathscr{A}. \tag{4}$$

(b) *Assume that* $U_1 \ge 0$ *and* $U_1 \not\equiv 0$. *Then*

$$P\{\mathbf{X} \le \mathbf{c}\} \le P\{\mathbf{Y} \le \mathbf{c}\}, \quad \text{for all } \mathbf{c} \in R^n, \tag{5}$$

implies that

$$\sum_{A:\, A \cap B \ne \varnothing} t(A) \ge \sum_{A:\, A \cap B \ne \varnothing} t^*(A), \quad \text{for all } B \ne \varnothing,\ B \in \mathscr{A}.$$

PROOF OF THEOREM 1(a). Let B be any nonempty set in $\mathscr{A}$ and let α_i, $i \in B$, be positive numbers. Then

$$\sum_{i \in B} \alpha_i X_i = \sum_{i \in B} \alpha_i \sum_{A:\, i \in A} Z_A = \sum_{A:\, A \cap B \ne \varnothing} \left(\sum_{i \in A \cap B} \alpha_i \right) Z_A. \tag{6}$$

Note that for each set A satisfying $A \supseteq B$, the corresponding term in the sum contains all of the α_i, $i \in B$; for each set A satisfying $A \cap B \ne \varnothing$, $A \not\supseteq B$, the corresponding term in the sum contains only a proper subset of the α_i, $i \in B$. For each nonempty set A in $\mathscr{A}$,

$$Z_A = U_1 + \cdots + U_{N(t(A))}, \tag{7}$$

and so for $\alpha > 0$,

$$\log E\{\exp(\alpha Z_A)\} = \lambda t(A)[m(\alpha) - 1], \quad \text{where } m(\alpha) = E\{\exp(\alpha U_1)\}. \tag{8}$$

Since $\{Z_A,\ A \in \mathscr{A}\}$ are independent,

$$\log E\left\{\exp\left(\sum_{i \in B} \alpha_i X_i\right)\right\} = \sum_{A:\, A \cap B \ne \varnothing} \lambda t(A) \left[m\left(\sum_{i \in A \cap B} \alpha_i \right) - 1 \right]. \tag{9}$$

If we set each α_i, $i \in B$, equal to $k > 0$, then

$$\begin{aligned} \log E\left\{\exp\left(k \sum_{i \in B} X_i\right)\right\} &= \sum_{A:\, A \supseteq B} \lambda t(A)[m(|B|k) - 1] \\ &\quad + \sum_{\substack{A:\, A \cap B \ne \varnothing \\ A \not\supseteq B}} \lambda t(A)[m(|A \cap B|k) - 1]. \end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned} \log E\left\{\exp\left(k \sum_{i \in B} Y_i\right)\right\} &= \sum_{A:\, A \supseteq B} \lambda t^*(A)[m(|B|k) - 1] \\ &\quad + \sum_{\substack{A:\, A \cap B \ne \varnothing \\ A \not\supseteq B}} \lambda t^*(A)[m(|A \cap B|k) - 1]. \end{aligned} \tag{11}$$

If $|B| = 1$, then the second terms on the right-hand sides of (10) and (11) are absent.

By hypothesis $P\{\mathbf{X} \geq \mathbf{c}\} \leq P\{\mathbf{Y} \geq \mathbf{c}\}$ for all $\mathbf{c} \in R^n$. Thus,

$$P\Big\{ \bigcap_{i \in B} \{X_i \geq c_i\} \Big\} \leq P\Big\{ \bigcap_{i \in B} \{Y_i \geq c_i\} \Big\}, \tag{12}$$

for all real c_i, $i \in B$. This implies that [see Brown and Rinott (1988), Section 1, or integrate by parts]

$$\log E\Big\{ \exp\Big(k \sum_{i \in B} X_i \Big) \Big\} \leq \log E\Big\{ \exp\Big(k \sum_{i \in B} Y_i \Big) \Big\}. \tag{13}$$

