

ASYMPTOTIC BEHAVIOR OF THE FINITE-SIZE MAGNETIZATION AS A FUNCTION OF THE SPEED OF APPROACH TO CRITICALITY

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The main focus of this paper is to determine whether the thermodynamic magnetization is a physically relevant estimator of the finite-size magnetization. This is done by comparing the asymptotic behaviors of these two quantities along parameter sequences converging to either a second-order point or the tricritical point in the mean-field Blume–Capel model. We show that the thermodynamic magnetization and the finite-size magnetization are asymptotic when the parameter α governing the speed at which the sequence approaches criticality is below a certain threshold α_0 . However, when α exceeds α_0 , the thermodynamic magnetization converges to 0 much faster than the finite-size magnetization. The asymptotic behavior of the finite-size magnetization is proved via a moderate deviation principle when $0 < \alpha < \alpha_0$ and via a weak-convergence limit when $\alpha > \alpha_0$. To the best of our knowledge, our results are the first rigorous confirmation of the statistical mechanical theory of finite-size scaling for a mean-field model.

1. Introduction. For the mean-field Blume–Capel model, as for other mean-field spin systems, the magnetization in the thermodynamic limit is well understood within the theory of large deviations. In this framework the thermodynamic magnetization arises as the unique, positive, global minimum point of the rate function in a large deviation principle. The question answered in this paper is whether, in a neighborhood of criticality, the thermodynamic magnetization is a physically relevant estimator of the finite-size magnetization, which is the expected value of the spin per site. A similar question is answered by the heuristic, statistical mechanical theory of finite-size scaling. This paper is both motivated by the theory of finite-size scaling and puts that theory on a firm foundation in the context of mean-field spin systems. It is hoped that our results suggest how this question can be addressed in the context of much more complicated, short-range spin systems.

Our approach is to evaluate the asymptotic behaviors of the thermodynamic magnetization and the physically relevant, finite-size magnetization along para-

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meter sequences converging to either a second-order point or the tricritical point in the mean-field Blume–Capel model. The thermodynamic magnetization is then considered to be a physically relevant estimator of the finite-size magnetization when these two quantities have the same asymptotic behavior. Our main finding is that the value of the parameter α governing the speed at which the sequence approaches criticality determines whether or not the asymptotic behaviors of these two quantities are the same. Specifically, we show in Theorem 4.1 that the thermodynamic magnetization and the finite-size magnetization are asymptotic when α is below a certain threshold α_0 and that therefore the thermodynamic magnetization is a physically relevant estimator when $0 < \alpha < \alpha_0$. However, when α exceeds α_0 , then according to Theorem 4.2, the thermodynamic magnetization converges to 0 much faster than the finite-size magnetization, and therefore the thermodynamic magnetization is not a physically relevant estimator when $\alpha > \alpha_0$. An advantage of using the thermodynamic magnetization as an estimator of the finite-state magnetization when $0 < \alpha < \alpha_0$ is that the asymptotic behavior of the former quantity is much easier to derive than the asymptotic behavior of the latter quantity [see the discussion at the end of the paragraph after (1.5)].

The investigation is carried out for a mean-field version of an important lattice spin model due to Blume and Capel, to which we refer as the B–C model [4, 6–8]. This mean-field model is equivalent to the B–C model on the complete graph on N vertices. It is one of the simplest models that exhibits the following intricate phase-transition structure: a curve of second-order points, a curve of first-order points and a tricritical point, which separates the two curves. A generalization of the B–C model is studied in [5].

The mean-field B–C model is defined by a canonical ensemble that we denote by $P_{N,\beta,K}$; N equals the number of spins, β is the inverse temperature and K is the interaction strength. $P_{N,\beta,K}$ is defined in (2.1) in terms of the Hamiltonian

$$H_{N,K}(\omega) = \sum_{j=1}^N \omega_j^2 - \frac{K}{N} \left(\sum_{j=1}^N \omega_j \right)^2,$$

in which ω_j represents the spin at site $j \in \{1, 2, \dots, N\}$ and takes values in $\Lambda = \{1, 0, -1\}$. The configuration space for the model is the set Λ^N containing all sequences $\omega = (\omega_1, \omega_2, \dots, \omega_N)$ with each $\omega_j \in \Lambda$. Expectation with respect to $P_{N,\beta,K}$ is denoted by $E_{N,\beta,K}$. The finite-size magnetization is defined by $E_{N,\beta,K}\{|S_N/N|\}$, where S_N equals the total spin $\sum_{j=1}^N \omega_j$.

Before introducing the results in this paper, we summarize the phase-transition structure of the model. For $\beta > 0$ and $K > 0$ we denote by $\mathcal{M}_{\beta,K}$ the set of equilibrium values of the magnetization. $\mathcal{M}_{\beta,K}$ coincides with the set of global minimum points of the free-energy functional $G_{\beta,K}$, which is defined in (2.5). It is known from heuristic arguments and is proved in [16] that there exists a critical inverse temperature $\beta_c = \log 4$ and that for $0 < \beta \leq \beta_c$ there exists a quantity $K(\beta)$ and for $\beta > \beta_c$ there exists a quantity $K_1(\beta)$ having the following properties. The

positive quantity $m(\beta, K)$ appearing in the following list is the thermodynamic magnetization.

1. Fix $0 < \beta \leq \beta_c$. Then for $0 < K \leq K(\beta)$, $\mathcal{M}_{\beta,K}$ consists of the unique pure phase 0, and for $K > K(\beta)$, $\mathcal{M}_{\beta,K}$ consists of two nonzero values $\pm m(\beta, K)$.
2. For $0 < \beta \leq \beta_c$, $\mathcal{M}_{\beta,K}$ undergoes a continuous bifurcation at $K = K(\beta)$, changing continuously from $\{0\}$ for $K \leq K(\beta)$ to $\{\pm m(\beta, K)\}$ for $K > K(\beta)$. This continuous bifurcation corresponds to a second-order phase transition.
3. Fix $\beta > \beta_c$. Then for $0 < K < K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of the unique pure phase 0, for $K = K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of 0 and two nonzero values $\pm m(\beta, K_1(\beta))$ and for $K > K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of two nonzero values $\pm m(\beta, K)$.
4. For $\beta > \beta_c$, $\mathcal{M}_{\beta,K}$ undergoes a discontinuous bifurcation at $K = K_1(\beta)$, changing discontinuously from $\{0\}$ for $K < K_1(\beta)$ to $\{0, \pm m(\beta, K)\}$ for $K = K_1(\beta)$ to $\{\pm m(\beta, K)\}$ for $K > K_1(\beta)$. This discontinuous bifurcation corresponds to a first-order phase transition.

Because of items 2 and 4, we refer to the curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$ as the second-order curve and to the curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$ as the first-order curve. Points on the second-order curve are called second-order points, and points on the first-order curve first-order points. The point $(\beta_c, K(\beta_c)) = (\log 4, 3/2 \log 4)$ separates the second-order curve from the first-order curve and is called the tricritical point. The two-phase region consists of all points in the positive β - K quadrant for which $\mathcal{M}_{\beta,K}$ consists of two values. Thus this region consists of all (β, K) above the second-order curve, above the tricritical point and above the first-order curve; that is, all (β, K) satisfying $0 < \beta \leq \beta_c$ and $K > K(\beta)$ and satisfying $\beta > \beta_c$ and $K > K_1(\beta)$. The sets that describe the phase-transition structure of the model are shown in Figure 1.

For fixed (β, K) lying in the two-phase region the finite-size magnetization $E_{N,\beta,K}\{|S_N/n|\}$ converges to the thermodynamic magnetization $m(\beta, K)$ as $N \rightarrow \infty$. In order to see this, we use the large deviation principle (LDP) for S_N/N with respect to $P_{N,\beta,K}$ in [16], Theorem 3.3, and the fact that the set of global minimum points of the rate function in that LDP coincides with the set $\mathcal{M}_{\beta,K}$ [16], Proposition 3.4, the structure of which has just been described. Since for (β, K) lying in the two-phase region $\mathcal{M}_{\beta,K} = \{\pm m(\beta, K)\}$, the LDP implies that the $P_{N,\beta,K}$ -distributions of S_N/N put an exponentially small mass on the complement of any open set containing $\pm m(\beta, K)$. Symmetry then yields the weak-convergence limit

$$(1.1) \quad P_{N,\beta,K}\{S_N/N \in dx\} \implies \left(\frac{1}{2}\delta_{m(\beta,K)} + \frac{1}{2}\delta_{-m(\beta,K)}\right)(dx).$$

This implies the desired result

$$(1.2) \quad \lim_{N \rightarrow \infty} E_{N,\beta,K}\{|S_N/N|\} = m(\beta, K).$$

The limit in the last display is closely related to the main focus of this paper. It shows that because the thermodynamic magnetization is the limit, as the number

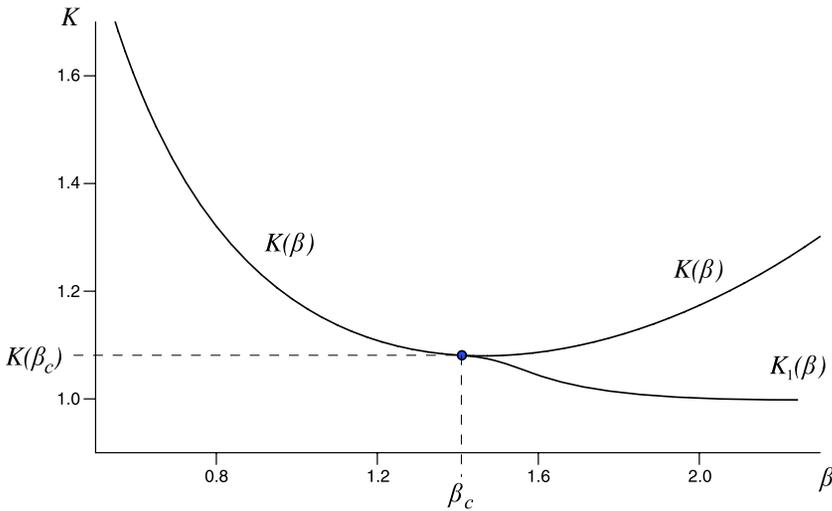


FIG. 1. The sets that describe the phase-transition structure of the mean-field B–C model: the second-order curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$, the first-order curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$ and the tricritical point $(\beta_c, K(\beta_c))$. The phase-coexistence region consists of all (β, K) above the second-order curve, above the tricritical point, on the first-order curve and above the first-order curve. The extension of the second-order curve to $\beta > \beta_c$ is called the spinodal curve.

of spins goes to ∞ , of the finite-size magnetization, the thermodynamic magnetization $m(\beta, K)$ is a physical relevant estimator of the finite-size magnetization, at least when evaluated at fixed (β, K) in the two-phase region.

The main focus of this paper is to determine whether the thermodynamic magnetization is a physically relevant estimator of the finite-size magnetization in a more general sense, namely, when evaluated along a class of sequences (β_n, K_n) that converge to a second-order point $(\beta, K(\beta))$ or the tricritical point $(\beta_c, K(\beta_c))$. The criterion for determining whether $m(\beta_n, K_n)$ is a physically relevant estimator is that as $n \rightarrow \infty$, $m(\beta_n, K_n)$ is asymptotic to the finite-size magnetization $E_{n, \beta_n, K_n}\{|S_n/n|\}$, both of which converge to 0. In this formulation we let $N = n$ in the finite-size magnetization; that is, we let the number of spins N coincide with the index n parametrizing the sequence (β_n, K_n) . As summarized in Theorems 4.1 and 4.2, our main finding is that $m(\beta_n, K_n)$ is a physically relevant estimator if the parameter α governing the speed at which (β_n, K_n) approaches criticality is below a certain threshold α_0 ; however, this is not true if $\alpha > \alpha_0$. For the sequences under consideration the parameter α determines the limits

$$b = \lim_{n \rightarrow \infty} n^\alpha (\beta_n - \beta) \quad \text{and} \quad k = \lim_{n \rightarrow \infty} n^\alpha (K_n - K(\beta)),$$

which are assumed to exist and not to be both 0. The value of α_0 depends on the type of the phase transition—first-order, second-order or tricritical—that influences the sequence, an issue addressed in Section 5 of [13].

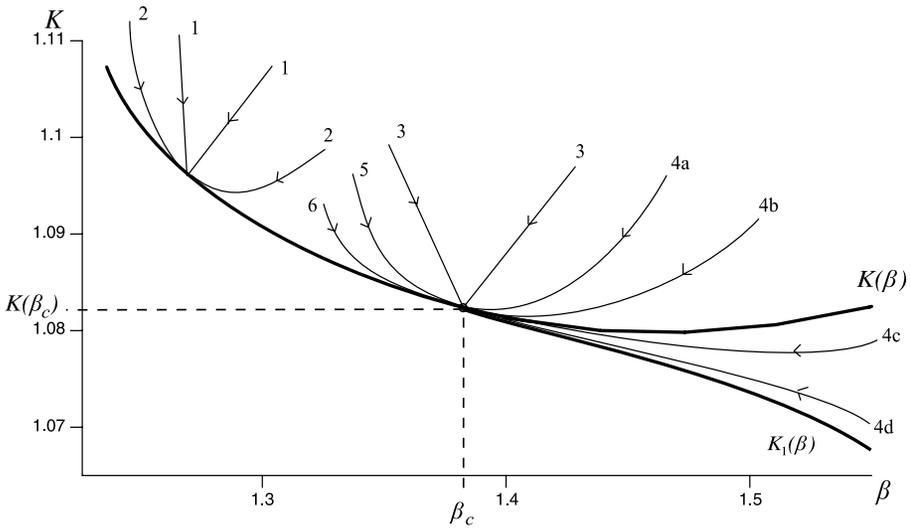


FIG. 2. Possible paths for the six sequences converging to a second-order point and to the tricritical point. The asymptotic results for the sequences converging on the paths labeled 1, 2, 3, 4a–4d, 5 and 6 are discussed in the respective Theorems 5.1–5.6. The sequences on the paths labeled 4a–4d are defined in (5.4) and in the second paragraph after that equation.

We illustrate the results contained in these two theorems by applying them to six types of sequences. In the case of second-order points two such sequences are considered in Theorems 5.1 and 5.2, and in the case of the tricritical point four such sequences are considered in Theorems 5.3–5.6. Possible paths followed by these sequences are shown in Figure 2. We believe that modulo uninteresting scale changes, irrelevant higher order terms and other inconsequential modifications, these are all the sequences of the form $\beta_n = \beta + b/n^\alpha$ and K_n equal to $K(\beta)$ plus a polynomial in $1/n^\alpha$, where $(\beta, K(\beta))$ is either a second-order point or the tricritical point and $m(\beta_n, K_n) \sim c/n^\delta$ for some $c > 0$ and $\delta > 0$.

We next summarize our main results on the asymptotic behaviors of the thermodynamic magnetization and the finite-size magnetization, first for small values of α and then for large values of α . The relevant information is given, respectively, in Theorems 3.1, 4.1 and 4.2. These theorems are valid for suitable positive sequences (β_n, K_n) parametrized by $\alpha > 0$, lying in the two-phase region for all sufficiently large n , and converging either to a second-order point or to the tricritical point. The hypotheses of these three theorems overlap but do not coincide. The hypotheses of Theorem 3.1 are satisfied by all six sequences considered in Section 5 while the hypotheses of each of the Theorems 4.1 and 4.2 are satisfied by all six sequences with one exception. For each of the six sequences the quantities θ and α_0 appearing in these asymptotic results are specified in Table 1.

The difference in the asymptotic behaviors of the thermodynamic magnetization and the finite-size magnetization for $\alpha > \alpha_0$ is described in item 3. As we discuss

TABLE 1

The equations where each of the six sequences is defined, the theorems where the asymptotic results in (1.3), (1.4) and (1.5) are stated for each sequence, and the values of α_0 , θ and $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ [see (1.6)] for each sequence

Seq.	Defn.	Thm.	α_0	θ	κ
1	(5.1)	Theorem 5.1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}(1 - \alpha)$
2	(5.2)	Theorem 5.2	$\frac{1}{2p}$	$\frac{p}{2}$	$\frac{1}{2}(1 - p\alpha)$
3	(5.3)	Theorem 5.3	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{2}(1 - \alpha)$
4	(5.4)	Theorem 5.4	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}(1 - 2\alpha)$
5	(5.5)	Theorem 5.5	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}(1 - 2\alpha)$
6	(5.6)	Theorem 5.6	$\frac{1}{2p-1}$	$\frac{1}{2}(p - 1)$	$\frac{1}{2}(1 - p\alpha)$

in Section 6, the difference is explained by the statistical mechanical theory of finite-size scaling.

1. According to Theorem 3.1, there exists positive quantities \bar{x} and θ such that for all $\alpha > 0$

$$(1.3) \quad m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}.$$

2. ($0 < \alpha < \alpha_0$). According to Theorem 4.1, there exists a threshold value $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$

$$(1.4) \quad E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{x}/n^{\theta\alpha} \quad \text{and} \quad E_{n,\beta_n,K_n}\{|S_n/n|\} \sim m(\beta_n, K_n).$$

Because $m(\beta_n, K_n)$ is asymptotic to the finite-size magnetization, $m(\beta_n, K_n)$ is a physically relevant estimator of the finite-size magnetization. In this case (β_n, K_n) converges to criticality slowly, and we are in the two-phase region, where the system is effectively infinite. Formally the first index n parametrizing the finite-size magnetization can be sent to ∞ before the index n parametrizing the sequence (β_n, K_n) is sent to ∞ , and so we have

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \approx \lim_{N \rightarrow \infty} E_{N,\beta_n,K_n}\{|S_N/N|\} = m(\beta_n, K_n).$$

3. ($\alpha > \alpha_0$). According to Theorem 4.2, there exists a positive quantity \bar{y} such that for all $\alpha > \alpha_0$

$$(1.5) \quad \begin{aligned} E_{n,\beta_n,K_n}\{|S_n/n|\} &\sim \bar{y}/n^{\theta\alpha_0} \quad \text{and} \\ E_{n,\beta_n,K_n}\{|S_n/n|\} &\gg m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}. \end{aligned}$$

Because $m(\beta_n, K_n)$ converges to 0 much faster than the finite-size magnetization, $m(\beta_n, K_n)$ is not a physically relevant estimator of the finite-size magnetization. In this case (β_n, K_n) converges to criticality quickly, and we are in the critical regime, where finite-size scaling effects are important.

