J. Phys. A: Math. Gen. 16 (1983) 195–199. Printed in Great Britain

Continuous symmetry breaking in a mean-field model

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Received 14 June 1982

Abstract. A magnetic system on the sites $\{j/n; j=1,\ldots,n\}$ of the circle $T \doteq \mathbb{R}$ (mod 1) is studied in the limit $n \to \infty$. The interaction is defined in terms of a continuous function $J(x, y)$, $x, y \in T$. For any ferromagnetic J ($J>0$) which satisfies a normalisation condition, the thermodynamic behaviour is identical to that of the Curie–Weiss model ($J \equiv 1$). This simple case is in contrast to the behaviour for a class of translation invariant, non-ferromagnetic J, for which a continuum of equilibrium states exists for sufficiently low temperatures. In both cases a probabilistic interpretation of the equilibrium states is given.

1. Introduction

For each $n \in \{1, 2, \ldots\}$ we define a magnetic system on the circle $T \doteq \mathbb{R}$ (mod 1). Let $J(x, y)$ be an arbitrary continuous function of $x, y \in T$ and let $\beta > 0$ denote the inverse absolute temperature. Our model is defined by the partition functions

$$Z(n, \beta) \doteq \sum_{\sigma_1,\ldots,\sigma_n=\pm 1} \exp\left[\frac{\beta}{2n} \sum_{j,k=1}^{n} J\left(\frac{j}{n}, \frac{k}{n}\right)\sigma_j\sigma_k\right]. \tag{1}$$

Each σ_j denotes the spin at the site j/n, and the exponent in (1) equals $-\beta$ times the energy of the spin configuration $(\sigma_1, \ldots, \sigma_n)$. The case $J \equiv 1$ defines the well known Curie–Weiss (or mean field) model (Thompson 1972, § 4.5).

In this paper we contrast the relatively simple thermodynamic behaviour for ferromagnetic J ($J>0$) with the much more complicated behaviour for a class of translation invariant, non-ferromagnetic J. The latter are given by

$$J(x, y) = -b + \nu \cos(2\pi p(x-y)) \tag{2}$$

for some $b \geq 0$, $\nu \neq 0$, and $p \in \{1, 2, \ldots\}$. If b exceeds $|\nu|$, then J in (2) is antiferromagnetic ($J \leq 0$). Basically, in the thermodynamic limit the general ferromagnetic case behaves exactly like the Curie–Weiss model while in the non-ferromagnetic case we have continuous symmetry breaking. Full details plus generalisations are given in Eisele and Ellis (1981).

In § 2 we describe a Gibbs variational formula for the specific free energy for general J. Sections 3 and 4 list the equilibrium states and give a probabilistic interpretation of these states in the non-ferromagnetic and ferromagnetic cases, respectively.

§ Research supported by National Science Foundation Grant MCS80–02149.

The probabilistic interpretation involves spin random variables $\bar{\sigma}_1^{(n)}, \ldots, \bar{\sigma}_n^{(n)}$. These are defined by the joint density

$$P_{n,\beta}\{\bar{\sigma}_1^{(n)}=\sigma_1,\ldots,\bar{\sigma}_n^{(n)}=\sigma_n\} \doteq \left[\exp\left(\frac{\beta}{2n}\sum_{j,k=1}^{n} J(j/n,k/n)\sigma_j\sigma_k\right)\right]\Big/Z(n,\beta), \tag{3}$$

where $(\sigma_1, \ldots, \sigma_n)$ is any configuration of spins. The density (3) defines the Gibbs measure corresponding to the partition function $Z(n, \beta)$. The probabilistic behaviour of the Curie–Weiss model has been studied extensively in Ellis and Newman (1978a, b) and Ellis *et al* (1980).

2. Gibbs variational formula

The specific free energy $\psi(\beta)$ is defined by the formula

$$-\beta\psi(\beta) \doteq \lim_{n\to\infty}\frac{1}{n}\log Z(n,\beta). \tag{4}$$

Our first result is a variational formula for $\psi(\beta)$. We define B to be the space of measurable functions f on T for which

$$-1 \leqslant \operatorname{ess\,inf} f \leqslant \operatorname{ess\,sup} f \leqslant 1, \tag{5}$$

where ess inf and ess sup denote essential infimum and essential supremum, respectively. For each $f \in B$ we define the functional

$$u(f) \doteq -\tfrac{1}{2}\int_T\int_T J(x,y)f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{6}$$

Let $i(z)$ denote the non-negative, strictly convex function

$$i(z) \doteq \begin{cases} \tfrac{1}{2}(1+z)\log(1+z)+\tfrac{1}{2}(1-z)\log(1-z) & \text{if } |z|\leqslant 1,\\ \infty & \text{if } |z|>1, \end{cases} \tag{7}$$

and $s(f)$ the functional

$$s(f) \doteq -\int_T i(f(x))\,\mathrm{d}x. \tag{8}$$

Theorem 1. Let $J(x, y)$ be an arbitrary continuous function of $x, y \in T$. Then for $\beta > 0$

$$\psi(\beta) = \inf\{u(f) - (1/\beta)s(f) : f \in B\}. \tag{9}$$

We think of B as the set of all possible states of the system in the thermodynamic limit. Then in the state f, $u(f)$ gives the energy, $s(f)$ the entropy, and $u(f) - \beta^{-1}s(f)$ the free energy.