Lemma 2 below together with (10) and (11) implies that

$$\log E\Big\{ \exp\Big(k \sum_{i \in B} X_i \Big) \Big\} = m(|B|k)\Big(\lambda \sum_{A:\, A \supseteq B} t(A) + o(1) \Big), \quad \text{as } k \to \infty, \tag{14}$$

$$\log E\Big\{ \exp\Big(k \sum_{i \in B} Y_i \Big) \Big\} = m(|B|k)\Big(\lambda \sum_{A:\, A \supseteq B} t^*(A) + o(1) \Big), \quad \text{as } k \to \infty. \tag{15}$$

Part (a) of Theorem 1 is a consequence of (13), (14) and (15). □

Lemma 2. *Assume that* $P\{U_1 > 0\} > 0$ *and that* $m(\alpha) = E\{\exp(\alpha U_1)\} < \infty$ *for all* $\alpha > 0$. *Then* $m(|B|k) \to \infty$ *as* $k \to \infty$ *and*

$$\lim_{k \to \infty} \frac{m(|A \cap B|k)}{m(|B|k)} = 0, \tag{16}$$

for each set A *satisfying* $A \cap B \neq \varnothing$, $A \not\supseteq B$.

Proof. There exists $\varepsilon > 0$ such that $P\{U_1 \geq \varepsilon\} > 0$. Thus,

$$m(|B|k) \geq E\big\{ \exp(|B|kU_1) \cdot 1_{\{U_1 \geq \varepsilon\}} \big\} \geq e^{|B|k\varepsilon} P\{U_1 \geq \varepsilon\}, \tag{17}$$

which tends to ∞ as $k \to \infty$. If A satisfies $A \cap B \neq \varnothing$, $A \not\supseteq B$, then $|A \cap B| < |B|$ and

$$0 \leq \frac{m(|A \cap B|k)}{m(|B|k)} \leq \frac{e^{|A \cap B|k\varepsilon} + E\big\{ \exp(|A \cap B|kU_1) \cdot 1_{\{U_1 \geq \varepsilon\}} \big\}}{E\big\{ \exp(|B|kU_1) \cdot 1_{\{U_1 \geq \varepsilon\}} \big\}}$$
$$\leq \frac{e^{|A \cap B|k\varepsilon}}{e^{|B|k\varepsilon} P\{U_1 \geq \varepsilon\}} + \frac{1}{e^{(|B| - |A \cap B|)k\varepsilon}},$$

which tends to 0 as $k \to \infty$. □

Proof of Theorem 1(b). Let B be any nonempty set in $\mathscr{A}$ and let α_i, $i \in B$, be positive numbers. By hypothesis, $P\{\mathbf{X} \leq \mathbf{c}\} \leq P\{\mathbf{Y} \leq \mathbf{c}\}$ for all $\mathbf{c} \in R^n$. Thus,

$$P\Big\{ \bigcap_{i \in B} \{X_i \leq c_i\} \Big\} \leq P\Big\{ \bigcap_{i \in B} \{Y_i \leq c_i\} \Big\}, \tag{18}$$

for all real c_i, $i \in B$. This implies that [see Brown and Rinott (1988), Section 1, or integrate by parts]

$$\log E\left\{\exp\left(-k \sum_{i \in B} X_i\right)\right\} \le \log E\left\{\exp\left(-k \sum_{i \in B} Y_i\right)\right\}, \quad \text{for } k > 0. \tag{19}$$

As in the proof of part (a),

$$\log E\left\{\exp\left(-k \sum_{i \in B} X_i\right)\right\} = \sum_{A:\, A \cap B \neq \varnothing} \lambda t(A)[m(-|A \cap B|k) - 1], \tag{20}$$

$$\log E\left\{\exp\left(-k \sum_{i \in B} Y_i\right)\right\} = \sum_{A:\, A \cap B \neq \varnothing} \lambda t^*(A)[m(-|A \cap B|k) - 1], \tag{21}$$

where $m(\alpha) = E\{\exp(\alpha U_1)\}$. For each set A satisfying $A \cap B \neq \varnothing$,

$$\lim_{k \to \infty} m(-|A \cap B|k) = P\{U_1 = 0\} < 1. \tag{22}$$

Hence part (b) follows from (19), (20) and (21). □

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DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
AMHERST, MASSACHUSETTS 01033