The asymptotic behavior of the thermodynamic magnetization $m(\beta_n, K_n) \rightarrow 0$ stated in (1.3) holds for all $\alpha > 0$. It is derived in Theorem 3.2 in [13] and is summarized in Theorem 3.1 in the present paper. In (1.4) we state the asymptotic behavior of the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ for $0 < \alpha < \alpha_0$. This result is proved in part (a) of Theorem 4.1 as a consequence of the moderate deviation principle (MDP) for the spin in Theorem 7.1, the weak-convergence limit in Corollary 7.3, and the uniform integrability estimate in Lemma 7.4. The asymptotic behavior of $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ stated in (1.5) for $\alpha > \alpha_0$ is proved in part (a) of Theorem 4.2 as a consequence of the weak-convergence limit for the spin in Theorem 8.1 and the uniform-integrability-type estimate in Proposition 8.3. In part (a) of Theorem 4.3 we state the asymptotic behavior of $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ for $\alpha = \alpha_0$. That result is a consequence of a weak-convergence limit analogous to the limit in Theorem 8.1 and the uniform-integrability-type estimate in Proposition 8.3. With changes in notation only, Theorem 3.1 and Theorems 4.1–4.3 also apply to other mean-field models including the Curie–Weiss model [12] and the Curie–Weiss–Potts model [17]. The proof of the asymptotic behavior of the thermodynamic magnetization in [13], Theorem 3.2, is purely analytic and is much more straightforward than the probabilistic proofs of the asymptotic behaviors of the finite-size magnetization in Theorems 4.1–4.3.

Figure 3 gives a pictorial representation of the phenomena that are summarized in (1.4) for $0 < \alpha < \alpha_0$ and in (1.5) for $\alpha > \alpha_0$. As we discuss in Section 2, for the sequences (β_n, K_n) under consideration the thermodynamic magnetization $m(\beta_n, K_n)$ can be characterized as the unique, positive, global minimum point in an LDP or, equivalently, as the unique, positive, global minimum point of the dual, free-energy functional G_{β_n, K_n} defined in (2.5). According to graph (a) in Figure 3, for $0 < \alpha < \alpha_0$, G_{β_n, K_n} has two deep, global minimum points at $\pm m(\beta_n, K_n)$.

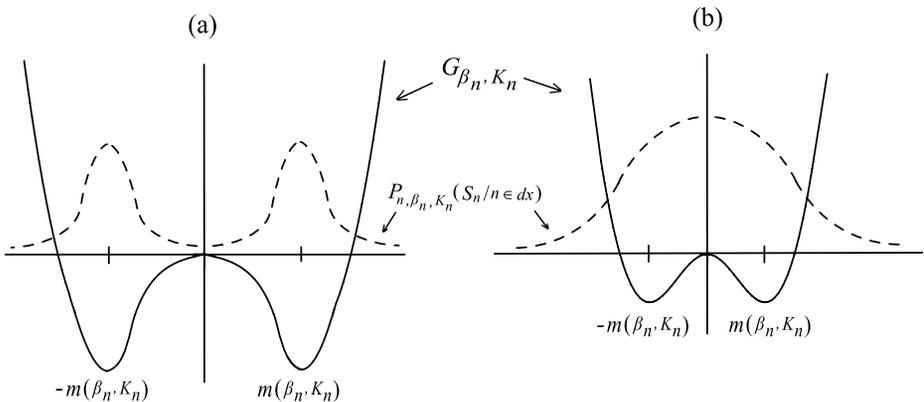


FIG. 3. G_{β_n, K_n} and $P_{n, \beta_n, K_n}\{S_n/n \in dx\}$ for (a) $0 < \alpha < \alpha_0$, (b) $\alpha > \alpha_0$. Graph (b) is not shown to scale. In fact, for $\alpha > \alpha_0$ the global minimum points $\pm m(\beta_n, K_n)$ of G_{β_n, K_n} are much closer to the origin and are much shallower than shown in graph (b).

Graph (b) in Figure 3, which is not shown to scale, exhibits the contrasting situation for $\alpha > \alpha_0$. In this case the global minimum points of G_{β_n, K_n} at $\pm m(\beta_n, K_n)$ are shallow and close to the origin. In the two graphs we also show the form of the distribution $P_{n, \beta_n, K_n}\{S_n/n \in dx\}$. For $0 < \alpha < \alpha_0$ this probability distribution is sharply peaked at $\pm m(\beta_n, K_n)$ as $n \rightarrow \infty$. In contrast, for $\alpha > \alpha_0$ the probability distribution is peaked at 0 and its standard deviation is much larger than $m(\beta_n, K_n)$.

In a work in progress we refine the asymptotic result in (1.4), which states that for $0 < \alpha < \alpha_0$, $m(\beta_n, K_n)$ is asymptotic to $E_{n, \beta_n, K_n}\{|S_n/n|\}$ as $n \rightarrow \infty$. Define $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$, which exceeds $\theta\alpha$ since $1 - \alpha/\alpha_0 > 0$. We conjecture that for a class of suitable sequences (β_n, K_n) that includes the first five sequences considered in Section 5, there exists a positive quantity \bar{v} such that for all $0 < \alpha < \alpha_0$

$$(1.6) \quad E_{n, \beta_n, K_n}\{||S_n/n| - m(\beta_n, K_n)|\} \sim \bar{v}/n^\kappa.$$

This refined asymptotic result would extend part (b) of Theorem 4.1. It is a consequence of the conjecture that when S_n/n is conditioned to lie in a suitable neighborhood of $m(\beta_n, K_n)$, the P_{n, β_n, K_n} -distributions of $n^\kappa(S_n/n - m(\beta_n, K_n))$ converge in distribution to a Gaussian.

For easy reference we list in Table 1 information about the six sequences considered in Section 5. The first two columns list, respectively, the equation in which each sequence is defined and the theorem in which the asymptotic results in equations (1.3), (1.4) and (1.5) are stated for each sequence. In these theorems the quantities \bar{x} and \bar{y} appearing in the three asymptotic results are defined. The three asymptotic results involve the quantities $\alpha_0, \theta, \theta\alpha$ and $\theta\alpha_0$, the values of the first two of which are listed in the next two columns of the table. In the last column of the table we list the values of $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. Through the factor $n^{-\kappa}$, κ governs the conjectured asymptotics of $E_{n, \beta_n, K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$ stated in (1.6).

The contents of this paper are as follows. In Section 2 we summarize the phase-transition structure of the mean-field B–C model. Theorem 3.1 in Section 3 gives the asymptotic behavior of the thermodynamic magnetization $m(\beta_n, K_n) \rightarrow 0$ for suitable sequences (β_n, K_n) converging either to a second-order point or to the tricritical point. The heart of the paper is Section 4. In this section Theorems 4.1, 4.2 and 4.3 give the asymptotic behavior of the finite-size magnetization $E_{n, \beta_n, K_n}\{|S_n/n|\} \rightarrow 0$ for three respective ranges of α : $0 < \alpha < \alpha_0$, $\alpha > \alpha_0$ and $\alpha = \alpha_0$. The quantity α_0 is a threshold value that depends on the type of the phase transition—first-order, second-order or tricritical—that influences the associated sequence (β_n, K_n) . These theorems also compare the asymptotic behaviors of the thermodynamic magnetization and the finite-size magnetization, showing that they are the same for $0 < \alpha < \alpha_0$ but not the same for $\alpha > \alpha_0$. In Section 5 the three theorems in the preceding section are applied to six specific sequences (β_n, K_n) , the first two of which converge to a second-order point and the last four of which converge to the tricritical point. Section 6 gives an overview of the statistical mechan-

ical theory of finite-size scaling, which gives insight into the physical phenomena underlying our mathematical results. Part (a) of Theorem 4.1 is derived in Section 7 from the MDP for the spin in Theorem 7.1, the weak-convergence limit for the spin in Corollary 7.3, and the uniform integrability estimate in Lemma 7.4. Finally, part (a) of Theorem 4.2 is derived in Section 8 from the weak-convergence limit for the spin in Theorem 8.1 and the uniform-integrability-type estimate in Proposition 8.3.

2. Phase-transition structure of the mean-field B–C model. After defining the mean-field B–C model, we introduce a function $G_{\beta,K}$, called the free-energy functional. The global minimum points of this function define the equilibrium values of the magnetization. The phase-transition structure of the model is summarized in Theorems 2.1 and 2.2. The first theorem shows that the model exhibits a second-order phase transition for $\beta \in (0, \beta_c]$, where $\beta_c = \log 4$ is the critical inverse temperature of the model. The second theorem shows that the model exhibits a first-order phase transition for $\beta > \beta_c$.

For $N \in \mathbb{N}$ the mean-field B–C model is a lattice-spin model defined on the complete graph on N vertices $1, 2, \dots, N$. The spin at site $j \in \{1, 2, \dots, N\}$ is denoted by ω_j , a quantity taking values in $\Lambda = \{1, 0, -1\}$. The configuration space for the model is the set Λ^N containing all sequences $\omega = (\omega_1, \omega_2, \dots, \omega_N)$ with each $\omega_j \in \Lambda$. In terms of a positive parameter K representing the interaction strength, the Hamiltonian is defined by

$$H_{N,K}(\omega) = \sum_{j=1}^N \omega_j^2 - \frac{K}{N} \left(\sum_{j=1}^N \omega_j \right)^2$$

for each $\omega \in \Lambda^N$. Let P_N be the product measure on Λ^N with identical one-dimensional marginals $\rho = \frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$. Thus P_N assigns the probability 3^{-N} to each $\omega \in \Lambda^N$. For inverse temperature $\beta > 0$ and for $K > 0$, the canonical ensemble for the mean-field B–C model is the sequence of probability measures that assign to each subset B of Λ^N the probability

$$\begin{aligned} P_{N,\beta,K}(B) &= \frac{1}{Z_N(\beta, K)} \cdot \int_B \exp[-\beta H_{N,K}] dP_N \\ (2.1) \qquad &= \frac{1}{Z_N(\beta, K)} \cdot \sum_{\omega \in B} \exp[-\beta H_{N,K}(\omega)] \cdot 3^{-N}. \end{aligned}$$

In this formula $Z_N(\beta, K)$ is the partition function equal to

$$\int_{\Lambda^N} \exp[-\beta H_{N,K}] dP_N = \sum_{\omega \in \Lambda^N} \exp[-\beta H_{N,K}(\omega)] \cdot 3^{-N}.$$

Expectation with respect to $P_{N,\beta,K}$ is denoted by $E_{N,\beta,K}$.

The analysis of the canonical ensemble $P_{N,\beta,K}$ is facilitated by absorbing the noninteracting component of the Hamiltonian into the product measure P_N , obtaining

$$(2.2) \quad P_{N,\beta,K}(d\omega) = \frac{1}{\tilde{Z}_N(\beta, K)} \cdot \exp\left[N\beta K \left(\frac{S_N(\omega)}{N}\right)^2\right] P_{N,\beta}(d\omega).$$

In this formula $S_N(\omega)$ equals the total spin $\sum_{j=1}^N \omega_j$, $P_{N,\beta}$ is the product measure on Λ^N with identical one-dimensional marginals

$$(2.3) \quad \rho_\beta(d\omega_j) = \frac{1}{Z(\beta)} \cdot \exp(-\beta\omega_j^2)\rho(d\omega_j),$$

$Z(\beta)$ is the normalization equal to $\int_\Lambda \exp(-\beta\omega_j^2)\rho(d\omega_j) = (1 + 2e^{-\beta})/3$ and $\tilde{Z}_N(\beta, K)$ is the normalization equal to $Z_N(\beta, K)/[Z(\beta)]^N$.

We denote by $\mathcal{M}_{\beta,K}$ the set of equilibrium macrostates of the mean-field B–C model. In order to describe this set, we introduce the cumulant generating function c_β of the measure ρ_β defined in (2.3); for $t \in \mathbb{R}$ this function is defined by

$$(2.4) \quad \begin{aligned} c_\beta(t) &= \log \int_\Lambda \exp(t\omega_1)\rho_\beta(d\omega_1) \\ &= \log\left(\frac{1 + e^{-\beta}(e^t + e^{-t})}{1 + 2e^{-\beta}}\right). \end{aligned}$$

For $x \in \mathbb{R}$ we define

$$(2.5) \quad G_{\beta,K}(x) = \beta K x^2 - c_\beta(2\beta K x).$$

As shown in Proposition 3.4 in [16], the set $\mathcal{M}_{\beta,K}$ of equilibrium macrostates of the mean-field B–C model can be characterized as the set of global minimum points of $G_{\beta,K}$:

$$(2.6) \quad \mathcal{M}_{\beta,K} = \{x \in [-1, 1] : x \text{ is a global minimum point of } G_{\beta,K}(x)\}.$$

In [16] the set $\mathcal{M}_{\beta,K}$ was denoted by $\tilde{\mathcal{E}}_{\beta,K}$.

We also define the canonical free energy

$$\varphi(\beta, K) = - \lim_{N \rightarrow \infty} \frac{1}{\beta N} \log \tilde{Z}_N(\beta, K),$$

where $\tilde{Z}_N(\beta, K)$ is the normalizing constant in (2.2). This limit exists and equals $\min_{x \in \mathbb{R}} \beta^{-1} G_{\beta,K}(x)$. Because of this property of $G_{\beta,K}$, we call $G_{\beta,K}$ the free-energy functional of the mean-field B–C model.

The next two theorems use (2.6) to determine the structure of $\mathcal{M}_{\beta,K}$ for $0 < \beta \leq \beta_c = \log 4$ and for $\beta > \beta_c$. The positive quantity $m(\beta, K)$ appearing in these theorems is called the thermodynamic magnetization. The first theorem, proved in Theorem 3.6 in [16], describes the continuous bifurcation in $\mathcal{M}_{\beta,K}$ for $0 < \beta \leq \beta_c$

as K crosses a curve $\{(\beta, K(\beta)) : 0 < \beta < \beta_c\}$. This bifurcation corresponds to a second-order phase transition, and this curve is called the second-order curve. The quantity $K(\beta)$, defined in (2.7), is denoted by $K_c^{(2)}(\beta)$ in [16].

THEOREM 2.1. *For $0 < \beta \leq \beta_c$, we define*

$$(2.7) \quad K(\beta) = 1/[2\beta c''(0)] = (e^\beta + 2)/(4\beta).$$

For these values of β , $\mathcal{M}_{\beta,K}$ has the following structure:

- (a) *For $0 < K \leq K(\beta)$, $\mathcal{M}_{\beta,K} = \{0\}$.*
- (b) *For $K > K(\beta)$, there exists $m(\beta, K) > 0$ such that $\mathcal{M}_{\beta,K} = \{\pm m(\beta, K)\}$.*
- (c) *$m(\beta, K)$ is a positive, increasing, continuous function for $K > K_c(\beta)$, and as $K \rightarrow (K(\beta))^+$, $m(\beta, K) \rightarrow 0$. Therefore, $\mathcal{M}_{\beta,K}$ exhibits a continuous bifurcation at $K(\beta)$.*

The next theorem, proved in Theorem 3.8 in [16], describes the discontinuous bifurcation in $\mathcal{M}_{\beta,K}$ for $\beta > \beta_c$ as K crosses a curve $\{(\beta, K_1(\beta)) : \beta > \beta_c\}$. This bifurcation corresponds to a first-order phase transition, and this curve is called the first-order curve. As shown in Theorem 3.8 in [16], for all $\beta > \beta_c$, $K_1(\beta) < K(\beta)$. The quantity $K_1(\beta)$ is denoted by $K_c^{(1)}(\beta)$ in [16].

THEOREM 2.2. *For $\beta > \beta_c$, $\mathcal{M}_{\beta,K}$ has the following structure in terms of the quantity $K_1(\beta)$, denoted by $K_c^{(1)}(\beta)$ in [16] and defined implicitly for $\beta > \beta_c$ on page 2231 of [16]:*

- (a) *For $0 < K < K_1(\beta)$, $\mathcal{M}_{\beta,K} = \{0\}$.*
- (b) *For $K = K_1(\beta)$ there exists $m(\beta, K_1(\beta)) > 0$ such that $\mathcal{M}_{\beta,K_1(\beta)} = \{0, \pm m(\beta, K_1(\beta))\}$.*
- (c) *For $K > K_1(\beta)$ there exists $m(\beta, K) > 0$ such that $\mathcal{M}_{\beta,K} = \{\pm m(\beta, K)\}$.*
- (d) *$m(\beta, K)$ is a positive, increasing, continuous function for $K \geq K_1(\beta)$, and as $K \rightarrow K_1(\beta)^+$, $m(\beta, K) \rightarrow m(\beta, K_1(\beta)) > 0$. Therefore, $\mathcal{M}_{\beta,K}$ exhibits a discontinuous bifurcation at $K_1(\beta)$.*

The phase-coexistence region is defined as the set of all points in the positive β - K quadrant for which $\mathcal{M}_{\beta,K}$ consists of more than one value. According to Theorems 2.1 and 2.2, the phase-coexistence region consists of all points above the second-order curve, above the tricritical point, on the first-order curve and above the first-order curve; that is,

$$\{(\beta, K) : 0 < \beta \leq \beta_c, K > K(\beta) \text{ and } \beta > \beta_c, K \geq K_1(\beta)\}.$$

Our derivation of the asymptotic behavior of the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ in this paper is valid for a class of sequences (β_n, K_n) lying in the phase-coexistence region for all sufficiently large n and converging either to a second-order point or to the tricritical point. In the next section we state an asymptotic formula for $m(\beta_n, K_n) \rightarrow 0$ for a general class of such sequences. That asymptotic formula will be used later in the paper when we study the asymptotic behavior of the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$.