Definition 2. A function $\bar{f} \in B$ is called an equilibrium state at inverse temperature β if

$$u(\bar{f}) - (1/\beta)s(\bar{f}) = \inf\{u(f) - (1/\beta)s(f) : f \in B\}. \tag{10}$$

We denote by $G(\beta)$ the set of all equilibrium states at inverse temperature β.

In order to motivate the results that follow, we point out that the equilibrium states in the limits $\beta \uparrow \infty$ and $\beta \downarrow 0$ are easy to find explicitly. By theorem 1 the totally ordered states, which are defined to be the equilibrium states in the limit $\beta \uparrow \infty$, are the functions $\tilde{f}$ which minimise $u(f)$. For any $J > 0$, we have $\tilde{f} \equiv 1$ or $\tilde{f} \equiv -1$. Now let J be given by (2) and define the function

$$g(x) \doteq \begin{cases} 1 & \text{if } \cos(2\pi px) > 0, \\ -1 & \text{if } \cos(2\pi px) < 0, \\ 0 & \text{if } \cos(2\pi px) = 0. \end{cases} \tag{11}$$

Then either $\tilde{f}(x) = g(x)$, $x \in T$, or (since J is translation-invariant) $\tilde{f}(x) = g(x + \lambda)$, where the phase shift λ is some number in T. Thus for J given by (2) we have continuous symmetry breaking in the limit $\beta \uparrow \infty$. In the limit $\beta \downarrow 0$, (9) does not make sense, but it is consistent with (9) to define the equilibrium states to be the functions $\tilde{f}$ which maximise $s(f)$. Since $s(f)$ is non-positive for all f, we have $\tilde{f} \equiv 0$ for any J.

3. Non-ferromagnetic J

Let J be given by (2). We first describe $G(\beta)$ for all $\beta > 0$. For each $\beta > 2/|\nu|$, one checks that the equation

$$\mu = \int_T \cos(2\pi px) \tanh[\beta\nu\mu \cos(2\pi px)] \, dx \tag{12}$$

has a unique positive root $\mu = \mu(\beta, \nu, p)$. We define

$$g_\beta(x) \doteq \tanh[\beta\nu\mu \cos(2\pi px)], \qquad x \in T. \tag{13}$$

This is an odd function of $\cos(2\pi px)$, and so it has the same periodicity properties as J.

Theorem 3. For J given by (2),

$$G(\beta) = \begin{cases} \{0\} & \text{if } 0 < \beta \leq 2/|\nu|, \\ \{g_\beta(+\lambda); \lambda \in T\} & \text{if } \beta > 2/|\nu|. \end{cases} \tag{14}$$

Theorem 3 is consistent with the discussion at the end of § 2 since g_β in (13) tends to the function g in (11) as β tends to ∞.

For the probabilistic interpretation, we recall the spin random variables $\tilde{\sigma}_1^{(n)}, \ldots, \tilde{\sigma}_n^{(n)}$ with joint density (3). Given an interval Δ on T, we define the total spin in Δ, $W_n(\Delta)$, by the formula

$$W_n(\Delta) \doteq \frac{1}{|\Delta|} \sum_{\{j:\, j/n \in \Delta\}} \tilde{\sigma}_j^{(n)}, \tag{15}$$

where $|\Delta|$ denotes the Lebesgue measure of Δ. If Δ is all of T, then we write W_n instead of $W_n(T)$. We consider a global law of large numbers and local laws of large numbers for the spin. The former describes the limiting distribution of the total spin in T, W_n/n, as $n \to \infty$. The latter describe the limiting joint distribution of the vector of local spins $(W_n(\Delta_1)/n, \ldots, W_n(\Delta_r)/n)$, where $\Delta_1, \ldots, \Delta_r$ are r intervals in T ($r \in \{1, 2, \ldots\}$). Although the global law follows from the local laws for $r \doteq 1$, $\Delta_1 \doteq T$, it is useful to discuss both. We write $E_{n,\beta}\{-\}$ for the expectation with respect to the measure $P_{n,\beta}$ in (3).

Theorem 4. Let J be given by (2). Then for any continuous function h mapping $\mathbb{R}$ to $\bar{\mathbb{R}}$, we have

$$\lim_{n\to\infty} E_{n,\beta}\left\{h\left(\frac{W_n}{n}\right)\right\} = h(0) \text{ for all } \beta > 0. \tag{16}$$

More generally, for any $r \in \{1, 2, \ldots\}$. any r intervals $\Delta_1, \ldots, \Delta_r$ in T, and any continuous function h mapping $\mathbb{R}^r$ to $\mathbb{R}$,

$$\lim_{n\to\infty} E_{n,\beta}\{h(W_n(\Delta_1)/n, \ldots, W_n(\Delta_r)/n)\} \tag{17}$$

$$= \begin{cases} h(\mathbf{0}) & \text{if } \quad 0 < \beta \leqslant 2/|\nu|, \\ \displaystyle\int_T h(g_\beta(\lambda\,;\Delta_1), \ldots, g_\beta(\lambda\,;\Delta_r))\,\mathrm{d}\lambda & \text{if } \quad \beta > 2/|\nu|. \end{cases}$$

Here $\mathbf{0}$ is the constant vector $(0, \ldots, 0) \in \mathbb{R}^r$ and $g_\beta(\lambda\,;\Delta_j)$ is defined as $|\Delta_j|^{-1}\int_{\Delta_j} g_\beta(x+\lambda)\,\mathrm{d}x$.

