Continuous symmetry breaking in a mean-field model

Theodor Eisele† and Richard S Ellis‡§

† Institut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 294, D-6900 Heidelberg 1, BRD
‡ Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts, 01003, USA

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Abstract. A magnetic system on the sites \( \{j/n; j = 1, \ldots, n\} \) of the circle \( T = \mathbb{R} \) (mod 1) is studied in the limit \( n \to \infty \). The interaction is defined in terms of a continuous function \( J(x, y), x, y \in T \). For any ferromagnetic \( J(J > 0) \) which satisfies a normalisation condition, the thermodynamic behaviour is identical to that of the Curie-Weiss model \( (J = 1) \). This simple case is in contrast to the behaviour for a class of translation invariant, non-ferromagnetic \( J \), for which a continuum of equilibrium states exists for sufficiently low temperatures. In both cases a probabilistic interpretation of the equilibrium states is given.

1. Introduction

For each \( n \in \{1, 2, \ldots\} \) we define a magnetic system on the circle \( T = \mathbb{R} \) (mod 1). Let \( J(x, y) \) be an arbitrary continuous function of \( x, y \in T \) and let \( \beta > 0 \) denote the inverse absolute temperature. Our model is defined by the partition functions

\[
Z(n, \beta) = \sum_{\sigma_1, \ldots, \sigma_n} \exp \left[ \frac{\beta}{2n} \sum_{j,k=1}^n J\left( \frac{j}{n}, \frac{k}{n} \right) \sigma_j \sigma_k \right].
\]  

(1)

Each \( \sigma_j \) denotes the spin at the site \( j/n \), and the exponent in (1) equals \( -\beta \) times the energy of the spin configuration \( (\sigma_1, \ldots, \sigma_n) \). The case \( J = 1 \) defines the well known Curie-Weiss (or mean field) model (Thompson 1972, § 4.5).

In this paper we contrast the relatively simple thermodynamic behaviour for ferromagnetic \( J \) \( (J > 0) \) with the much more complicated behaviour for a class of translation invariant, non-ferromagnetic \( J \). The latter are given by

\[
J(x, y) = -b + \nu \cos(2\pi p(x - y))
\]

(2)

for some \( b \geq 0, \nu \neq 0, \) and \( p \in \{1, 2, \ldots\} \). If \( b \) exceeds \( |\nu| \), then \( J \) in (2) is antiferromagnetic \( (J \leq 0) \). Basically, in the thermodynamic limit the general ferromagnetic case behaves exactly like the Curie-Weiss model while in the non-ferromagnetic case we have continuous symmetry breaking. Full details plus generalisations are given in Eisele and Ellis (1981).

In § 2 we describe a Gibbs variational formula for the specific free energy for general \( J \). Sections 3 and 4 list the equilibrium states and give a probabilistic interpretation of these states in the non-ferromagnetic and ferromagnetic cases, respectively.

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The probabilistic interpretation involves spin random variables \( \sigma_1^{(n)}, \ldots, \sigma_n^{(n)} \). These are defined by the joint density

\[
P_{n,\beta}\{\sigma_1^{(n)} = \sigma_1, \ldots, \sigma_n^{(n)} = \sigma_n\} \doteq \frac{\exp\left(\frac{\beta}{2n} \sum_{j,k=1}^n J(j/n, k/n)\sigma_j\sigma_k\right)}{Z(n, \beta)},
\]

where \((\sigma_1, \ldots, \sigma_n)\) is any configuration of spins. The density (3) defines the Gibbs measure corresponding to the partition function \(Z(n, \beta)\). The probabilistic behaviour of the Curie–Weiss model has been studied extensively in Ellis and Newman (1978a, b) and Ellis et al (1980).

2. Gibbs variational formula

The specific free energy \( \psi(\beta) \) is defined by the formula

\[
-\beta \psi(\beta) \doteq \lim_{n \to \infty} \frac{1}{n} \log Z(n, \beta).
\]

Our first result is a variational formula for \( \psi(\beta) \). We define \( B \) to be the space of measurable functions \( f \) on \( T \) for which

\[
-1 \leq \text{ess inf } f \leq \text{ess sup } f \leq 1,
\]

where ess inf and ess sup denote essential infimum and essential supremum, respectively. For each \( f \in B \) we define the functional

\[
u(f) = -\frac{1}{2} \int_T \int_T J(x, y)f(x)f(y) \, dx \, dy.
\]

Let \( i(z) \) denote the non-negative, strictly convex function

\[
i(z) = \begin{cases} 
\frac{1}{2}(1+z) \log(1+z) + \frac{1}{2}(1-z) \log(1-z) & \text{if } |z| \leq 1, \\
\infty & \text{if } |z| > 1,
\end{cases}
\]

and \( s(f) \) the functional

\[
s(f) = -\int_T i(f(x)) \, dx.
\]

**Theorem 1.** Let \( J(x, y) \) be an arbitrary continuous function of \( x, y \in T \). Then for \( \beta > 0 \)

\[
\psi(\beta) = \inf\{\nu(f) - (1/\beta)s(f): f \in B\}.
\]

We think of \( B \) as the set of all possible states of the system in the thermodynamic limit. Then in the state \( f \), \( u(f) \) gives the energy, \( s(f) \) the entropy, and \( u(f) - \beta^{-1}s(f) \) the free energy.