3. Asymptotic behavior of $m(\beta_n, K_n)$. The main result in this section is Theorem 3.1. It states the asymptotic behavior of the thermodynamic magnetization $m(\beta_n, K_n) \rightarrow 0$ for sequences (β_n, K_n) lying in the phase-coexistence region for all sufficiently large n and converging either to a second-order point or to the tricritical point. The asymptotic behavior is expressed in terms of the unique positive, global minimum point of an associated polynomial that is introduced in hypothesis (iii) of the theorem. With several modifications the hypotheses of the next theorem are also the hypotheses under which we derive the rates at which $E_{n, \beta_n, K_n} \{|S_n/n|\} \rightarrow 0$ later in the paper.

As shown in part (iii) of Theorem 3.1, the asymptotics of $m(\beta_n, K_n)$ depend on the asymptotics of the scaled free-energy function $n^{\alpha/\alpha_0} G_{\beta_n, K_n}(x/n^{\theta\alpha})$. Because of Lemma 7.2, the asymptotics of the finite-size magnetization in Theorems 4.1–4.3 depend on precisely the same asymptotics. Lemma 7.2 coincides with Lemma 4.1 in [9]. In that paper the connections among the asymptotics of the scaled free-energy functional, the limit theorems underlying the asymptotics of the finite-size magnetization and Lemma 4.1 are described in detail. These limit theorems are analogues of the MDP in Theorem 7.1 and of the weak convergence limit in Theorem 8.1.

Theorem 3.1 restates the main theorem in [13], Theorem 3.2. Hypotheses (iii)(a) and (iv) in the next theorem coincide with hypotheses (iii)(a) and (iv) in Theorem 3.2 in [13] except that the latter hypotheses are expressed in terms of $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$ while here we have substituted the formulas for u and γ . Hence u and γ no longer appear.

THEOREM 3.1. *Let (β_n, K_n) be a positive sequence that converges either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. We assume that (β_n, K_n) satisfies the following four hypotheses:*

- (i) (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n .
- (ii) The sequence (β_n, K_n) is parametrized by $\alpha > 0$. This parameter regulates the speed of approach of (β_n, K_n) to the second-order point or the tricritical point in the following sense:

$$b = \lim_{n \rightarrow \infty} n^\alpha (\beta_n - \beta) \quad \text{and} \quad k = \lim_{n \rightarrow \infty} n^\alpha (K_n - K(\beta))$$

both exist, and b and k are not both 0; if $b \neq 0$, then b equals 1 or -1 .

- (iii) There exists an even polynomial g of degree 4 or 6 satisfying $g(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ together with the following two properties; g is called the Ginzburg–Landau polynomial.

- (a) $\exists \alpha_0 > 0$ and $\exists \theta > 0$ such that for all $\forall \alpha > 0$

$$\lim_{n \rightarrow \infty} n^{\alpha/\alpha_0} G_{\beta_n, K_n}(x/n^{\theta\alpha}) = g(x)$$

uniformly for x in compact subsets of \mathbb{R} .

- (b) g has a unique, positive global minimum point \bar{x} ; thus the set of global minimum points of g equals $\{\pm \bar{x}\}$ or $\{0, \pm \bar{x}\}$.

(iv) *There exists a polynomial H satisfying $H(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ together with the following property: $\forall \alpha > 0 \exists R > 0$ such that $\forall n \in \mathbb{N}$ sufficiently large and $\forall x \in \mathbb{R}$ satisfying $|x/n^{\theta\alpha}| < R$, $n^{\alpha/\alpha_0} G_{\beta_n, K_n}(x/n^{\theta\alpha}) \geq H(x)$.*

Under hypotheses (i)–(iv), for any $\alpha > 0$

$$m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}, \quad \text{that is, } \lim_{n \rightarrow \infty} n^{\theta\alpha} m(\beta_n, K_n) = \bar{x}.$$

If $b \neq 0$, then this becomes $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^\theta$.

It is clear from the proof of the theorem that if hypotheses (iii) and (iv) are valid for a specific value of $\alpha > 0$, then we obtain the asymptotic formula $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}$ for that value of α .

In the next section, we state the main results on the rates at which $E_{n, \beta_n, K_n}\{|S_n/n|\} \rightarrow 0$ for small α satisfying $0 < \alpha < \alpha_0$, for large α satisfying $\alpha > \alpha_0$, and for intermediate α satisfying $\alpha = \alpha_0$. We also compare these rates with the asymptotic behavior of the thermodynamic magnetization $m(\beta_n, K_n) \rightarrow 0$.

4. Main results on rates at which $E_{n, \beta_n, K_n}\{|S_n/n|\} \rightarrow 0$. Let a_n be a positive sequence converging to 0. In stating the three results on the rates at which the finite-size magnetization $E_{n, \beta_n, K_n}\{|S_n/n|\} \rightarrow 0$, we write

$$E_{n, \beta_n, K_n}\{|S_n/n|\} \sim a_n \quad \text{if } \lim_{n \rightarrow \infty} E_{n, \beta_n, K_n}\{|S_n/n|\}/a_n = 1,$$

and we write

$$E_{n, \beta_n, K_n}\{|S_n/n|\} \gg a_n \quad \text{if } \lim_{n \rightarrow \infty} E_{n, \beta_n, K_n}\{|S_n/n|\}/a_n = \infty.$$

Let α be the quantity parametrizing the sequences (β_n, K_n) as explained in hypothesis (ii) of Theorem 3.1. We begin with Theorem 4.1, which gives the rate at which $E_{n, \beta_n, K_n}\{|S_n/n|\} \rightarrow 0$ for small α satisfying $0 < \alpha < \alpha_0$. Theorem 4.2 gives the rate at which $E_{n, \beta_n, K_n}\{|S_n/n|\} \rightarrow 0$ for large α satisfying $\alpha > \alpha_0$ while Theorem 4.3 gives the rate at which $E_{n, \beta_n, K_n}\{|S_n/n|\} \rightarrow 0$ for intermediate α satisfying $\alpha = \alpha_0$. In all three cases we compare these rates with the rate at which $m(\beta_n, K_n) \rightarrow 0$. In the next section we specialize these theorems to the six sequences mentioned in the [Introduction](#).

Part (a) of the next theorem gives the rate at which $E_{n, \beta_n, K_n}\{|S_n/n|\} \rightarrow 0$ for $0 < \alpha < \alpha_0$, and part (b) shows that for these values of α , $E_{n, \beta_n, K_n}\{|S_n/n|\} \sim m(\beta_n, K_n)$. It follows that for $0 < \alpha < \alpha_0$, $m(\beta_n, K_n)$ is a physically relevant estimator of the finite-size magnetization $E_{n, \beta_n, K_n}\{|S_n/n|\}$ because it has the same asymptotic behavior as that quantity.

The next theorem is valid under hypotheses (i) and (ii) of Theorem 3.1, hypotheses (iii)(a) and (iv) of that theorem for all $0 < \alpha < \alpha_0$, the inequality $0 < \theta\alpha_0 < 1/2$, and a new hypothesis (iii')(b). The inequality $0 < \theta\alpha_0 < 1/2$ is satisfied by all six sequences considered in Section 5. The new hypothesis (iii')(b)

restricts hypothesis (iii)(b) of Theorem 3.1 by assuming that the set of global minimum points of the Ginzburg–Landau polynomial g equals $\{\pm\bar{x}\}$ for some \bar{x} . As we remark after the statement of the theorem, this restriction is needed in order to prove part (a). The proof does not cover the case where the set of global minimum points of g equals $\{0, \pm\bar{x}\}$ for some $\bar{x} > 0$. The conjecture is that in this case there exists $0 < \lambda < 1/2$ such that $E_{n,\beta_n,K_n}\{|S_n/n|\} \sim 2\lambda\bar{x}/n^{\theta\alpha}$ (see the discussion before Corollary 7.3). An example of a sequence for which the set of global minimum points of g contains three points is given in case (d) of sequence 4 in the next section. By contrast, all the other sequences considered in the next section satisfy the new hypothesis that the set of global minimum points of g equals $\{\pm\bar{x}\}$ for some \bar{x} .

THEOREM 4.1 ($0 < \alpha < \alpha_0$). *Let (β_n, K_n) be a positive sequence parametrized by $\alpha > 0$ and converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta_c, K(\beta_c))$. We assume hypotheses (i) and (ii) of Theorem 3.1 together with hypotheses (iii)(a) and (iv) of that theorem for all $0 < \alpha < \alpha_0$. We also assume the inequality $0 < \theta\alpha_0 < 1/2$ and the following hypothesis, which restricts hypothesis (iii)(b) of Theorem 3.1:*

(iii')(b) *The set of global minimum points of the Ginzburg–Landau polynomial g equals $\{\pm\bar{x}\}$ for some $\bar{x} > 0$.*

The following conclusions hold:

(a) *For all $0 < \alpha < \alpha_0$*

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{x}/n^{\theta\alpha}, \quad \text{that is, } \lim_{n \rightarrow \infty} n^{\theta\alpha} E_{n,\beta_n,K_n}\{|S_n/n|\} = \bar{x}.$$

(b) *For all $0 < \alpha < \alpha_0$, $E_{n,\beta_n,K_n}\{|S_n/n|\} \sim m(\beta_n, K_n)$.*

Part (a) of the theorem is proved from the moderate deviation principle (MDP) for the P_{n,β_n,K_n} -distributions of $S_n/n^{1-\theta\alpha}$ in Theorem 7.1, which shows that the rate function equals $g - \inf_{y \in \mathbb{R}} g(y)$. The inequality $0 < \theta\alpha_0 < 1/2$ is used to control an error term in the proof of the MDP. According to hypothesis (iii')(b), the set of global minimum points of g equals $\{\pm\bar{x}\}$ for some $\bar{x} > 0$. It quickly follows from the MDP that the sequence of P_{n,β_n,K_n} -distributions of $S_n/n^{1-\theta\alpha}$ converges weakly to $\frac{1}{2}\delta_{\bar{x}} + \frac{1}{2}\delta_{-\bar{x}}$. The uniform integrability of $S_n/n^{1-\theta\alpha}$, derived in Lemma 7.4 from the MDP, yields the limit $E_{n,\beta_n,K_n}\{|S_n/n^{1-\theta\alpha}|\} \rightarrow \bar{x}$ as $n \rightarrow \infty$. This is the asymptotic formula for $E_{n,\beta_n,K_n}\{|S_n/n|\}$ in part (a) of Theorem 4.1. Part (b) of the theorem follows from part (a) and the asymptotic formula $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}$, which is the conclusion of Theorem 3.1.

We next state Theorem 4.2, which in part (a) gives the rate at which $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ for $\alpha > \alpha_0$. Part (b) shows that for these values of α , $E_{n,\beta_n,K_n}\{|S_n/n|\} \gg m(\beta_n, K_n)$. Because $m(\beta_n, K_n) \rightarrow 0$ at an asymptotically faster rate than the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\}$, $m(\beta_n, K_n)$ is not a physically relevant estimator of that quantity for $\alpha > \alpha_0$.

In order to prove part (a) of the next theorem, we need hypothesis (iv) of Theorem 3.1 for $\alpha = \alpha_0$, the inequality $0 < \theta\alpha_0 < 1/2$ and a new hypothesis (v), in which we assume that for all $\alpha > \alpha_0$, $nG_{\beta_n, K_n}(x/n^{\theta\alpha_0})$ convergence pointwise to a polynomial $\tilde{g}(x)$ that goes to ∞ as $|x| \rightarrow \infty$. As we will see for the first five of the six sequences considered in the next section, \tilde{g} in hypothesis (v) equals the highest order term of the Ginzburg–Landau polynomial g . We omit the analysis showing that this description of \tilde{g} can, in fact, be validated in general if the uniform convergence in hypothesis (iii)(a) of Theorem 3.1 on compact subsets of \mathbb{R} is strengthened to uniform convergence on compact subsets of an appropriate open set in \mathbb{C} containing the origin and if $\theta\alpha_0$ equals a certain value depending on the degree of g . This stronger convergence is valid for the six sequences considered in the next section. However, the additional condition on $\theta\alpha_0$, valid for the first five sequences, is not satisfied by the sixth sequence.

In part (b) of the next theorem the rates at which $E_{n, \beta_n, K_n}\{|S_n/n|\} \rightarrow 0$ and $m(\beta_n, K_n) \rightarrow 0$ are compared. In order to prove part (b), we also need hypotheses (i) and (ii) of Theorem 3.1 and hypotheses (iii) and (iv) of that theorem for all $\alpha > \alpha_0$. These hypotheses allow us to apply Theorem 3.1 for all $\alpha > \alpha_0$.

THEOREM 4.2 ($\alpha > \alpha_0$). *Let (β_n, K_n) be a positive sequence parametrized by $\alpha > 0$ and converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta_c, K(\beta_c))$. We assume hypotheses (i) and (ii) of Theorem 3.1, hypothesis (iii) of Theorem 3.1 for all $\alpha > \alpha_0$ and hypothesis (iv) of Theorem 3.1 for all $\alpha \geq \alpha_0$. We also assume the inequality $0 < \theta\alpha_0 < 1/2$ and the following hypothesis:*

(v) *There exists an even polynomial \tilde{g} of degree 4 or 6 satisfying $\tilde{g}(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ together with the following property: $\exists \alpha_0 > 0$ and $\exists \theta > 0$ such that $\forall \alpha > \alpha_0$ and $\forall x \in \mathbb{R}$*

$$\lim_{n \rightarrow \infty} nG_{\beta_n, K_n}(x/n^{\theta\alpha_0}) = \tilde{g}(x).$$

The following conclusions hold:

(a) We define

$$\bar{y} = \frac{1}{\int_{\mathbb{R}} \exp[-\tilde{g}(x)] dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tilde{g}(x)] dx.$$

Then for all $\alpha > \alpha_0$

$$E_{n, \beta_n, K_n}\{|S_n/n|\} \sim \bar{y}/n^{\theta\alpha_0}, \quad \text{that is, } \lim_{n \rightarrow \infty} n^{\theta\alpha_0} E_{n, \beta_n, K_n}\{|S_n/n|\} = \bar{y}.$$

(b) For all $\alpha > \alpha_0$, $E_{n, \beta_n, K_n}\{|S_n/n|\} \gg m(\beta_n, K_n)$.

Part (a) of the theorem is proved from the weak convergence of the sequence of P_{n, β_n, K_n} -distributions of $S_n/n^{1-\theta\alpha}$ to a probability measure having a density proportional to $\exp[-\tilde{g}]$, which is shown in Theorem 8.1. The proof of this weak

convergence relies on hypothesis (v) of Theorem 4.2 and the lower bound in hypothesis (iv) of Theorem 3.1 for $\alpha = \alpha_0$. The inequality $0 < \theta\alpha_0 < 1/2$ is used to control an error term in the proof. The uniform-integrability-type estimate in Proposition 8.3 yields the limit $E_{n,\beta_n,K_n}\{|S_n/n|^{1-\theta\alpha_0}\} \rightarrow \bar{y}$ as $n \rightarrow \infty$. This is the asymptotic formula for $E_{n,\beta_n,K_n}\{|S_n/n|\}$ in part (a) of Theorem 4.2. Part (b) of the theorem follows from part (a), the asymptotic formula $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}$ and the fact that since $\alpha > \alpha_0$, the decay rate $n^{-\theta\alpha}$ of $m(\beta_n, K_n) \rightarrow 0$ is asymptotically larger than the decay rate $n^{-\theta\alpha_0}$ of $E_{n,\beta_n,K_n}\{|S_n/n|\}$.

We end this section by stating Theorem 4.3. Part (a) gives the rate at which $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ for $\alpha = \alpha_0$, and part (b) compares this rate with the rate at which $m(\beta_n, K_n) \rightarrow 0$. The theorem is valid under hypotheses (i) and (ii) of Theorem 3.1, hypotheses (iii) and (iv) of Theorem 3.1 for $\alpha = \alpha_0$ and the inequality $0 < \theta\alpha_0 < 1/2$.

THEOREM 4.3 ($\alpha = \alpha_0$). *Let (β_n, K_n) be a positive sequence parametrized by $\alpha > 0$ and converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta_c, K(\beta_c))$. We assume hypotheses (i) and (ii) of Theorem 3.1, hypotheses (iii) and (iv) of Theorem 3.1 for $\alpha = \alpha_0$ and the inequality $0 < \theta\alpha_0 < 1/2$. The following conclusions hold:*

(a) *We define*

$$\bar{z} = \frac{1}{\int_{\mathbb{R}} \exp[-g(x)] dx} \cdot \int_{\mathbb{R}} |x| \exp[-g(x)] dx.$$

Then for all $\alpha = \alpha_0$

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{z}/n^{\theta\alpha_0}, \quad \text{that is, } \lim_{n \rightarrow \infty} n^{\theta\alpha_0} E_{n,\beta_n,K_n}\{|S_n/n|\} = \bar{z}.$$

(b) *For $\alpha = \alpha_0$, $E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{z} \cdot m(\beta_n, K_n)/\bar{x}$.*

We omit the proof of part (a) of the theorem, which can be derived like part (a) of Theorem 4.2. According to hypothesis (iii)(a) of Theorem 3.1, $nG_{\beta_n,K_n}(x/n^{\theta\alpha_0})$ converges to $g(x)$ uniformly for x in compact subsets of \mathbb{R} . The pointwise convergence of $nG_{\beta_n,K_n}(x/n^{\theta\alpha_0})$ to $g(x)$ and the lower bound in hypothesis (iv) of Theorem 3.1 for $\alpha = \alpha_0$ allow us to prove that the sequence of P_{n,β_n,K_n} -distributions of $S_n/n^{1-\theta\alpha_0}$ converges weakly to a probability measure having a density proportional to $\exp[-g]$. The inequality $0 < \theta\alpha_0 < 1/2$ is used to control an error term in the proof. The asymptotic formula for $E_{n,\beta_n,K_n}\{|S_n/n|\}$ in part (a) of Theorem 4.3 follows from this weak-convergence limit and the uniform-integrability-type estimate in Proposition 8.3, the hypotheses of which can be verified in the context of Theorem 4.3 as they are verified at the end of Section 8 in the context of Theorem 4.2. When $\alpha = \alpha_0$, $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha_0}$ [Theorem 3.1(b)]. Hence part (a) of Theorem 4.3 implies that

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{z}/n^{\theta\alpha_0} \sim \bar{z} \cdot m(\beta_n, K_n)/\bar{x}.$$

This is the conclusion of part (b) of the theorem.