In order to interpret the limit (17), we assume that each Δ_j is a small interval with centre $x_j \in T$. Then for $\beta > 2/|\nu|$, the right-hand side of (17) is close to $\int_T h[g_\beta(x_1+\lambda), \ldots, g_\beta(x_r+\lambda)]\,\mathrm{d}\lambda$. The latter is the expectation of the random variable $h[g_\beta(x_1+\lambda(\omega)), \ldots, g_\beta(x_r+\lambda(\omega))]$, where $\lambda(\omega)$ is a random phase shift, uniformly distributed in T. Theorem 4 implies that for all $\beta > 0$ we have zero magnetisation per site as $n \to \infty$ (because of (16)) but for $\beta > 2/|\nu|$ the spins cluster into $2p$ alternating islands of plus spins and minus spins as $n \to \infty$. The spins are described locally by a wave with shape g_β but with random phase shift.

4. Ferromagnetic J

We assume that $J(x, y) > 0$ is a continuous function of $x, y \in T$ which satisfies the normalisation conditions

$$\int_T J(x, y)\mathrm{d}y = 1 = \int_T J(x, y)\,\mathrm{d}x \qquad \text{for each } x, y \in T. \tag{18}$$

We show that the thermodynamic behaviour for such J is identical to that for the case $J \equiv 1$, which defines the Curie–Weiss model.

For $\beta > 1$ the Curie–Weiss model exhibits spontaneous magnetisation. The value of the latter is a number $m(\beta)$ which is the unique positive solution of the equation

$$\tanh(\beta m) = m. \tag{19}$$

For $0 < \beta \leqslant 1$, there is no spontaneous magnetisation.

The next two theorems are the analogues of theorems 3 and 4, respectively. We write $\mathbf{1}$ for the constant function 1 on T.

Theorem 5. Let $J(x, y) > 0$ be a continuous function of $x, y \in T$ which satisfies (18). We have

$$G(\beta) = \begin{cases} \{0\} & \text{if } 0 < \beta \leqslant 1, \\ \{m(\beta)\mathbf{1}, \ -m(\beta)\mathbf{1}\} & \text{if } \beta > 1. \end{cases} \tag{20}$$

Theorem 6. Let J be as in theorem 5. Then for any continuous function h mapping $\mathbb{R}$ to $\mathbb{R}$

$$\lim_{n\to\infty} E_{n,\beta}\{h(W_n/n)\} = \begin{cases} h(0) & \text{if } 0<\beta\leqslant 1, \\ \tfrac{1}{2}[h(m(\beta))+h(-m(\beta))] & \text{if } \beta>1. \end{cases} \tag{21}$$

More generally, for any $r\in\{1, 2, \ldots\}$, any r intervals $\Delta_1, \ldots, \Delta_r$ in T, and any continuous function h mapping $\mathbb{R}^r$ to $\mathbb{R}$,

$$\lim_{n\to\infty} E_{n,\beta}\{h(W_n(\Delta_1)/n), \ldots, (W_n(\Delta_r)/n)\}$$
$$= \begin{cases} h(\mathbf{0}) & \text{if } 0<\beta\leqslant 1, \\ \tfrac{1}{2}[h(\boldsymbol{m}(\beta))+h(-\boldsymbol{m}(\beta))] & \text{if } \beta>1. \end{cases} \tag{22}$$

Here $\boldsymbol{m}(\beta)$ is the constant vector $(m(\beta), \ldots, m(\beta))\in\mathbb{R}^r$.

We refer to the states $m(\beta)\mathbf{1}$ and $-m(\beta)\mathbf{1}$ in (20) as the plus state and the minus state, respectively. In contrast to the situation in theorem 4, theorem 6 shows that for ferromagnetic interactions, the local structure of both the plus state and the minus state completely mimics the global structure.

References

Eisele T and Ellis R S 1981 *Symmetry breaking and random waves for magnetic systems on a circle.* Univ. Heidelberg/Univ. Massachusetts preprint *Z. Wahrscheinlichkeitsth. u. verw. Geb.* (to appear)

Ellis R S and Newman C M 1978a *Z. Wahrscheinlichkeitsth. u. verw. Geb.* **44** 117–39

—— 1978b *J. Stat. Phys.* **19** 149–61

Ellis R S, Newman C M and Rosen J S 1980 *Z. Wahrscheinlichkeitsth. u. verw. Geb.* **51** 153–69

Thompson C 1972 *Mathematical Statistical Mechanics* (New York: Macmillan) § 4.5