**Definition 2.** A function \( \tilde{f} \in B \) is called an equilibrium state at inverse temperature \( \beta \) if

\[
u(\tilde{f}) - (1/\beta)s(\tilde{f}) = \inf\{\nu(f) - (1/\beta)s(f): f \in B\}.
\]

We denote by \( G(\beta) \) the set of all equilibrium states at inverse temperature \( \beta \).
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In order to motivate the results that follow, we point out that the equilibrium states in the limits \( \beta \uparrow \infty \) and \( \beta \downarrow 0 \) are easy to find explicitly. By theorem 1 the totally ordered states, which are defined to be the equilibrium states in the limit \( \beta \uparrow \infty \), are the functions \( \tilde{f} \) which minimise \( u(f) \). For any \( J > 0 \), we have \( \tilde{f} = 1 \) or \( \tilde{f} = -1 \). Now let \( J \) be given by (2) and define the function

\[
   g(x) = \begin{cases} 
   1 & \text{if } \cos(2\pi px) > 0, \\
   -1 & \text{if } \cos(2\pi px) < 0, \\
   0 & \text{if } \cos(2\pi px) = 0.
   \end{cases}
\]

Then either \( \tilde{f}(x) = g(x), x \in T \), or (since \( J \) is translation-invariant) \( \tilde{f}(x) = g(x + \lambda) \), where the phase shift \( \lambda \) is some number in \( T \). Thus for \( J \) given by (2) we have continuous symmetry breaking in the limit \( \beta \uparrow \infty \). In the limit \( \beta \downarrow 0 \), (9) does not make sense, but it is consistent with (9) to define the equilibrium states to be the functions \( \tilde{f} \) which maximise \( s(f) \). Since \( s(f) \) is non-positive for all \( f \), we have \( \tilde{f} = 0 \) for any \( J \).

3. Non-ferromagnetic \( J \)

Let \( J \) be given by (2). We first describe \( G(\beta) \) for all \( \beta > 0 \). For each \( \beta > 2/|\nu| \), one checks that the equation

\[
   \mu = \int_T \cos(2\pi px) \tanh[\beta \nu \mu \cos(2\pi px)] \, dx
\]

has a unique positive root \( \mu = \mu(\beta, \nu, p) \). We define

\[
   g_\beta(x) = \tanh[\beta \nu \mu \cos(2\pi px)], \quad x \in T.
\]

This is an odd function of \( \cos(2\pi px) \), and so it has the same periodicity properties as \( J \).

**Theorem 3.** For \( J \) given by (2),

\[
   G(\beta) = \begin{cases} 
   \{0\} & \text{if } 0 < \beta \leq 2/|\nu|, \\
   \{g_\beta(\lambda) ; \lambda \in T\} & \text{if } \beta > 2/|\nu|.
   \end{cases}
\]

Theorem 3 is consistent with the discussion at the end of § 2 since \( g_\beta \) in (13) tends to the function \( g \) in (11) as \( \beta \) tends to \( \infty \).

For the probabilistic interpretation, we recall the spin random variables \( \sigma_1^{(n)}, \ldots, \sigma_n^{(n)} \) with joint density (3). Given an interval \( \Delta \) on \( T \), we define the total spin in \( \Delta \), \( W_n(\Delta) \), by the formula

\[
   W_n(\Delta) = \frac{1}{|\Delta|} \sum_{i/n \in \Delta} \sigma_i^{(n)},
\]