In numerical calculations we studied the relative size of \bar{z} and \bar{x} . Depending on the magnitude of the coefficient of the quadratic term in the Ginzburg–Landau polynomial g , \bar{z}/\bar{x} can be less than 1, can equal 1 and can exceed 1.

In the next section we specialize Theorems 4.1, 4.2 and 4.3 to the six sequences mentioned in the Introduction.

5. Results for six sequences. In [13] we apply Theorem 3.1 to determine the asymptotic behavior of the thermodynamic magnetization $m(\beta_n, K_n) \rightarrow 0$ for six sequences (β_n, K_n) parametrized by $\alpha > 0$. The first two sequences converge to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, and the last four sequences converge to the tricritical point $(\beta_c, K(\beta_c))$. In the present section we specialize to the first five sequences the results in Theorems 4.1, 4.2 and 4.3 concerning the asymptotic behaviors of $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ for $0 < \alpha < \alpha_0$, $\alpha > \alpha_0$ and $\alpha = \alpha_0$. We also compare these asymptotic behaviors with the asymptotic behavior of $m(\beta_n, K_n) \rightarrow 0$. In addition we state the results of Theorems 3.1, 4.1 and 4.3 for the sixth sequence. However, for this sequence, one of the hypotheses of Theorem 4.2 is not valid, and so that theorem cannot be applied.

In order to be able to apply these four theorems, we must verify the validity of their hypotheses, which are the following:

- *Theorem 3.1.* Hypotheses (i) and (ii) and hypotheses (iii) and (iv) for all $\alpha > 0$.
- *Theorem 4.1.* Hypotheses (i) and (ii) of Theorem 3.1, hypotheses (iii)(a) and (iv) of Theorem 3.1 for all $0 < \alpha < \alpha_0$, the inequality $0 < \theta\alpha_0 < 1/2$ and the new hypothesis (iii')(b).
- *Theorem 4.2.* Hypotheses (i) and (ii) of Theorem 3.1, hypothesis (iii) of Theorem 3.1 for all $\alpha > \alpha_0$, hypothesis (iv) of Theorem 3.1 for all $\alpha \geq \alpha_0$, the inequality $0 < \theta\alpha_0 < 1/2$ and the new hypothesis (v).
- *Theorem 4.3.* Hypotheses (i) and (ii) of Theorem 3.1, hypotheses (iii) and (iv) of Theorem 3.1 for $\alpha = \alpha_0$ and the inequality $0 < \theta\alpha_0 < 1/2$.

Thus, in order to verify the hypotheses of the four theorems, it suffices to verify hypotheses (i) and (ii) of Theorem 3.1, hypotheses (iii)(a) and (iv) of Theorem 3.1 for all $\alpha > 0$, hypothesis (iii')(b) for all $0 < \alpha < \alpha_0$, hypothesis (iii)(b) for all $\alpha \geq \alpha_0$, the inequality $0 < \theta\alpha_0 < 1/2$, and hypothesis (v) of Theorem 4.2.

The quantities \bar{y} and \bar{z} appearing in the asymptotic formulas in Theorems 4.2 and 4.3 are defined as follows in terms of the polynomial \tilde{g} , introduced in hypothesis (v) of Theorem 4.2, and in terms of the Ginzburg–Landau polynomial g :

$$\bar{y} = \frac{1}{\int_{\mathbb{R}} \exp[-\tilde{g}(x)] dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tilde{g}(x)] dx$$

and

$$\bar{z} = \frac{1}{\int_{\mathbb{R}} \exp[-g(x)] dx} \cdot \int_{\mathbb{R}} |x| \exp[-g(x)] dx.$$

For the first five sequences, \tilde{g} equals the highest-order term in g . For the sixth sequence, Theorem 4.2 cannot be applied because hypothesis (v) of that theorem is not valid. In each sequence $K(\beta) = (e^\beta + 2)/(4\beta)$ for $\beta > 0$. The curve $\{(\beta, K(\beta)) : 0 < \beta < \beta_c\}$ is the second-order curve, $(\beta_c, K(\beta_c))$ is the tricritical point and the curve $\{(\beta, K(\beta)) : \beta > \beta_c\}$ is the spinodal curve.

Sequence 1.

Definition of sequence 1. Given $0 < \beta < \beta_c$, $\alpha > 0$, $b \in \{1, 0, -1\}$ and $k \in \mathbb{R}$, $k \neq 0$, the sequence is defined by

$$(5.1) \quad \beta_n = \beta + b/n^\alpha \quad \text{and} \quad K_n = K(\beta) + k/n^\alpha.$$

This sequence converges to the second-order point $(\beta, K(\beta))$ along a ray with slope k/b if $b \neq 0$.

Hypotheses (i) and (ii) in Theorem 3.1. Hypothesis (i) states that (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n . In order to guarantee this, we assume that $K'(\beta)b - k < 0$. This inequality is equivalent to $K_n > K(\beta_n)$ for all sufficiently large n and thus guarantees that (β_n, K_n) lies in the phase-coexistence region above the second-order curve for all sufficiently large n . Hypothesis (ii) is also satisfied.

Other hypotheses.

1. Define $\alpha_0 = 1/2$ and $\theta = 1/2$. As shown in Theorem 4.1 in [13], the uniform convergence in hypothesis (iii)(a) of Theorem 3.1 is valid for all $\alpha > 0$ with the Ginzburg–Landau polynomial

$$g(x) = \beta(K'(\beta)b - k)x^2 + c_4(\beta)x^4$$

where $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/8 \cdot 4!$.

Since $\theta\alpha_0 = 1/4$, we have $0 < \theta\alpha_0 < 1/2$, which is one of the hypotheses of Theorems 4.1–4.3.

2. We assume that $K'(\beta)b - k < 0$. Then, as required by hypothesis (iii)(b) of Theorem 3.1 and hypothesis (iii')(b) of Theorem 4.1, the set of global minimum points of g is $\{\pm\bar{x}\}$, where $\bar{x} > 0$ is defined in (4.6) in [13].
3. Hypothesis (iv) of Theorem 3.1 is valid for all $\alpha > 0$ with the polynomial H given on page 113 of [13].
4. The pointwise convergence in hypothesis (v) of Theorem 4.2 holds with \tilde{g} equal to the highest order term in g ; namely, $\tilde{g}(x) = c_4(\beta)x^4$. This is easily verified using equation (4.4) in [13].

We now specialize to sequence 1 the results in Theorems 3.1, 4.1, 4.2 and 4.3 concerning the asymptotic behavior of $m(\beta_n, K_n) \rightarrow 0$ and the asymptotic behaviors of $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ for $0 < \alpha < \alpha_0$, for $\alpha > \alpha_0$ and for $\alpha = \alpha_0$.

THEOREM 5.1. *Let (β_n, K_n) be sequence 1 that is defined in (5.1) and converges to a second-order point $(\beta, K(\beta))$ for $0 < \beta < \beta_c$. Assume that $K'(\beta)b - k < 0$. The following conclusions hold:*

(a) *For all $\alpha > 0$,*

$$m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2}.$$

If $b \neq 0$ in the definition of β_n , then $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^{1/2}$.

(b) *For all $0 < \alpha < \alpha_0 = 1/2$,*

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{x}/n^{\alpha/2} \sim m(\beta_n, K_n).$$

(c) *For all $\alpha > \alpha_0 = 1/2$,*

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{y}/n^{1/4} \gg m(\beta_n, K_n).$$

(d) *For $\alpha = \alpha_0 = 1/2$,*

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{z}/n^{1/4} \sim \bar{z} \cdot m(\beta_n, K_n)/\bar{x}.$$

Sequence 2.

Definition of sequence 2. Given $0 < \beta_0 < \beta_c$, $\alpha > 0$, $b \in \{1, -1\}$, an integer $p \geq 2$ and a real number $\ell \neq K^{(p)}(\beta)$, the sequence is defined by

$$(5.2) \quad \begin{aligned} \beta_n &= \beta_0 + b/n^\alpha \quad \text{and} \\ K_n &= K(\beta_0) + \sum_{j=1}^{p-1} K^{(j)}(\beta_0)b^j/(j!n^{j\alpha}) + \ell b^p/(p!n^{p\alpha}). \end{aligned}$$

This sequence converges to the second-order point $(\beta_0, K(\beta_0))$ along a curve that coincides with the second-order curve to order $p - 1$ in powers of $\beta - \beta_0$.

Hypotheses (i) and (ii) in Theorem 3.1. Hypothesis (i) states that (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n . In order to guarantee this, we assume that $(K^{(p)}(\beta_0) - \ell)b^p < 0$. This inequality is equivalent to $K_n > K(\beta_n)$ for all sufficiently large n and thus guarantees that (β_n, K_n) lies in the phase-coexistence region above the second-order curve for all sufficiently large n . Hypothesis (ii) is also satisfied.

Other hypotheses.

1. Define $\alpha_0 = 1/2p$ and $\theta = p/2$. As shown in Theorem 4.2 in [13], the uniform convergence in hypothesis (iii)(a) of Theorem 3.1 is valid for all $\alpha > 0$ with the Ginzburg–Landau polynomial

$$g(x) = \frac{1}{p!}\beta_0(K^{(p)}(\beta_0) - \ell)b^p x^2 + c_4(\beta_0)x^4$$

$$\text{where } c_4(\beta_0) = (e^{\beta_0} + 2)^2(4 - e^{\beta_0})/8 \cdot 4!.$$

Since $\theta\alpha_0 = 1/4$, we have $0 < \theta\alpha_0 < 1/2$, which is one of the hypotheses of Theorems 4.1–4.3.

2. We assume that $(K^{(p)}(\beta_0) - \ell)b^p < 0$. Then, as required by hypothesis (iii)(b) of Theorem 3.1 and hypothesis (iii')(b) of Theorem 4.1, the set of global minimum points of g is $\{\pm\bar{x}\}$, where $\bar{x} > 0$ is defined in (4.9) in [13].
3. Hypothesis (iv) of Theorem 3.1 is valid for all $\alpha > 0$ with the polynomial H given on page 115 of [13].
4. The pointwise convergence in hypothesis (v) of Theorem 4.2 holds with \tilde{g} equal to the highest order term in g ; namely, $\tilde{g}(x) = c_4(\beta_0)x^4$. This is easily verified using (4.8) in [13].

We now specialize to sequence 2 the results in Theorems 3.1, 4.1, 4.2 and 4.3 concerning the asymptotic behavior of $m(\beta_n, K_n) \rightarrow 0$ and the asymptotic behaviors $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ for $0 < \alpha < \alpha_0$, for $\alpha > \alpha_0$ and for $\alpha = \alpha_0$.

THEOREM 5.2. *Let (β_n, K_n) be sequence 2 that is defined in (5.2) and converges to a second-order point $(\beta_0, K(\beta_0))$ for $0 < \beta_0 < \beta_c$. Assume that $(K^{(p)}(\beta_0) - \ell)b^p < 0$. The following conclusions hold:*

(a) For all $\alpha > 0$,

$$m(\beta_n, K_n) \sim \bar{x}/n^{p\alpha/2} = \bar{x}|\beta_0 - \beta_n|^{p/2}.$$

(b) For all $0 < \alpha < \alpha_0 = 1/2p$,

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{x}/n^{p\alpha/2} \sim m(\beta_n, K_n).$$

(c) For all $\alpha > \alpha_0 = 1/2p$,

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{y}/n^{1/4} \gg m(\beta_n, K_n).$$

(d) For $\alpha = \alpha_0 = 1/2p$,

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{z}/n^{1/4} \sim \bar{z} \cdot m(\beta_n, K_n)/\bar{x}.$$

Sequence 3.

Definition of sequence 3. This sequence is defined as in (5.1) with β replaced by β_c . Thus given $\alpha > 0$, $b \in \{1, 0, -1\}$, and $k \in \mathbb{R}, k \neq 0$, the sequence is defined by

$$(5.3) \quad \beta_n = \beta_c + b/n^\alpha \quad \text{and} \quad K_n = K(\beta_c) + k/n^\alpha.$$

This sequence converges to the tricritical point $(\beta_c, K(\beta_c))$ along a ray with slope k/b if $b \neq 0$.

Hypotheses (i) and (ii) in Theorem 3.1. Hypothesis (i) states that (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n . In order to guarantee this, we assume that $K'(\beta_c)b - k < 0$. This inequality is equivalent to $K_n > K(\beta_n)$ for all sufficiently large n and thus guarantees that for all sufficiently large n , (β_n, K_n) lies in the phase-coexistence region above the spinodal curve if $b = 1$, above the second-order curve if $b = -1$ and above the tricritical point if $b = 0$. Hypothesis (ii) is also satisfied.

Other hypotheses.

1. Define $\alpha_0 = 2/3$ and $\theta = 1/4$. As shown in Theorem 4.3 in [13], the uniform convergence in hypothesis (iii)(a) of Theorem 3.1 is valid for all $\alpha > 0$ with the Ginzburg–Landau polynomial

$$g(x) = \beta_c(K'(\beta_c)b - k)x^2 + c_6x^6 \quad \text{where } c_6 = 9/40.$$

Since $\theta\alpha_0 = 1/6$, we have $0 < \theta\alpha_0 < 1/2$, which is one of the hypotheses of Theorems 4.1–4.3.

2. We assume that $K'(\beta_c)b - k < 0$. Then, as required by hypothesis (iii)(b) of Theorem 3.1 and hypothesis (iii')(b) of Theorem 4.1, the set of global minimum points of g is $\{\pm\bar{x}\}$, where $\bar{x} > 0$ is defined in (4.14) in [13].
3. Hypothesis (iv) of Theorem 3.1 is valid for all $\alpha > 0$ with the polynomial H given on page 117 of [13].
4. The pointwise convergence in hypothesis (v) of Theorem 4.2 holds with \tilde{g} equal to the highest order term in g ; namely, $\tilde{g}(x) = c_6x^6$. This is easily verified using (4.13) in [13].

We now specialize to sequence 3 the results in Theorems 3.1, 4.1, 4.2 and 4.3 concerning the asymptotic behavior of $m(\beta_n, K_n) \rightarrow 0$ and the asymptotic behaviors of $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ for $0 < \alpha < \alpha_0$, for $\alpha > \alpha_0$ and for $\alpha = \alpha_0$.

THEOREM 5.3. *Let (β_n, K_n) be sequence 3 that is defined in (5.3) and converges to the tricritical point $(\beta_c, K(\beta_c))$. Assume that $K'(\beta_c)b - k < 0$. The following conclusions hold:*

- (a) *For all $\alpha > 0$,*

$$m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/4}.$$

If $b \neq 0$ in the definition of β_n , then $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^{1/4}$.

- (b) *For all $0 < \alpha < \alpha_0 = 2/3$,*

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{x}/n^{\alpha/4} \sim m(\beta_n, K_n).$$

- (c) *For all $\alpha > \alpha_0 = 2/3$,*

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{y}/n^{1/6} \gg m(\beta_n, K_n).$$

- (d) *For $\alpha = \alpha_0 = 2/3$,*

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{z}/n^{1/6} \sim \bar{z} \cdot m(\beta_n, K_n)/\bar{x}.$$

Sequence 4.

Of the six sequences this sequence exhibits the most complicated behavior, the description of which is divided into four cases (a)–(d) described in the third paragraph below. In addition, for cases (c) and (d) the validity of hypothesis (i) of Theorem 3.1 involves the validity of two conjectures. For cases (a), (b) and (c)

all the other hypotheses of Theorems 3.1, 4.1, 4.2 and 4.3 are valid. However, for case (d) hypothesis (iii')(b) of Theorem 4.1 is not valid, and therefore that theorem cannot be applied in that case.

Definition of sequence 4. Given $\alpha > 0$, a curvature parameter $\ell \in \mathbb{R}$, and another parameter $\tilde{\ell} \in \mathbb{R}$, sequence 4 is defined by

$$(5.4) \quad \beta_n = \beta_c + 1/n^\alpha \quad \text{and} \quad K_n = K(\beta_c) + K'(\beta_c)/n^\alpha + \ell/(2n^\alpha) + \tilde{\ell}/(6n^{3\alpha}).$$

Since $\beta_n - \beta_c = 1/n^\alpha$ the sequence converges from the right to the tricritical point $(\beta_c, K(\beta_c))$ along the curve $(\beta, \tilde{K}(\beta))$, where for $\beta > \beta_c$

$$\tilde{K}(\beta) = K(\beta_c) + K'(\beta_c)(\beta - \beta_c) + \ell(\beta - \beta_c)^2/2 + \tilde{\ell}(\beta - \beta_c)^3/6.$$

At the tricritical point this curve is tangent to the spinodal curve, which is the extension of the second-order curve to $\beta > \beta_c$. As shown in [16], Theorem 3.8, the spinodal curve lies above the first-order curve for all $\beta > \beta_c$.