where \( |\Delta| \) denotes the Lebesgue measure of \( \Delta \). If \( \Delta \) is all of \( T \), then we write \( W_n \) instead of \( W_n(\Delta) \). We consider a global law of large numbers and local laws of large numbers for the spin. The former describes the limiting distribution of the total spin in \( T \), \( W_n/n \), as \( n \to \infty \). The latter describe the limiting joint distribution of the vector of local spins \( \{W_n(\Delta_1)/n, \ldots, W_n(\Delta_r)/n\} \), where \( \Delta_1, \ldots, \Delta_r \) are \( r \) intervals in \( T \) (\( r \in \{1, 2, \ldots\} \)). Although the global law follows from the local laws for \( r = 1, \Delta_1 = T \), it is useful to discuss both. We write \( E_{n,\beta}\{-\} \) for the expectation with respect to the measure \( P_{n,\beta} \) in (3).
Theorem 4. Let $J$ be given by (2). Then for any continuous function $h$ mapping $\mathbb{R}$ to $\mathbb{R}$, we have
\[
\lim_{n \to \infty} E_{n, \beta} \left\{ h \left( \frac{W_n}{n} \right) \right\} = h(0) \text{ for all } \beta > 0. \tag{16}
\]
More generally, for any $r \in \{1, 2, \ldots\}$, any $r$ intervals $\Delta_1, \ldots, \Delta_r$ in $T$, and any continuous function $h$ mapping $\mathbb{R}'$ to $\mathbb{R}$,
\[
\lim_{n \to \infty} E_{n, \beta} \left\{ h \left( W_n(\Delta_1)/n, \ldots, W_n(\Delta_r)/n \right) \right\} \tag{17}
\]
\[
= \begin{cases} 
  h(0) & \text{if } 0 < \beta \leq 2/|\nu|, \\
  \int_T h(g_\beta(\lambda; \Delta_1), \ldots, g_\beta(\lambda; \Delta_r)) \, d\lambda & \text{if } \beta > 2/|\nu|.
\end{cases}
\]
Here $0$ is the constant vector $(0, \ldots, 0) \in \mathbb{R}'$ and $g_\beta(\lambda; \Delta_i)$ is defined as $|\Delta_i|^{-1} \int_{\Delta_i} g_\beta(x + \lambda) \, dx$.

In order to interpret the limit (17), we assume that each $\Delta_i$ is a small interval with centre $x_i \in T$. Then for $\beta > 2/|\nu|$, the right-hand side of (17) is close to $\int_T h[g_\beta(x_1 + \lambda), \ldots, g_\beta(x + \lambda)] \, d\lambda$. The latter is the expectation of the random variable $h[g_\beta(x_1 + \lambda(\omega), \ldots, g_\beta(x + \lambda(\omega))]$, where $\lambda(\omega)$ is a random phase shift, uniformly distributed in $T$. Theorem 4 implies that for all $\beta > 0$ we have zero magnetisation per site as $n \to \infty$ (because of (16)) but for $\beta > 2/|\nu|$ the spins cluster into $2p$ alternating islands of plus spins and minus spins as $n \to \infty$. The spins are described locally by a wave with shape $g_\beta$ but with random phase shift.

4. Ferromagnetic $J$

We assume that $J(x, y) > 0$ is a continuous function of $x, y \in T$ which satisfies the normalisation conditions
\[
\int_T J(x, y) \, dy = 1 = \int_T J(x, y) \, dx \quad \text{for each } x, y \in T. \tag{18}
\]
We show that the thermodynamic behaviour for such $J$ is identical to that for the case $J = 1$, which defines the Curie–Weiss model.

For $\beta > 1$ the Curie–Weiss model exhibits spontaneous magnetisation. The value of the latter is a number $m(\beta)$ which is the unique positive solution of the equation
\[
\tanh(\beta m) = m. \tag{19}
\]
For $0 < \beta \leq 1$, there is no spontaneous magnetisation.

The next two theorems are the analogues of theorems 3 and 4, respectively. We write $1$ for the constant function $1$ on $T$.

Theorem 5. Let $J(x, y) > 0$ be a continuous function of $x, y \in T$ which satisfies (18). We have
\[
G(\beta) = \begin{cases} 
  \{0\} & \text{if } 0 < \beta \leq 1, \\
  \{m(\beta)1, -m(\beta)1\} & \text{if } \beta > 1.
\end{cases} \tag{20}
\]
Theorem 6. Let \( J \) be as in theorem 5. Then for any continuous function \( h \) mapping \( \mathbb{R} \) to \( \mathbb{R} \)

\[
\lim_{n \to \infty} E_{n, \beta} \{ h \left( \frac{W_n}{n} \right) \} = \begin{cases} h(0) & \text{if } 0 < \beta \leq 1, \\ \frac{1}{2} [h(m(\beta)) + h(-m(\beta))] & \text{if } \beta > 1. \end{cases}
\]

More generally, for any \( r \in \{1, 2, \ldots\} \), any \( r \) intervals \( \Delta_1, \ldots, \Delta_r \) in \( T \), and any continuous function \( h \) mapping \( \mathbb{R}' \) to \( \mathbb{R} \),

\[
\lim_{n \to \infty} E_{n, \beta} \{ h \left( \frac{W_n(\Delta_1)}{n} \right), \ldots, \left( \frac{W_n(\Delta_r)}{n} \right) \} = \begin{cases} h(0) & \text{if } 0 < \beta \leq 1, \\ \frac{1}{2} [h(m(\beta)) + h(-m(\beta))] & \text{if } \beta > 1. \end{cases}
\]

Here \( m(\beta) \) is the constant vector \((m(\beta), \ldots, m(\beta)) \in \mathbb{R}'\).

We refer to the states \( m(\beta)1 \) and \(-m(\beta)1\) in (20) as the plus state and the minus state, respectively. In contrast to the situation in theorem 4, theorem 6 shows that for ferromagnetic interactions, the local structure of both the plus state and the minus state completely mimics the global structure.

References