Hypotheses (i) and (ii) in Theorem 3.1. The discussion of hypothesis (i) for this sequence involves four cases (a), (b), (c) and (d) that are presented in the next paragraph. The validity of this hypotheses for the last two of these four cases depends on the validity of conjectures 1 and 2 stated at the end of this paragraph. These conjectures are supported by partial proofs, numerical evidence and properties of the Ginzburg–Landau polynomials and are discussed in detail in Section 6 of [14]. The two conjectures involve the behavior, in a neighborhood of the tricritical point, of the first-order curve defined by $K_1(\beta)$ for $\beta > \beta_c$. Since $\lim_{\beta \rightarrow \beta_c^+} K_1(\beta) = K(\beta_c)$ [16], Sections 3.1 and 3.3, by continuity we extend the definition of $K_1(\beta)$ to β_c by defining $K_1(\beta_c) = K(\beta_c)$. We assume that the first three right-hand derivatives of $K_1(\beta)$ exist at β_c and denote them by $K_1'(\beta_c)$, $K_1''(\beta_c)$ and $K_1'''(\beta_c)$. We also define $\ell_c = K''(\beta_c) - 5/(4\beta_c)$. Conjectures 1 and 2 state the following: (1) $K_1'(\beta_c) = K'(\beta_c)$, (2) $K_1''(\beta_c) = \ell_c < 0 < K''(\beta_c)$.

The choices of ℓ and $\tilde{\ell}$ defining the four cases of sequence 4 are as follows. Cases (a)–(c) correspond to $\ell > \ell_c$ and suitable values of $\tilde{\ell}$, and case (d) corresponds to $\ell = \ell_c$ and suitable values of $\tilde{\ell}$.

- (a) $\ell > K''(\beta_c)$ and any $\tilde{\ell} \in \mathbb{R}$.
- (b) $\ell = K''(\beta_c)$ and any $\tilde{\ell} > K'''(\beta_c)$.
- (c) $K''(\beta_c) > \ell > \ell_c$ and any $\tilde{\ell} \in \mathbb{R}$.
- (d) $\ell = \ell_c$ and any $\tilde{\ell} > K_1'''(\beta_c)$.

For all four cases hypothesis (ii) is satisfied. For cases (a) and (b) and for all sufficiently large n , (β_n, K_n) lies in the phase-coexistence region above the spinodal curve, and so hypothesis (i) is satisfied. If conjectures 1 and 2 are valid, then for cases (c) and (d) (β_n, K_n) lies in the phase-coexistence region between the spinodal and first-order curves for all sufficiently large n , and again hypothesis (i) is valid. In the discussion of the validity of the hypotheses of Theorems 3.1 and

4.1–4.3 for sequence 4, conjectures 1 and 2 are needed only for the last assertion. For all four cases (β_n, K_n) converges to the tricritical point along the curve $\{(\beta, \tilde{K}(\beta)) : \beta > \beta_c\}$, where $\tilde{K}(\beta)$ is defined in the display after (5.4). If conjectures 1 and 2 are valid, then for cases (a)–(c) this curve coincides with the first-order curve to order 1 in powers of $\beta - \beta_c$, while for case (d) this curve coincides with the first-order curve to order 2 in powers of $\beta - \beta_c$.

Other hypotheses.

The validity of these hypotheses for cases (c) and (d) does not depend on conjectures 1 and 2. A major difference between cases (a)–(c) and case (d) appears in item 2.

1. Define $\alpha_0 = 1/3$ and $\theta = 1/2$. As shown in Theorem 4.4 in [13], for all four cases the uniform convergence in hypothesis (iii)(a) of Theorem 3.1 is valid for all $\alpha > 0$ with the Ginzburg–Landau polynomial

$$g(x) = \frac{1}{2}\beta_c(K''(\beta_c) - \ell)x^2 - 4c_4x^4 + c_6x^6$$

where $c_4 = 3/16$ and $c_6 = 9/40$.

Since $\theta\alpha_0 = 1/6$, we have $0 < \theta\alpha_0 < 1/2$, which is one of the hypotheses of Theorems 4.1–4.3.

2. We assume that $\ell > \ell_c = K''(\beta_c) - 5/(4\beta_c)$. Then, as required by hypothesis (iii)(b) of Theorem 3.1 and hypothesis (iii')(b) of Theorem 4.1, for cases (a)–(c) the set of global minimum points of g equals $\{\pm\bar{x}\}$, where $\bar{x} = \bar{x}(\ell) > 0$ is defined in (4.19) in [13]. If $\ell = \ell_c$, then for case (d) the set of global minimum points of g equals $\{0, \pm\bar{x}\}$, where $\bar{x} = \bar{x}(\ell_c) > 0$ is defined in (4.19) in [13]. Hence for case (d) hypothesis (iii)(b) of Theorem 3.1 is valid, but hypothesis (iii')(b) of Theorem 4.1 is not valid.
3. For all four cases, hypothesis (iv) of Theorem 3.1 is valid for all $\alpha > 0$ with the polynomial H given on page 120 of [13].
4. For all four cases, the pointwise convergence in hypothesis (v) of Theorem 4.2 holds with \tilde{g} equal to the highest order term in g ; namely, $\tilde{g}(x) = c_6x^6$. This is easily verified using (4.16) in [13].

We now specialize to sequence 4 the results in Theorems 3.1, 4.1, 4.2 and 4.3 concerning the asymptotic behavior of $m(\beta_n, K_n) \rightarrow 0$ and the asymptotic behaviors of $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ for $0 < \alpha < \alpha_0$, for $\alpha > \alpha_0$ and for $\alpha = \alpha_0$. Parts (a), (c) and (d) of the theorem are valid for all four cases of the sequence. However, part (b) is valid only for cases (a), (b) and (c) because, as we point out in item 2 above, for case (d) hypothesis (iii')(b) of Theorem 4.1 does not hold.

THEOREM 5.4. *Let (β_n, K_n) be sequence 4 that is defined in (5.4) and converges to the tricritical point $(\beta_c, K(\beta_c))$. Assume that ℓ and $\bar{\ell}$ are defined as in one of the four cases (a)–(d) and that for cases (c)–(d) conjectures 1 and 2 are valid. The following conclusions hold:*

(a) For cases (a)–(d), for all $\alpha > 0$,

$$m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2} = \bar{x}(\beta_n - \beta_c)^{1/2}.$$

(b) For cases (a)–(c), for all $0 < \alpha < \alpha_0 = 1/3$,

$$E_{n, \beta_n, K_n}\{|S_n/n|\} \sim \bar{x}/n^{\alpha/2} \sim m(\beta_n, K_n).$$

(c) For cases (a)–(d), for all $\alpha > \alpha_0 = 1/3$,

$$E_{n, \beta_n, K_n}\{|S_n/n|\} \sim \bar{y}/n^{1/6} \gg m(\beta_n, K_n).$$

(d) For cases (a)–(d), for $\alpha = \alpha_0 = 1/3$,

$$E_{n, \beta_n, K_n}\{|S_n/n|\} \sim \bar{z}/n^{1/6} \sim \bar{z} \cdot m(\beta_n, K_n)/\bar{x}.$$

Sequence 5.

Definition of sequence 5. This sequence is defined as in (5.2) with $b = -1$, $p = 2$ and β_0 replaced by β_c . Thus given $\alpha > 0$ and a real number $\ell \neq K''(\beta_c)$, the sequence is defined by

$$(5.5) \quad \beta_n = \beta_c - 1/n^\alpha \quad \text{and} \quad K_n = K(\beta_c) - K'(\beta_c)/n^\alpha + \ell/2n^{2\alpha}.$$

This sequence converges to the tricritical point $(\beta_c, K(\beta_c))$ from the left along a curve that coincides with the second-order curve to order 2 in powers of $\beta - \beta_c$.

Hypotheses (i) and (ii) in Theorem 3.1. Hypothesis (i) states that (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n . In order to guarantee this, we assume that $\ell > K''(\beta_c)$. This inequality is equivalent to $K_n > K(\beta_n)$ for all sufficiently large n and thus guarantees that (β_n, K_n) lies in the phase-coexistence region above the second-order curve for all sufficiently large n . Hypothesis (ii) is also satisfied.

Other hypotheses.

1. Define $\alpha_0 = 1/3$ and $\theta = 1/2$. As shown in Theorem 4.5 in [13], the uniform convergence in hypothesis (iii)(a) of Theorem 3.1 is valid for all $\alpha > 0$ with the Ginzburg–Landau polynomial

$$g(x) = \frac{1}{2}\beta_c(K''(\beta_c) - \ell)x^2 + 4c_4x^4 + c_6x^6$$

$$\text{where } c_4 = 3/16 \text{ and } c_6 = 9/40.$$

Since $\theta\alpha_0 = 1/6$, we have $0 < \theta\alpha_0 < 1/2$, which is one of the hypotheses of Theorems 4.1–4.3.

2. We assume that $\ell > K''(\beta_c)$. Then, as required by hypothesis (iii)(b) of Theorem 3.1 and hypothesis (iii')(b) of Theorem 4.1, the set of global minimum points of g is $\{\pm\bar{x}\}$, where $\bar{x} > 0$ is defined in (4.23) in [13].
3. Hypothesis (iv) of Theorem 3.1 is valid for all $\alpha > 0$ with the polynomial H given on page 121 of [13].

4. The pointwise convergence in hypothesis (v) of Theorem 4.2 holds with \tilde{g} equal to the highest order term in g ; namely, $\tilde{g}(x) = c_6 x^6$. This is easily verified using (4.21) in [13].

We now specialize to sequence 5 the results in Theorems 3.1, 4.1, 4.2 and 4.3 concerning the asymptotic behavior of $m(\beta_n, K_n) \rightarrow 0$ and the asymptotic behaviors of $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ for $0 < \alpha < \alpha_0$, for $\alpha > \alpha_0$ and for $\alpha = \alpha_0$.

THEOREM 5.5. *Let (β_n, K_n) be sequence 5 that is defined in (5.5) and converges to the tricritical point $(\beta_c, K(\beta_c))$. Assume that $\ell > K''(\beta_c)$. The following conclusions hold:*

- (a) For all $\alpha > 0$,

$$m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2} = \bar{x}(\beta_c - \beta_n)^{1/2}.$$

- (b) For all $0 < \alpha < \alpha_0 = 1/3$,

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{x}/n^{\alpha/2} \sim m(\beta_n, K_n).$$

- (c) For all $\alpha > \alpha_0 = 1/3$,

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{y}/n^{1/6} \gg m(\beta_n, K_n).$$

- (d) For $\alpha = \alpha_0 = 1/3$,

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{z}/n^{1/6} \sim \bar{z} \cdot m(\beta_n, K_n)/\bar{x}.$$

Sequence 6.

For this sequence the hypotheses of Theorems 3.1, 4.1 and 4.3 are all valid. However, Theorem 4.2 cannot be applied because hypothesis (v) of that theorem is not valid.

Definition of sequence 6. This sequence is defined as in (5.2) with $b = -1$, an integer $p \geq 3$, and β_0 replaced by β_c . Thus given $\alpha > 0$ and a real number $\ell \neq K^{(p)}(\beta_c)$, the sequence is defined by

$$(5.6) \quad \beta_n = \beta_c - 1/n^\alpha \quad \text{and} \\ K_n = K(\beta_c) + \sum_{j=1}^{p-1} K^{(j)}(\beta_c)(-1)^j/(j!n^{j\alpha}) + \ell(-1)^p/(p!n^{p\alpha}).$$

This sequence converges to the tricritical point $(\beta_c, K(\beta_c))$ from the left along a curve that coincides with the second-order curve to order $p - 1$ in powers of $\beta - \beta_c$.

Hypotheses (i) and (ii) in Theorem 3.1. Hypothesis (i) states that (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n . In order to guarantee this, we assume that $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$. This inequality is equivalent to

$K_n > K(\beta_n)$ for all sufficiently large n and thus guarantees that (β_n, K_n) lies in the phase-coexistence region above the second-order curve for all sufficiently large n . Hypothesis (ii) is also satisfied.

Other hypotheses.

1. Define $\alpha_0 = 1/(2p - 1)$ and $\theta = (p - 1)/2$. As shown in Theorem 4.6 in [13], the uniform convergence in hypothesis (iii)(a) of Theorem 3.1 is valid for all $\alpha > 0$ with the Ginzburg–Landau polynomial

$$g(x) = \frac{1}{p!} \beta_c (K^{(p)}(\beta_c) - \ell) (-1)^p x^2 + 4c_4 x^4 \quad \text{where } c_4 = 3/16.$$

Since $\theta\alpha_0 = (p - 1)/[2(2p - 1)]$ and $p \geq 3$, we have $0 < \theta\alpha_0 < 1/2$, which is one of the hypotheses of Theorems 4.1–4.3.

2. We assume that $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$. Then, as required by hypothesis (iii)(b) of Theorem 3.1 and hypothesis (iii')(b) of Theorem 4.1, the set of global minimum points of g is $\{\pm \bar{x}\}$, where $\bar{x} > 0$ is defined in (4.25) in [13].
3. Hypothesis (iv) of Theorem 3.1 is valid for all $\alpha > 0$ with the polynomial H given on page 122 of [13].
4. The only problem arises in hypothesis (v) of Theorem 4.2, which is not valid for all $\alpha > \alpha_0$ with the values of α_0 and θ in item 1. In fact, one uses equation (4.21) in [13] to verify that with these values of α_0 , θ and α , $nG_{\beta_n, K_n}(x/n^{\theta\alpha_0}) \rightarrow 0$ for all $x \in \mathbb{R}$. Hence with these values of α_0 , θ and α , Theorem 4.2 cannot be applied.

We now specialize to sequence 6 the results in Theorems 3.1, 4.1 and 4.3 concerning the asymptotic behavior of $m(\beta_n, K_n) \rightarrow 0$ and the asymptotic behaviors of $E_{n, \beta_n, K_n}\{|S_n/n|\} \rightarrow 0$ for $0 < \alpha < \alpha_0$ and for $\alpha = \alpha_0$.

THEOREM 5.6. *Let (β_n, K_n) be sequence 6 that is defined in (5.5) and converges to the tricritical point $(\beta_c, K(\beta_c))$. Assume that $(K^{(p)}(\beta) - \ell)(-1)^p < 0$. The following conclusions hold:*

- (a) For all $\alpha > 0$,

$$m(\beta_n, K_n) \sim \bar{x}/n^{(p-1)\alpha/2} = \bar{x}(\beta_c - \beta_n)^{(p-1)/2}.$$

- (b) For all $0 < \alpha < \alpha_0 = 1/(2p - 1)$,

$$E_{n, \beta_n, K_n}\{|S_n/n|\} \sim \bar{x}/n^{(p-1)\alpha/2} \sim m(\beta_n, K_n).$$

- (c) For $\alpha = \alpha_0 = 1/(2p - 1)$,

$$E_{n, \beta_n, K_n}\{|S_n/n|\} \sim \bar{z}/n^{(p-1)/[2(2p-1)]} \sim \bar{z} \cdot m(\beta_n, K_n)/\bar{x}.$$

The one gap in Theorem 5.6 is the failure of hypothesis (v) of Theorem 4.2 for all $\alpha > \alpha_0$. We omit the analysis that gives a variation of Theorem 4.2 describing

a subset of $\alpha > \alpha_0$ for which the asymptotics of $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ can be determined.

This completes our description, in the context of the six sequences, of the three theorems in Section 4 on how the asymptotic behaviors of the thermodynamic magnetization $m(\beta_n, K_n) \rightarrow 0$ and the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$ compare for $0 < \alpha < \alpha_0$, $\alpha > \alpha_0$ and $\alpha = \alpha_0$. In the next section we outline the theory of finite-size scaling, which gives insight into the physical phenomena underlying the theorems in Section 4.

6. The theory of finite-size scaling. In Theorems 4.1 and 4.2 we compare the asymptotic behavior of the thermodynamic magnetization $m(\beta_n, K_n) \rightarrow 0$ with the asymptotic behavior of the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\} \rightarrow 0$, first for $0 < \alpha < \alpha_0$ and then for $\alpha > \alpha_0$. The results described in these two theorems are intimately connected with the theory of finite-size scaling. This nonrigorous but highly suggestive theory was developed in statistical mechanics in order to understand phase transitions in finite systems. In fact, our work in this paper was motivated by the theory of finite-size scaling and can be understood in that context. At the same time, our results put ideas of finite-size scaling on a firm mathematical footing for the mean-field B–C model. To the best of our knowledge, this is the first time that the theory of finite-size scaling has been rigorously derived for a mean-field model. After sketching the theory of finite-size scaling, we show that its predictions are consistent with those in Theorem 5.1. That theorem specializes Theorems 4.1 and 4.2 to sequence 1, which is defined in (5.1).

The theory of finite-size scaling is a generalization of scaling theory to apply to finite systems [2]. Scaling theory gives a methodology for analyzing the singularities of thermodynamic quantities such as the magnetization in a neighborhood of criticality. One formulation of scaling theory emphasizes the fundamental role of the correlation length ξ by expressing the singularities in thermodynamic quantities in terms of ξ . For example, in a neighborhood of criticality the thermodynamic magnetization behaves like $\xi^{-\tilde{\beta}/\nu}$, where $\tilde{\beta}$ is the magnetization exponent and ν is the correlation-length exponent [22]. The singularity in the correlation length as a function of the distance to criticality is controlled by the exponent $-\nu$.

The theory of finite-size scaling asserts that in a neighborhood of criticality quantities such as the finite-size magnetization behave like functions of the linear system size L and the ratio of the correlation length ξ to the linear system size. When $\xi/L \ll 1$, the system is effectively infinite so that finite-size quantities are independent of L , and the critical singularities are the same as those in the thermodynamic limit. On the other hand, when $\xi/L \gg 1$, critical fluctuations are instead limited by the system size. In this regime, the theory of finite-size scaling asserts that the power-law singularities as a function of ξ are replaced by power-law singularities as a function of L . For example, in the case of the finite-size magnetization the theory of finite-size scaling asserts that in a neighborhood of criticality it behaves like $L^{-\tilde{\beta}/\nu} f(\xi/L)$. The function $f(x)$ interpolates continuously between

the two regimes. Thus, as $x = \xi/L \rightarrow 0$, $f(x) \approx x^{-\tilde{\beta}/\nu}$. In this case the finite-size magnetization behaves like $L^{-\tilde{\beta}/\nu}(\xi/L)^{-\tilde{\beta}/\nu} = \xi^{-\tilde{\beta}/\nu}$ and so is independent of L . As discussed in the preceding paragraph, the thermodynamic magnetization also behaves like the same function $\xi^{-\tilde{\beta}/\nu}$. On the other hand, as $x = \xi/L \rightarrow \infty$, $f(x) \rightarrow 1$ and the finite-size magnetization behaves like $L^{-\tilde{\beta}/\nu}$.

These ideas cannot be directly applied to the mean-field B–C model or other mean-field spin systems since neither the system length L nor the correlation length ξ are defined. Appropriate quantities for mean-field spin systems are N , the number of spins, and Ξ , the size of the giant cluster in the Fortuin–Kasteleyn representation [18, 19]. For such systems the mappings $N = L^{d_c}$ and $\Xi = \xi^{d_c}$ are expected to yield, in a neighborhood of criticality, correct scaling relations for thermodynamic quantities such as the magnetization and correct finite-size scaling relations for quantities such as the finite-size magnetization. In these equations d_c denotes the upper critical dimension. This is defined as the dimension above which short-range spin systems such as the B–C model [4, 6–8] have the same critical exponents as the associated mean-field models. Thus in the case of the thermodynamic magnetization the scaling expression $\xi^{-\tilde{\beta}/\nu}$, which is appropriate for short-range models, is replaced by $\Xi^{-\tilde{\beta}/d_c\nu}$. In addition, in the case of the finite-size magnetization, the finite-size scaling expression $L^{-\tilde{\beta}/\nu} f(\xi/L)$, which is appropriate for short-range models, is replaced by $N^{-\tilde{\beta}/d_c\nu} f((\Xi/N)^{1/d_c})$.

In order to apply the ideas of finite-size scaling to the mean-field B–C model, we consider a sequence (β_n, K_n) converging to criticality—that is, a second-order point or the tricritical point—from the phase-coexistence region. We also identify the number of spins N with the index n parametrizing the sequence (β_n, K_n) . Thus the finite-size scaling expression for the finite-size magnetization takes the form $n^{-\tilde{\beta}/d_c\nu} f((\Xi/n)^{1/d_c})$. As in Section 5 of [13], we bring in the quantity $\mu_1(\beta_n, K_n)$ representing the distance of (β_n, K_n) to criticality. According to scaling theory, Ξ behaves like $\mu_1^{-d_c\nu}$.

We now specialize these ideas to sequence 1. Defined in (5.1), this sequence converges to a second-order point and $\mu_1 \approx n^{-\alpha}$. Thus for this sequence the correlation volume Ξ behaves like $\mu_1^{-d_c\nu} = n^{d_c\alpha\nu}$, and so the ratio Ξ/n appearing in the argument of f behaves like $n^{d_c\alpha\nu-1}$. Since for mean-field second-order points $\nu = \tilde{\beta} = 1/2$ and $d_c = 4$ [20], we see that Ξ and Ξ/n behave, respectively, like $n^{2\alpha}$ and $n^{2\alpha-1}$. The conclusion is that for sequence 1 the scaling relation for the thermodynamic magnetization takes the form

$$(6.1) \quad \Xi^{-\tilde{\beta}/d_c\nu} \approx n^{-\alpha/2},$$

and the finite-size scaling expression for the finite-size magnetization takes the form

$$(6.2) \quad n^{-\tilde{\beta}/d_c\nu} f((\Xi/n)^{1/d_c}) \approx n^{-1/4} f(n^{(2\alpha-1)/4}).$$

The next step is to relate this phenomenology with the conclusions of Theorem 5.1, which specializes Theorems 4.1 and 4.2 to sequence 1. The key is to recall that $f(x) \approx x^{-\beta/\nu} = x^{-1}$ as $x \rightarrow 0$ and $f(x) \rightarrow 1$ as $x \rightarrow \infty$. According to the formula in (6.2), the theory of finite-size scaling predicts a change in behavior in the finite-size magnetization when $\alpha = 1/2$. This agrees with Theorem 5.1, which states that for sequence 1 the threshold value α_0 equals $1/2$. For $0 < \alpha < 1/2$, the ratio $\Xi/n = n^{2\alpha-1}$ is much less than 1, and the finite-size magnetization behaves like $n^{-1/4}n^{-(2\alpha-1)/4} = n^{-\alpha/2}$. This behavior coincides with the behavior of the thermodynamic magnetization given in (6.1), making this prediction of the theory of finite-size scaling consistent with part (b) of Theorem 5.1. On the other hand, for $\alpha > 1/2$, since the ratio $\Xi/n = n^{2\alpha-1}$ is much bigger than 1, we have $f(n^{(2\alpha-1)/4}) \approx 1$, and so the finite-size magnetization behaves like $n^{-1/4}$. This converges to 0 much more slowly than the thermodynamic magnetization, which behaves like $n^{-\alpha/2}$. Again this prediction of the theory of finite-size scaling is consistent with part (c) of Theorem 5.1.

Similar heuristic arguments based on the theory of finite-size scaling can be applied to the other sequences discussed in Section 5. They yield the correct asymptotic behaviors for the finite-size magnetization for $0 < \alpha < \alpha_0$ and $\alpha > \alpha_0$, in agreement with Theorems 5.2–5.6. However, the tricritical region presents additional difficulties because of the cross-over from the second-order regime to the tricritical regime. The correct treatment of these sequences in the scaling regime is discussed in the context of scaling theory in Section 5 of [13].

This completes our discussion of the theory of finite-size scaling and its relationship with the main mathematical results given in Theorems 4.1 and 4.2 and specialized to the six sequences in Theorems 5.1–5.6. In the next section we discuss how part (a) of Theorem 4.1 follows from the MDP in Theorem 7.1. These two theorems describe the asymptotic behavior of suitably scaled versions of the spin per site for small values of α .

7. Proof of part (a) of Theorem 4.1. We start by sketching how we will prove part (a) of Theorem 4.1. When the quantity α parametrizing the sequence (β_n, K_n) satisfies $0 < \alpha < \alpha_0$, Theorem 7.1 states the MDP for $S_n/n^{1-\theta\alpha}$ under the hypotheses of Theorem 4.1 except for hypothesis (iii')(b). The rate function in this MDP is $g(x) - \inf_{y \in \mathbb{R}} g(y)$, which under the latter hypothesis has global minimum points at $\pm \bar{x}$. The MDP implies that the P_{n, β_n, K_n} -distributions of $S_n/n^{1-\theta\alpha}$ put an exponentially small mass on the complement of any open set containing the global minimum points $\pm \bar{x}$ of the rate function. Symmetry then yields the following weak limit, stated in Corollary 7.3:

$$P_{n, \beta_n, K_n} \{S_n/n^{1-\theta\alpha} \in dx\} \implies \left(\frac{1}{2}\delta_{\bar{x}} + \frac{1}{2}\delta_{-\bar{x}}\right)(dx);$$

that is, if f is any bounded, continuous function, then

$$\lim_{n \rightarrow \infty} \int_{\Lambda_n} f(S_n/n^{1-\theta\alpha}) dP_{n, \beta_n, K_n} = \int_{\mathbb{R}} f d\left(\frac{1}{2}\delta_{\bar{x}} + \frac{1}{2}\delta_{-\bar{x}}\right) = \frac{1}{2}f(\bar{x}) + \frac{1}{2}f(-\bar{x}).$$

In Lemma 7.4 we verify that with respect to P_{n,β_n,K_n} , the sequence $S_n/n^{1-\theta\alpha}$ is uniformly integrable. The uniform integrability allows us to replace the bounded, continuous function f in the last display by the absolute value function, yielding

$$\lim_{n \rightarrow \infty} \int_{\Lambda^n} |S_n/n^{1-\theta\alpha}| dP_{n,\beta_n,K_n} = \lim_{n \rightarrow \infty} E_{n,\beta_n,K_n} |S_n/n^{1-\theta\alpha}| = \bar{x}.$$

This limit is the conclusion of part (a) of Theorem 4.1.

We next formulate the concept of an MDP for the mean-field B–C model. Let (β_n, K_n) be a positive sequence converging either to a second-order point or to the tricritical point. Also let γ and u be real numbers satisfying $\gamma \in (0, 1/2)$ and $u \in (0, 1)$, and let Γ be a continuous function on \mathbb{R} that satisfies $\Gamma(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. For any subset A of \mathbb{R} , $\Gamma(A)$ denotes the infimum of Γ over A . We say that with respect to P_{n,β_n,K_n} , $S_n/n^{1-\gamma}$ satisfies the MDP with exponential speed n^u and rate function Γ if for any closed set F in \mathbb{R}

$$(7.1) \quad \limsup_{n \rightarrow \infty} \frac{1}{n^u} \log P_{n,\beta_n,K_n} \{S_n/n^{1-\gamma} \in F\} \leq -\Gamma(F)$$

and for any open set Φ in \mathbb{R}

$$(7.2) \quad \liminf_{n \rightarrow \infty} \frac{1}{n^u} \log P_{n,\beta_n,K_n} \{S_n/n^{1-\gamma} \in \Phi\} \geq -\Gamma(\Phi).$$

While an MDP is also a large deviation principle, the term MDP is often used whenever the exponential speed a_n of the large deviation probabilities satisfies $a_n/n \rightarrow 0$ as $n \rightarrow \infty$; [10], Section 3.7.

For $0 < \alpha < \alpha_0$ we now state the MDP for $S_n/n^{1-\theta\alpha}$ with exponential speed $n^{1-\alpha/\alpha_0}$. The hypotheses of Theorem 4.1 are hypotheses (i) and (ii) of Theorem 3.1, hypotheses (iii)(a) and (iv) of that theorem for all $0 < \alpha < \alpha_0$, hypothesis (iii')(b) and the inequality $0 < \theta\alpha_0 < 1/2$. The MDP holds under the same hypotheses except for hypothesis (iii')(b), which requires that the set of global minimum points of the Ginzburg–Landau polynomial g equals $\{\pm\bar{x}\}$ for some $\bar{x} > 0$. Later in this section we will use the MDP together with this hypothesis on the set of global minimum points of g to prove Theorem 4.1. Since $0 < \alpha < \alpha_0$ and $0 < \theta\alpha_0 < 1/2$, the quantities appearing in the exponents of n in the MDP satisfy $0 < \theta\alpha < 1/2$ and $0 < 1 - \alpha/\alpha_0 < 1$. The latter inequality implies that the exponential speed satisfies $n^{1-\alpha/\alpha_0} \rightarrow \infty$ as $n \rightarrow \infty$.

THEOREM 7.1. *Let (β_n, K_n) be a positive sequence parametrized by $\alpha > 0$ and converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta_c, K(\beta_c))$. We assume hypotheses (i) and (ii) of Theorem 3.1, hypotheses (iii)(a) and (iv) of that theorem for all $0 < \alpha < \alpha_0$ and the inequality $0 < \theta\alpha_0 < 1/2$. Then for all $0 < \alpha < \alpha_0$, $S_n/n^{1-\theta\alpha}$ satisfies the MDP with respect to P_{n,β_n,K_n} with exponential speed $n^{1-\alpha/\alpha_0}$ and rate function $\Gamma(x) = g(x) - \inf_{y \in \mathbb{R}} g(y)$.*

The MDP in Theorem 7.1 is proved exactly like the MDP in part (a) of Theorem 8.1 in [9] with only changes in notation. Rather than repeat the proof, we motivate the MDP via the related Laplace principle. Given $\gamma \in (0, 1/2)$ and $u \in (0, 1)$, we say that with respect to P_{n,β_n,K_n} , $S_n/n^{1-\gamma}$ satisfies the Laplace principle with exponential speed n^u and rate function Γ if for any bounded, continuous function ψ

$$\lim_{n \rightarrow \infty} \frac{1}{n^u} \log \int_{\Lambda^n} \exp[n^u \psi(S_n/n^{1-\gamma})] dP_{n,\beta_n,K_n} = \sup_{x \in \mathbb{R}} \{\psi(x) - \Gamma(x)\}.$$

By Theorem 1.2.3 in [11], if $S_n/n^{1-\gamma}$ satisfies the Laplace principle with exponential speed n^u and rate function Γ , then $S_n/n^{1-\gamma}$ satisfies the MDP with the same exponential speed and the same rate function.

Under the hypotheses of Theorem 7.1 we now motivate the Laplace principle for $S_n/n^{1-\theta\alpha}$ with exponential speed $n^{1-\alpha/\alpha_0}$ and thus the MDP stated in that theorem. The main ideas are only sketched because full details of the proof of an analogous Laplace principle are given in the proof of Theorem 8.1 in [9]. Fix $u \in (0, 1)$. If b_n and c_n are two positive sequences, then we write $b_n \asymp c_n$ if

$$\lim_{n \rightarrow \infty} \frac{1}{n^u} \log b_n = \lim_{n \rightarrow \infty} \frac{1}{n^u} \log c_n.$$

We need the following lemma. It can be proved like Lemma 3.3 in [15], which applies to the Curie–Weiss model, or like Lemma 3.2 in [17], which applies to the Curie–Weiss–Potts model. In an equivalent form, the next lemma is well known in the literature as the Hubbard–Stratonovich transformation, where it is invoked to analyze models with quadratic Hamiltonians (see, e.g., [1], page 2363). The following lemma is also used in the proof of Theorem 4.2 in the next section.

LEMMA 7.2. *Given a positive sequence (β_n, K_n) , let W_n be a sequence of normal random variables with mean 0 and variance $(2\beta_n K_n)^{-1}$ defined on a probability space (Ω, \mathcal{F}, Q) . Then for any $\bar{\gamma} \in [0, 1)$ and any bounded, continuous function f ,*

$$\begin{aligned} (7.3) \quad & \int_{\Lambda^n \times \Omega} f(S_n/n^{1-\bar{\gamma}} + W_n/n^{1/2-\bar{\gamma}}) d(P_{n,\beta_n,K_n} \times Q) \\ &= \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^{\bar{\gamma}})] dx} \cdot \int_{\mathbb{R}} f(x) \exp[-nG_{\beta_n,K_n}(x/n^{\bar{\gamma}})] dx. \end{aligned}$$

Let ψ be any bounded, continuous function. We start our motivation of the proof of the Laplace principle for $S_n/n^{1-\theta\alpha}$ with exponential speed $n^{1-\alpha/\alpha_0}$ by substituting $\bar{\gamma} = \theta\alpha$ and $f = \exp(n^{1-\alpha/\alpha_0}\psi)$ into (7.3), obtaining

$$\begin{aligned} (7.4) \quad & \int_{\Lambda^n \times \Omega} \exp[n^u \psi(S_n/n^{1-\gamma} + W_n/n^{1/2-\gamma})] d(P_{n,\beta_n,K_n} \times Q) \\ &= \frac{1}{Z_{n,\gamma}} \cdot \int_{\mathbb{R}} \exp[n^u \{\psi(x) - n^{1-u} G_{\beta_n,K_n}(x/n^\gamma)\}] dx. \end{aligned}$$

In order to simplify the notation, we have written γ in place of $\theta\alpha$ and u in place of $1 - \alpha/\alpha_0$. In the last display $Z_{n,\gamma}$ is the normalization equal to

$$(7.5) \quad Z_{n,\gamma} = \int_{\mathbb{R}} \exp[-nG_{\beta_n, K_n}(x/n^\gamma)] dx.$$

Let us suppose that the limit of n^{-u} times the logarithm of the right-hand side of (7.4) exists. We then claim that since $0 < \alpha < \alpha_0$, the term $W_n/n^{1/2-\gamma}$ does not contribute to the asymptotic behavior of the left-hand side of (7.4). From this claim it follows that if the limit of n^{-u} times the logarithm of the right-hand side exists, then

$$(7.6) \quad \begin{aligned} & \int_{\Lambda^n} \exp[n^u \psi(S_n/n^{1-\gamma})] dP_{n,\beta_n, K_n} \\ & \asymp \frac{1}{Z_{n,\gamma}} \cdot \int_{\mathbb{R}} \exp[n^u \{\psi(x) - n^{1-u} G_{\beta_n, K_n}(x/n^\gamma)\}] dx. \end{aligned}$$

As on page 543 of [9], we justify the claim by showing that $W_n/n^{1/2-\gamma}$ is superexponentially small relative to n^u [11], Theorem 1.3.3. This holds provided $1 - 2\gamma = 1 - 2\theta\alpha > u = 1 - \alpha/\alpha_0$, which is valid since $0 < \theta\alpha_0 < 1/2$. This completes our justification of the claim.

We continue our motivation of the Laplace principle for $S_n/n^{1-\gamma}$. The uniform convergence of $n^{1-u}G_{\beta_n, K_n}(x/n^\gamma)$ to $g(x)$ in hypothesis (iii)(a) of Theorem 3.1 suggests that

$$(7.7) \quad \begin{aligned} & \int_{\Lambda^n} \exp[n^u \psi(S_n/n^{1-\gamma})] dP_{n,\beta_n, K_n} \\ & \asymp \frac{1}{Z_{n,\gamma}} \cdot \int_{\mathbb{R}} \exp[n^u \{\psi(x) - n^{1-u} G_{\beta_n, K_n}(x/n^\gamma)\}] dx \\ & \asymp \frac{1}{\int_{\mathbb{R}} \exp[-n^u g(x)] dx} \cdot \int_{\mathbb{R}} \exp[n^u \{\psi(x) - g(x)\}] dx. \end{aligned}$$

The proof of this asymptotic relationship is based on hypothesis (iii)(a) of Theorem 3.1 for $0 < \alpha < \alpha_0$, which states that $n^{1-u}G_{\beta_n, K_n}(x/n^\gamma) = n^{\alpha/\alpha_0}G_{\beta_n, K_n}(x/n^{\theta\alpha})$ converges to $g(x)$ uniformly on compact sets, and on several other steps, which depend in part on the lower bound in hypothesis (iv) of Theorem 3.1 for $0 < \alpha < \alpha_0$.

We define $\bar{g} = \inf_{y \in \mathbb{R}} g(y)$. According to Laplace's method, the asymptotic behavior of the integrals in the last line of (7.7) is governed by the maximum values of the respective integrands. Hence

$$(7.8) \quad \begin{aligned} & \int_{\mathbb{R}} \exp[n^u \{\psi(x) - n^{1-u} G_{\beta_n, K_n}(x/n^\gamma)\}] dx \\ & \asymp \int_{\mathbb{R}} \exp[n^u \{\psi(x) - g(x)\}] dx \\ & \asymp \exp\left[n^u \cdot \sup_{x \in \mathbb{R}} \{\psi(x) - g(x)\}\right] \end{aligned}$$

and

$$\begin{aligned}
 Z_{n,\gamma} &= \int_{\mathbb{R}} \exp[n^u \{-n^{1-u} G_{\beta_n, K_n}(x/n^\gamma)\}] dx \\
 (7.9) \quad &\asymp \int_{\mathbb{R}} \exp[-n^u g(x)] dx \\
 &\asymp \exp[-n^u \cdot \inf_{y \in \mathbb{R}} g(y)] = \exp[-n^u \bar{g}].
 \end{aligned}$$

Combining these two asymptotic relationships gives

$$\begin{aligned}
 &\int_{\Lambda^n} \exp[n^u \psi(S_n/n^{1-\gamma})] dP_{n,\beta_n, K_n} \\
 &\asymp \frac{1}{Z_{n,\gamma}} \int_{\mathbb{R}} \exp[n^u \{\psi(x) - n^{1-u} G_{\beta_n, K_n}(x/n^\gamma)\}] dx \\
 &\asymp \exp\left[n^u \cdot \sup_{x \in \mathbb{R}} \{\psi(x) - (g(x) - \bar{g})\}\right].
 \end{aligned}$$

These calculations complete the motivation that $S_n/n^{1-\gamma} = S_n/n^{1-\theta\alpha}$ satisfies the Laplace principle and thus the MDP with exponential speed $n^u = n^{1-\alpha/\alpha_0}$ and rate function $\Gamma(x) = g(x) - \bar{g}$.

The hypotheses of the MDP in Theorem 7.1 are the hypotheses of Theorem 4.1 except for hypothesis (iii')(b). We now bring in that hypothesis, which states that the set of global minimum points of the Ginzburg–Landau polynomial equals $\{\pm\bar{x}\}$ for some $\bar{x} > 0$. In conjunction with the MDP we use this hypothesis to prove part (a) of Theorem 4.1. The next step in that proof is contained in the following corollary, which states that the sequence of P_{n,β_n, K_n} -distributions of $S_n/n^{1-\theta\alpha}$ converges weakly to a symmetric sum of point masses at \bar{x} and $-\bar{x}$. This is almost immediate because up to an additive constant the rate function in the MDP equals g , and so the P_{n,β_n, K_n} -distributions of $S_n/n^{1-\theta\alpha}$ put an exponentially small mass on the complement of any open set containing the global minimum points $\pm\bar{x}$ of the rate function.

We saw in the last section that the hypotheses of Theorem 4.1 are valid for all six sequences defined in equations (5.1)–(5.6) except for case (d) of sequence 4, which is defined for $\ell = \ell_c$ and suitable values of $\bar{\ell}$. As noted in the discussion leading up to Theorem 5.4, when $\ell = \ell_c$, the set of global minimum points of g equals $\{0, \pm\bar{x}\}$ for some $\bar{x} > 0$. We are currently investigating the form of the weak limit replacing (7.10) in the next corollary when the set of global minimum points of g has this form. The conjecture is that in this case there exists $0 < \lambda < 1/2$ such that

$$P_{n,\beta_n, K_n}\{S_n/n^{1-\theta\alpha} \in dx\} \implies ((1 - 2\lambda)\delta_0 + \lambda\delta_{\bar{x}} + \lambda\delta_{-\bar{x}})(dx).$$

By the uniform integrability proved in Lemma 7.4, this weak limit, if true, would imply that

$$\lim_{n \rightarrow \infty} E_{n,\beta_n, K_n}\{|S_n/n^{1-\theta\alpha}|\} = 2\lambda\bar{x}.$$

COROLLARY 7.3. *Let (β_n, K_n) be a positive sequence parametrized by $\alpha > 0$ and converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta_c, K(\beta_c))$. We assume the hypotheses of Theorem 4.1. Then for all $0 < \alpha < \alpha_0$ we have the weak limit*

$$(7.10) \quad P_{n,\beta_n,K_n}\{S_n/n^{1-\theta\alpha} \in dx\} \implies (\frac{1}{2}\delta_{\bar{x}} + \frac{1}{2}\delta_{-\bar{x}})(dx),$$

where $\{\pm\bar{x}\}$ is the set of global minimum points of g as specified in hypothesis (iii')(b) of Theorem 4.1.

PROOF. We write γ for $\theta\alpha$ and u for $1 - \alpha/\alpha_0$. Since $0 < \alpha < \alpha_0$, we have $0 < u < 1$, and so $n^u \rightarrow \infty$ as $n \rightarrow \infty$. Let $\varepsilon > 0$ be given. There exists $M > 0$ such that the rate function $\Gamma(x) = g(x) - \inf_{y \in \mathbb{R}} g(y)$ in the MDP in Theorem 7.1 is an increasing function on the interval $[M, \infty)$ and $\Gamma(M) > 0$. Hence the moderate deviation upper bound and symmetry imply that

$$P_{n,\beta_n,K_n}\{|S_n/n^{1-\gamma}| \geq M\} \leq \exp[-n^u \Gamma(M)/2].$$

It follows that for all sufficiently large n , $P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \notin [-2M, 2M]\} \leq \varepsilon$. Thus the distributions $P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in dx\}$ are tight, and any subsequence has a weakly convergent subsubsequence [21], Theorem 1, Section III.2. We now apply the moderate deviation upper bound to any closed set F in \mathbb{R} not containing the global minimum points $\pm\bar{x}$ of Γ . Since $\Gamma(F) > 0$, we have for all sufficiently large n

$$P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in F\} \leq \exp[-n^u \Gamma(F)/2] \rightarrow 0.$$

Thus by symmetry, any subsequence of $P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in dx\}$ has a subsubsequence converging weakly to $(\frac{1}{2}\delta_{\bar{x}} + \frac{1}{2}\delta_{-\bar{x}})(dx)$. This yields the weak limit in (7.10). \square

We are now ready to prove part (a) of Theorem 4.1. If the sequence $S_n/n^{1-\theta\alpha}$ is uniformly integrable [3], Theorem 5.4, then by integrating both sides of (7.10) with respect to the absolute value function, we obtain for all $0 < \alpha < \alpha_0$

$$\lim_{n \rightarrow \infty} E_{n,\beta_n,K_n}\{|S_n/n^{1-\theta\alpha}|\} = \bar{x}.$$

This assertion is part (a) of Theorem 4.1. The required uniform integrability is proved in the next lemma from the MDP in Theorem 7.1.

LEMMA 7.4. *The random variables $S_n/n^{1-\theta\alpha}$ in Corollary 7.3 are uniformly integrable with respect to P_{n,β_n,K_n} ; that is,*

$$\lim_{M \rightarrow \infty} \sup_{n \in \mathbb{N}} E_{n,\beta_n,K_n}\{|S_n/n^{1-\theta\alpha}| \cdot 1_{\{|S_n/n^{1-\theta\alpha}| \geq M\}}\} = 0.$$

PROOF. We write γ for $\theta\alpha$ and u for $1 - \alpha/\alpha_0$. Γ denotes the rate function $g - \inf_{y \in \mathbb{R}} g(y)$ in the MDP in Theorem 7.1. Since $0 < \alpha < \alpha_0$, we have $0 < u < 1$, and so $n^u \rightarrow \infty$ as $n \rightarrow \infty$. Let $\varepsilon > 0$ be given. Since g is a polynomial and $g(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, $\exists M_0 \in (0, \infty)$ such that

$$\inf_{|x| \geq M_0} \Gamma(x) \geq g(M_0)/2 > 0 \quad \text{and} \quad \exp\left[-\frac{1}{8}g(M_0)\right] \leq \varepsilon.$$

The MDP in Theorem 7.1 implies that for all $M \geq M_0$ there exists $N_0 \in \mathbb{N}$ depending only on M_0 such that for all $n \geq N_0$

$$\begin{aligned} P_{n, \beta_n, K_n} \{|S_n/n^{1-\gamma}| \geq M\} &\leq P_{n, \beta_n, K_n} \{|S_n/n^{1-\gamma}| \geq M_0\} \\ &\leq \exp\left[-\frac{1}{2}n^u \inf_{\{|x| \geq M_0\}} \Gamma(x)\right]. \end{aligned}$$

Since $|S_n| \leq n$, it follows that for all $M \geq M_0$ and for all $n \geq N_0$

$$\begin{aligned} E_{n, \beta_n, K_n} \{|S_n/n^{1-\gamma}| \cdot 1_{\{|S_n/n^{1-\gamma}| \geq M\}}\} &\leq E_{n, \beta_n, K_n} \{|S_n/n^{1-\gamma}| \cdot 1_{\{|S_n/n^{1-\gamma}| \geq M_0\}}\} \\ &\leq n^\gamma \cdot \exp\left[-\frac{1}{2}n^u \inf_{|x| \geq M_0} \Gamma(x)\right] \\ &\leq n^\gamma \cdot \exp\left[-\frac{1}{4}n^u g(M_0)\right]. \end{aligned}$$

There exists $N_1 \geq N_0$ such that for all $n \geq N_1$, $n^\gamma \cdot \exp[-\frac{1}{8}n^u g(M_0)] \leq 1$. Hence for all $M \geq M_0$ and for all $n \geq N_1$,

$$E_{n, \beta_n, K_n} \{|S_n/n^{1-\gamma}| \cdot 1_{\{|S_n/n^{1-\gamma}| \geq M\}}\} \leq \exp[-\frac{1}{8}n^u g(M_0)] \leq \exp[-\frac{1}{8}g(M_0)] \leq \varepsilon,$$

which implies that for all $M \geq M_0$

$$\sup_{n \geq N_1} E_{n, \beta_n, K_n} \{|S_n/n^{1-\gamma}| \cdot 1_{\{|S_n/n^{1-\gamma}| \geq M\}}\} \leq \varepsilon.$$

In addition,

$$\begin{aligned} \max_{1 \leq n < N_1} E_{n, \beta_n, K_n} \{|S_n/n^{1-\gamma}| \cdot 1_{\{|S_n/n^{1-\gamma}| \geq M\}}\} \\ \leq N_1^\gamma \cdot \max_{1 \leq n < N_1} P_{n, \beta_n, K_n} \{|S_n/n^{1-\gamma}| \geq M\} \rightarrow 0 \quad \text{as } M \rightarrow \infty. \end{aligned}$$

The last two displays complete the proof of the desired uniform integrability. The proof of part (a) of Theorem 4.1 is complete. \square

In the next section we prove part (a) of Theorem 4.2. This theorem gives the asymptotics of $E_{n, \beta_n, K_n} \{|S_n/n^{1-\theta\alpha_0}|\}$ when the quantity α parametrizing the sequence (β_n, K_n) exceeds α_0 .

8. Proof of part (a) of Theorem 4.2. Under the assumption that the quantity α parametrizing the sequence (β_n, K_n) exceeds α_0 , part (a) of Theorem 4.2 states that

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{\theta\alpha_0} E_{n, \beta_n, K_n} \{|S_n/n|\} \\ &= \lim_{n \rightarrow \infty} E_{n, \beta_n, K_n} \{|S_n/n^{1-\theta\alpha_0}|\} \\ &= \bar{y} = \frac{1}{\int_{\mathbb{R}} \exp[-\tilde{g}(x)] dx} \cdot \int_{\mathbb{R}} |x| \exp[-\tilde{g}(x)] dx. \end{aligned}$$

Let Π_n and Π denote the probability measures on \mathbb{R} defined by

$$(8.1) \quad \Pi_n(dx) = \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n, K_n}(x/n^{\theta\alpha_0})] dx} \cdot \exp[-nG_{\beta_n, K_n}(x/n^{\theta\alpha_0})] dx$$

and

$$(8.2) \quad \Pi(x) = \frac{1}{\int_{\mathbb{R}} \exp[-\tilde{g}(x)] dx} \cdot \exp[-\tilde{g}(x)] dx.$$

The quantity \bar{y} can be written in terms of Π as $\int_{\mathbb{R}} |x| d\Pi$. Part (a) of Theorem 4.2 is proved in two steps, the weak-convergence limit in step 1 and the uniform-integrability-type limit in Proposition 8.3 that yields step 2.

Step 1. Prove that the sequence Π_n and the sequence of P_{n, β_n, K_n} -distributions of $S_n/n^{1-\theta\alpha_0}$ both converge weakly to Π ; that is, for any bounded, continuous function f ,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f d\Pi_n = \int_{\mathbb{R}} f d\Pi$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} E_{n, \beta_n, K_n} \{f(S_n/n^{1-\theta\alpha_0})\} &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(S_n/n^{1-\theta\alpha_0}) dP_{n, \beta_n, K_n} \\ &= \int_{\mathbb{R}} f d\Pi. \end{aligned}$$

Step 2. Prove

$$\begin{aligned} (8.3) \quad \lim_{n \rightarrow \infty} E_{n, \beta_n, K_n} \{|S_n/n^{1-\theta\alpha_0}|\} &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |S_n/n^{1-\theta\alpha_0}| dP_{n, \beta_n, K_n} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |x| d\Pi_n = \bar{y} \\ &= \int_{\mathbb{R}} |x| d\Pi. \end{aligned}$$

The key is to approximate the unbounded function $|x|$ by the sequence of bounded, continuous functions $f_j(x) = \min\{|x|, j\}$. The limits in the last display are a consequence of the limits in step 1 and the uniform-integrability-type limit in Proposition 8.3.

The proof of the weak-convergence limit in step 1 is given in the next theorem.

THEOREM 8.1. *We assume the hypotheses of Theorem 4.2. Then for all $\alpha > \alpha_0$ the following conclusions hold:*

- (a) *The sequence Π_n defined in (8.1) converges weakly to the probability measure Π defined in (8.2).*
- (b) *The P_{n,β_n,K_n} -distributions of $S_n/n^{1-\theta\alpha_0}$ converges weakly to Π .*

The proof of this theorem relies on the following technical lemma, which is proved in part (c) of Lemma 4.4 in [9].

LEMMA 8.2. *Let (β_n, K_n) be a positive sequence parametrized by $\alpha > 0$ and converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. Assume that there exists $\bar{\gamma} > 0$ and $\bar{R} > 0$ such that the sequence*

$$\xi_n = \int_{\{|x| < \bar{R}n^{\bar{\gamma}}\}} \exp[-nG_{\beta_n, K_n}(x/n^{\bar{\gamma}})] dx$$

is bounded. Then there exist constants $c_1 > 0$ and $c_2 > 0$ such that for all sufficiently large n

$$\int_{\{|x| \geq \bar{R}n^{\bar{\gamma}}\}} \exp[-nG_{\beta_n, K_n}(x/n^{\bar{\gamma}})] dx \leq c_1 \exp[-c_2n] \rightarrow 0.$$

We now prove Theorem 8.1.

PROOF OF THEOREM 8.1. We write γ_0 for $\theta\alpha_0$. The proof follows the same pattern as the proof of Theorem 6.1 in [9]. The starting point is Lemma 7.2 with $\bar{\gamma} = \gamma_0 = \theta\alpha_0$. That lemma states that for any bounded, continuous function f

$$\begin{aligned} & \int_{\Lambda^n \times \Omega} f(S_n/n^{1-\gamma_0} + W_n/n^{1/2-\gamma_0}) d(P_{n,\beta_n,K_n} \times Q) \\ (8.4) \quad &= \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx} \\ & \times \int_{\mathbb{R}} f(x) \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx, \end{aligned}$$

where W_n is a sequence of normal random variables with mean 0 and variance $(2\beta_n K_n)^{-1}$. Suppose that the limit of the right-hand side of (8.4) equals $\int_{\mathbb{R}} f d\Pi$. Since by hypothesis $0 < \gamma_0 = \theta\alpha_0 < 1/2$, rewriting the limit of the left-hand side of (8.4) in terms of characteristic functions shows that the term $W_n/n^{1/2-\gamma_0}$ does not contribute to this limit. It follows that if the limit of the right-hand side of (8.4)

equals $\int_{\mathbb{R}} f d\Pi$, then

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \int_{\Lambda^n} f(S_n/n^{1-\gamma_0}) dP_{n, \beta_n, K_n} \\
 (8.5) \quad &= \lim_{n \rightarrow \infty} \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx} \\
 & \quad \times \int_{\mathbb{R}} f(x) \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx \\
 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f d\Pi_n = \int_{\mathbb{R}} f d\Pi.
 \end{aligned}$$

In order to calculate the limit of the sequence $\int_{\mathbb{R}} f d\Pi_n$, we appeal to the point-wise convergence of $nG_{\beta_n, K_n}(x/n^{\gamma_0})$ to $\tilde{g}(x)$ in hypotheses (v) of Theorem 4.2 and the lower bound in hypotheses (iv) of Theorem 3.1 for $\alpha = \alpha_0$. This states that there exists a polynomial H satisfying $H(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ together with the following property: $\exists R > 0$ such that $\forall n \in \mathbb{N}$ sufficiently large and $\forall x \in \mathbb{R}$ satisfying $|x/n^{\gamma_0}| < R$, $nG_{\beta_n, K_n}(x/n^{\gamma_0}) \geq H(x)$. We then use the integrability of $\exp[-H]$ and the dominated convergence theorem to write

$$\begin{aligned}
 (8.6) \quad & \lim_{n \rightarrow \infty} \int_{\{|x| < Rn^{\gamma_0}\}} f(x) \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx \\
 &= \int_{\mathbb{R}} f(x) \exp[-\tilde{g}(x)] dx.
 \end{aligned}$$

In order to handle the integrals over the complementary sets $\{|x| \geq Rn^{\gamma_0}\}$, we appeal to Lemma 8.2, for which we must verify the hypothesis. Setting $f \equiv 1$ in (8.6), we see that

$$\lim_{n \rightarrow \infty} \int_{\{|x| < Rn^{\gamma_0}\}} \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx = \int_{\mathbb{R}} \exp[-\tilde{g}(x)] dx$$

and thus that the sequence $\int_{\{|x| < Rn^{\gamma_0}\}} \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx$ is bounded. Lemma 8.2 with $\bar{\gamma} = \gamma_0$ and $\bar{R} = R$ implies that

$$\lim_{n \rightarrow \infty} \int_{\{|x| \geq Rn^{\gamma_0}\}} \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx = 0.$$

It follows that

$$(8.7) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx = \int_{\mathbb{R}} f(x) \exp[-\tilde{g}(x)] dx$$

and

$$(8.8) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx = \int_{\mathbb{R}} \exp[-\tilde{g}(x)] dx.$$

Substituting into (8.5) the limits in the last two displays yields the weak convergence asserted in parts (a) and (b) of Theorem 8.1:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(S_n/n^{1-\gamma_0}) dP_{n,\beta_n,K_n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^{\gamma_0})] dx} \cdot \int_{\mathbb{R}} f(x) \exp[-nG_{\beta_n,K_n}(x/n^{\gamma_0})] dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f d\Pi_n = \frac{1}{\int_{\mathbb{R}} \exp[-\tilde{g}(x)] dx} \cdot \int_{\mathbb{R}} f(x) \exp[-\tilde{g}(x)] dx = \int_{\mathbb{R}} f d\Pi. \end{aligned}$$

The proof of the theorem is complete. \square

We now turn to the proof of the limit in (8.3) in step 2, writing γ_0 for $\theta\alpha_0$. The proof depends on an appropriate asymptotic formula for $E_{n,\beta_n,K_n}\{|S_n/n^{1-\gamma_0}|\}$, which we derive from Lemma 7.2. In that lemma let $\tilde{\gamma} = \gamma_0$, let the bounded, continuous function f equal $f_j(x) = \min\{|x|, j\}$, and send $j \rightarrow \infty$. The monotone convergence theorem implies that

$$\begin{aligned} & \int_{\Lambda^n \times \Omega} |S_n/n^{1-\gamma_0} + W_n/n^{1/2-\gamma_0}| d(P_{n,\beta_n,K_n} \times Q) \\ (8.9) \quad &= \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^{\gamma_0})] dx} \cdot \int_{\mathbb{R}} |x| \exp[-nG_{\beta_n,K_n}(x/n^{\gamma_0})] dx \\ &= \int_{\mathbb{R}} |x| d\Pi_n. \end{aligned}$$

In this formula W_n is a sequence of normal random variables with mean 0 and variance $(2\beta_n K_n)^{-1}$ defined on a probability space (Ω, \mathcal{F}, Q) , and Π_n is the probability measure defined in (8.1).

We write $\tilde{E}_{n,\beta_n,K_n}$ to denote expectation with respect to the product measure $P_{n,\beta_n,K_n} \times Q$. Since $(\beta_n, K_n) \rightarrow (\beta, K(\beta))$, there exists a positive constant c such that for all $n \in \mathbb{N}$,

$$\tilde{E}_{n,\beta_n,K_n}\{|W_n/n^{1/2-\gamma_0}|\} \leq c/n^{1/2-\gamma_0}.$$

Thus

$$\begin{aligned} & \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\gamma_0} + W_n/n^{1/2-\gamma_0}|\} + c/n^{1/2-\gamma_0} \\ & \geq E_n\{|S_n/n^{1-\gamma_0}|\} \geq \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\gamma_0} + W_n/n^{1/2-\gamma_0}|\} - c/n^{1/2-\gamma_0}. \end{aligned}$$

Suppose that we could prove

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |x| d\Pi_n = \bar{y} = \int_{\mathbb{R}} |x| d\Pi.$$

Since $0 < \gamma_0 = \theta\alpha_0 < 1/2$, we would then obtain from (8.9) the desired limit

$$(8.10) \quad \begin{aligned} & \lim_{n \rightarrow \infty} E_{n, \beta_n, K_n} \{|S_n/n^{1-\gamma_0}|\} \\ &= \lim_{n \rightarrow \infty} \tilde{E}_{n, \beta_n, K_n} \{|S_n/n^{1-\gamma_0} + W_n/n^{1/2-\gamma_0}|\} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |x| d\Pi_n = \bar{y} = \int_{\mathbb{R}} |x| d\Pi. \end{aligned}$$

We complete the proof of part (a) of Theorem 4.2 by showing the limit in the last line of (8.10). Part (a) of Theorem 8.1 shows that the sequence Π_n converges weakly to Π . According to a standard result, the limit in the last line of (8.10) would follow immediately from the weak convergence of Π_n to Π if we could prove that Π_n satisfies the following uniform-integrability estimate:

$$\lim_{j \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|x| > j\}} |x| d\Pi_n = 0.$$

The next proposition shows that the limit in the last line of (8.10) is a consequence of a condition that is weaker than uniform integrability.

PROPOSITION 8.3. *Let $\tilde{\Pi}_n$ be a sequence of probability measures on \mathbb{R} that converges weakly to a probability measure $\tilde{\Pi}$ on \mathbb{R} . Assume in addition that $\int_{\mathbb{R}} |x| d\tilde{\Pi} < \infty$ and that*

$$(8.11) \quad \lim_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\{|x| > j\}} |x| d\tilde{\Pi}_n = 0.$$

It then follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |x| d\tilde{\Pi}_n = \int_{\mathbb{R}} |x| d\tilde{\Pi}.$$

PROOF. For $j \in \mathbb{N}$, f_j denotes the bounded, continuous function that equals $|x|$ for $|x| \leq j$ and equals j for $|x| > j$. Then

$$\begin{aligned} & \left| \int_{\mathbb{R}} |x| d\tilde{\Pi}_n - \int_{\mathbb{R}} |x| d\tilde{\Pi} \right| \\ & \leq \int_{\mathbb{R}} ||x| - f_j| d\tilde{\Pi}_n + \left| \int_{\mathbb{R}} f_j d\tilde{\Pi}_n - \int_{\mathbb{R}} f_j d\tilde{\Pi} \right| + \int_{\mathbb{R}} ||x| - f_j| d\tilde{\Pi} \\ & \leq 2 \int_{\{|x| > j\}} |x| d\tilde{\Pi}_n + \left| \int_{\mathbb{R}} f_j d\tilde{\Pi}_n - \int_{\mathbb{R}} f_j d\tilde{\Pi} \right| + 2 \int_{\{|x| > j\}} |x| d\tilde{\Pi}. \end{aligned}$$

Since $\tilde{\Pi}_n \Rightarrow \tilde{\Pi}$, we have $\int_{\mathbb{R}} f_j d\tilde{\Pi}_n \rightarrow \int_{\mathbb{R}} f_j d\tilde{\Pi}$, and therefore

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}} |x| d\tilde{\Pi}_n - \int_{\mathbb{R}} |x| d\tilde{\Pi} \right| \\ & \leq 2 \limsup_{n \rightarrow \infty} \int_{\{|x| > j\}} |x| d\tilde{\Pi}_n + 2 \int_{\{|x| > j\}} |x| d\tilde{\Pi}. \end{aligned}$$

By the assumptions on $\tilde{\Pi}_n$ and $\tilde{\Pi}$, both terms on the right-hand side of this inequality converge to 0 as $j \rightarrow \infty$. This completes the proof. \square

In order to justify the limit in the last line of (8.10), we must verify the hypotheses of Proposition 8.3 for the measures Π_n and Π defined in (8.1) and (8.2). Clearly the measure Π defined in (8.2) satisfies $\int_{\mathbb{R}} |x| d\Pi < \infty$. We now verify the condition in (8.11) for the measures Π_n defined in (8.1). For any $j \in \mathbb{N}$ and all sufficiently large n we will find quantities A_j , B_n and C_n with the following properties:

$$\int_{\{|x|>j\}} |x| d\Pi_n \leq A_j + B_n + C_n,$$

$A_j \rightarrow 0$ as $j \rightarrow \infty$, $B_n \rightarrow 0$ as $n \rightarrow \infty$ and $C_n \rightarrow 0$ as $n \rightarrow \infty$. It follows from these properties that

$$(8.12) \quad \lim_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\{|x|>j\}} |x| d\Pi_n \leq \lim_{j \rightarrow \infty} A_j + \lim_{n \rightarrow \infty} B_n + \lim_{n \rightarrow \infty} C_n = 0.$$

This yields the limit in (8.11), proving step 2 and thus completing the proof of part (a) of Theorem 4.2.

We now specify the quantities A_j , B_n , and C_n having the properties in the preceding paragraph. Given positive integers j and n , let R and c be positive numbers that satisfy $c > R$ and that will be specified below. We then partition the set $\{|x| > j\}$ into the following three subsets:

$$\begin{aligned} \{|x| > j\} &= [\{|x| > j\} \cap \{|x/n^{\gamma_0}| < R\}] \\ &\cup [\{|x| > j\} \cap \{R \leq |x/n^{\gamma_0}| < c\}] \cup [\{|x| > j\} \cap \{|x/n^{\gamma_0}| \geq c\}]. \end{aligned}$$

Since for all n

$$\{|x| > j\} \subset [\{|x| > j\} \cap \{|x/n^{\gamma_0}| < R\}] \cup \{R \leq |x/n^{\gamma_0}| < c\} \cup \{|x/n^{\gamma_0}| \geq c\},$$

it follows that for all n

$$(8.13) \quad \begin{aligned} \int_{\{|x|>j\}} |x| d\Pi_n &\leq \int_{\{|x|>j\} \cap \{|x/n^{\gamma_0}|<R\}} |x| d\Pi_n + \int_{\{R \leq |x/n^{\gamma_0}| < c\}} |x| d\Pi_n \\ &\quad + \int_{\{|x/n^{\gamma_0}| \geq c\}} |x| d\Pi_n. \end{aligned}$$

We next estimate each of these three integrals. The convergence proved in (8.8) implies that the sequence $1/Z_n$ is positive and bounded. By hypothesis (iv) of Theorem 3.1 for $\alpha = \alpha_0$ there exists $R > 0$ such that for all sufficiently large $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ satisfying $|x/n^{\gamma_0}| < R$

$$nG_{\beta_n, K_n}(x/n^{\gamma_0}) \geq H(x),$$

where H is a polynomial satisfying $H(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. Since $\exp[-H(x)]$ is integrable, for all sufficiently large n we estimate the first integral on the right-hand side of (8.13) by

$$(8.14) \quad \int_{\{|x|>j\} \cap \{|x/n^{\gamma_0}|<R\}} |x| d\Pi_n \leq A_j = \text{const} \cdot \int_{\{|x|>j\}} |x| \exp[-H(x)] dx \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

By part (a) of Lemma 4.4 in [9], there exists $c > 0$ and $D > 0$ such that $G_{\beta_n, K_n}(x) \geq Dx^2$ for all $|x| \geq c$; thus for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ satisfying $|x/n^{\gamma_0}| \geq c$,

$$nG_{\beta_n, K_n}(x/n^{\gamma_0}) \geq nD(x/n^{\gamma_0})^2 = n^{1-2\gamma_0} Dx^2.$$

Without loss of generality c can be chosen to be larger than the quantity R specified in the preceding paragraph. Since the sequence $1/Z_n$ is bounded, we estimate the third integral on the right-hand side of (8.13) by

$$(8.15) \quad \int_{\{|x/n^{\gamma_0}| \geq c\}} |x| d\Pi_n \leq C_n = \frac{1}{Z_n} \cdot \int_{\{|x/n^{\gamma_0}| \geq c\}} |x| \exp[-n^{1-2\gamma_0} Dx^2] dx \leq \text{const} \cdot n^{2\gamma_0-1} \cdot \exp[-nc^2D] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

With these choices of R and c we estimate the middle integral on the right-hand side of (8.13) by

$$\int_{\{R \leq |x/n^{\gamma_0}| < c\}} |x| d\Pi_n \leq B_n = cn^{\gamma_0} \cdot \Pi_n\{|x/n^{\gamma_0}| \geq R\} = cn^{\gamma_0} \cdot \frac{1}{Z_n} \cdot \int_{\{|x/n^{\gamma_0}| \geq R\}} \exp[-nG_{\beta_n, K_n}(x/n^{\gamma_0})] dx.$$

Since the sequence $1/Z_n$ is bounded, the display after (8.6) and Lemma 8.2 with $\bar{R} = R$ and $\bar{\gamma} = \gamma_0$ imply the existence of constants $c_1 > 0$ and $c_2 > 0$ such that

$$(8.16) \quad \int_{\{R \leq |x/n^{\gamma_0}| < c\}} |x| d\Pi_n \leq B_n \leq cn^{\gamma_0} \cdot \text{const} \cdot c_1 \exp[-c_2n] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Together, equations (8.14), (8.16) and (8.15) prove (8.12), which completes the proof of part (a) of Theorem 4.2. \square

REFERENCES

- [1] ANTONI, M. and RUFFO, S. (1995). Clustering and relaxation in Hamiltonian long-range dynamics. *Phys. Rev. E* **52** 2361–2374.
- [2] BARBER, M. N. (1983). Finite-size scaling. In *Phase Transitions and Critical Phenomena* (C. Domb and J. Lebowitz, eds.) **8** 145–266. Academic Press, London. [MR794318](#)
- [3] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. [MR0233396](#)
- [4] BLUME, M. (1966). Theory of the first-order magnetic phase change in UO_2 . *Phys. Rev.* **141** 517–524.
- [5] BLUME, M., EMERY, V. J. and GRIFFITHS, R. B. (1971). Ising model for the λ transition and phase separation in He^3 - He^4 mixtures. *Phys. Rev. A* **4** 1071–1077.
- [6] CAPEL, H. W. (1966). On the possibility of first-order phase transitions in Ising systems of triplet ions with zero-field splitting. *Physica* **32** 966–988.
- [7] CAPEL, H. W. (1967). On the possibility of first-order phase transitions in Ising systems of triplet ions with zero-field splitting. II. *Physica* **33** 295–331.
- [8] CAPEL, H. W. (1967). On the possibility of first-order phase transitions in Ising systems of triplet ions with zero-field splitting. III. *Physica* **37** 423–441.
- [9] COSTENIUC, M., ELLIS, R. S. and OTTO, P. T.-H. (2007). Multiple critical behavior of probabilistic limit theorems in the neighborhood of a tricritical point. *J. Stat. Phys.* **127** 495–552. [MR2316196](#)
- [10] DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. *Applications of Mathematics (New York)* **38**. Springer, New York. [MR1619036](#)
- [11] DUPUIS, P. and ELLIS, R. S. (1997). *A Weak Convergence Approach to the Theory of Large Deviations*. Wiley, New York. [MR1431744](#)
- [12] ELLIS, R. S. (1985). *Entropy, Large Deviations, and Statistical Mechanics. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **271**. Springer, New York. Reprinted in *Classics in Mathematics* in 2006. [MR2189669](#)
- [13] ELLIS, R. S., MACHTA, J. and OTTO, P. T.-H. (2008). Asymptotic behavior of the magnetization near critical and tricritical points via Ginzburg–Landau polynomials. *J. Stat. Phys.* **133** 101–129. [MR2438899](#)
- [14] ELLIS, R. S., MACHTA, J. and OTTO, P. T.-H. (2008). Ginzburg–Landau polynomials and the asymptotic behavior of the magnetization near critical and tricritical points. Unpublished manuscript. Available at <http://arxiv.org/abs/0803.0178>.
- [15] ELLIS, R. S. and NEWMAN, C. M. (1978). Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Z. Wahrsch. Verw. Gebiete* **44** 117–139. [MR0503333](#)
- [16] ELLIS, R. S., OTTO, P. T. and TOUCHETTE, H. (2005). Analysis of phase transitions in the mean-field Blume–Emery–Griffiths model. *Ann. Appl. Probab.* **15** 2203–2254. [MR2152658](#)
- [17] ELLIS, R. S. and WANG, K. (1990). Limit theorems for the empirical vector of the Curie–Weiss–Potts model. *Stochastic Process. Appl.* **35** 59–79. [MR1062583](#)
- [18] FORTUIN, C. M. and KASTELEYN, P. W. (1972). On the random-cluster model. I. Introduction and relation to other models. *Physica* **57** 536–564. [MR0359655](#)
- [19] GRIMMETT, G. (2006). *The Random-Cluster Model. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **333**. Springer, Berlin. [MR2243761](#)
- [20] PLISCHKE, M. and BERGERSEN, B. (2006). *Equilibrium Statistical Physics*, 3rd ed. World Scientific, Hackensack, NJ. [MR2238776](#)
- [21] SHIRYAEV, A. N. (1996). *Probability*, 2nd ed. Translated by R. P. Boas. *Graduate Texts in Mathematics* **95**. Springer, New York. [MR1368405](#)

- [22] STANLEY, H. E. (1971). *Introduction to Phase Transitions and Critical Phenomena*. Oxford Univ. Press, New York.

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