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Conditional Gaussian Fluctuations and Refined Asymptotics of the Spin in the Phase-Coexistence Region

Richard S. Ellis · Jingran Li

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Abstract We present four results on the fluctuations of the spin per site around the thermodynamic magnetization in the mean-field Blume-Capel model. Our first two results refine the main theorem in a previous paper (Ellis et al. in Ann. Appl. Probab. 20:2118–2161, 2010), in which the first rigorous confirmation of the statistical mechanical theory of finite-size scaling for a mean-field model is given. Our first main result studies the asymptotics of the centered, finite-size magnetization, giving its precise rate of convergence to 0 along parameter sequences lying in the phase-coexistence region and converging sufficiently slowly to either a second-order point or the tricritical point of the model. A simple inequality yields our second main result, which generalizes the main theorem in Ellis et al. (Ann. Appl. Probab. 20:2118–2161, 2010) by giving an upper bound on the rate of convergence to 0 of the absolute value of the difference between the finite-size magnetization and the thermodynamic magnetization. These first two results have direct relevance to the theory of finite-size scaling. They are consequences of our third main result. This is a new conditional limit theorem for the spin per site, where the conditioning allows us to focus on a neighborhood of the pure states having positive thermodynamic magnetization. Our fourth main result is a conditional central limit theorem showing that the fluctuations of the spin per site are Gaussian in a neighborhood of the pure states having positive thermodynamic magnetization.

Keywords Finite-size magnetization · Thermodynamic magnetization · Second-order phase transition · First-order phase transition · Tricritical point · Conditional central limit theorem · Moderate deviation principle · Large deviation principle · Blume-Capel model · Finite-size scaling

R.S. Ellis (✉) · J. Li
Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003, USA
e-mail: rsellis@math.umass.edu

J. Li
e-mail: jingran@math.umass.edu

1 Introduction

The purpose of this paper is to analyze the asymptotic behavior of the fluctuations of the spin per site around the thermodynamic magnetization along parameter sequences having physical relevance in the mean-field Blume-Capel model. This analysis is presented in three limit theorems and an upper bound. Our research culminates a series of papers that study the phase-transition structure of the model via analytic techniques and probabilistic limit theorems [6, 8, 11, 12]. The mean-field Blume-Capel model is a mean-field version of an important lattice model due to Blume and Capel, to which we will refer as the B-C model [2–5]. The mean-field B-C model is an important object of study because it is one of the simplest models that exhibits the following complicated phase-transition structure: a curve of second-order points; a curve of first-order points; and a tricritical point, which separates the two curves.

Our first two main results refine the main theorem in [11]. The goal of [11] is to compare the asymptotic behaviors of the thermodynamic magnetization and the finite-size magnetization along parameter sequences lying in the phase coexistence region and converging to either a second-order point or the tricritical point of the mean-field B-C model. Theorem 4.1 in that paper shows that these two quantities are asymptotic when the parameter α controlling the speed at which the sequence approaches criticality is below a certain threshold α_0. However, when α exceeds α_0, the thermodynamic magnetization converges to 0 much faster than the finite-size magnetization. These results in [11] are the first rigorous confirmations of the statistical mechanical theory of finite-size scaling for a mean-field model [1], [11, §6].

The importance of both the theory of finite-size scaling and the mean-field B-C model motivate us in this paper to refine Theorem 4.1 in [11]. Our first main result studies the asymptotics, along appropriate parameter sequences, of the centered, finite-size magnetization for $0 < \alpha < \alpha_0$, obtaining its precise rate of convergence to 0. Our second main result applies a simple inequality that gives an upper bound on the rate of convergence to 0 of the absolute value of the difference between the finite-size magnetization and the thermodynamic magnetization. This upper bound yields the conclusion of Theorem 4.1 in [11] as a corollary and, along with the first main result, has additional relevance to the theory of finite-size scaling discussed below. Our first two main results are stated in (1.4) and (1.6). While Theorem 4.1 in [11] is obtained from a moderate deviation principle, the refinements of that theorem in this paper are obtained from our third main result, which is the conditional limit theorem stated in (1.7). Our fourth main result is a conditional central limit theorem stated in (1.9) and showing that the fluctuations of the spin per site are Gaussian in a neighborhood of the pure states having positive thermodynamic magnetization.

The mean-field B-C model is defined by a canonical ensemble that we denote by $P_{N,\beta,K}$; N is the number of vertices, $\beta > 0$ is the inverse temperature, and $K > 0$ is the interaction strength. $P_{N,\beta,K}$ is defined in (2.1) in terms of the Hamiltonian

$$H_{N,K}(\omega) = \sum_{j=1}^{N} \omega_j^2 - \frac{K}{N}\left(\sum_{j=1}^{N} \omega_j\right)^2.$$

In this formula ω_j is the spin at site $j \in \{1, 2, \ldots, N\}$ and takes values in $\Lambda = \{-1, 0, 1\}$. The configuration space for the model is the set Λ^N containing all sequences $\omega = (\omega_1, \ldots, \omega_N)$ with each $\omega_j \in \Lambda$. Expectation with respect to $P_{N,\beta,K}$ is denoted by $E_{N,\beta,K}$. The finite-size magnetization is defined by $E_{N,\beta,K}\{|S_N/N|\}$, where S_N equals the total spin $\sum_{j=1}^{N} \omega_j$.

Before discussing the results in this paper, we first summarize the phase-transition structure of the mean-field B-C model as derived in [12]. For $\beta > 0$ and $K > 0$, we denote by

$\mathcal{M}_{\beta,K}$ the set of equilibrium values of the magnetization. The set $\mathcal{M}_{\beta,K}$ coincides with the set of global minimum points of the free-energy function $G_{\beta,K}$, which is defined in (2.3)–(2.4). The critical inverse temperature of the mean-field B-C model is $\beta_c = \log 4$. For $0 < \beta \le \beta_c$ there exists a quantity $K(\beta)$ and for $\beta > \beta_c$ there exists a quantity $K_1(\beta)$ having the following properties. The positive quantity $m(\beta, K)$ appearing in this list is the thermodynamic magnetization.

1. Fix $0 < \beta \le \beta_c$. Then for $0 < K \le K(\beta)$, $\mathcal{M}_{\beta,K}$ consists of the unique pure phase 0, and for $K > K(\beta)$, $\mathcal{M}_{\beta,K}$ consists of two nonzero values $\pm m(\beta, K)$.
2. For $0 < \beta \le \beta_c$, $\mathcal{M}_{\beta,K}$ undergoes a continuous bifurcation at $K = K(\beta)$, changing continuously from $\{0\}$ for $K \le K(\beta)$ to $\{\pm m(\beta, K)\}$ for $K > K(\beta)$. This continuous bifurcation corresponds to a second-order phase transition.
3. Fix $\beta > \beta_c$. Then for $0 < K < K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of the unique pure phase 0; for $K = K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of 0 and two nonzero values $\pm m(\beta, K_1(\beta))$; and for $K > K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of two nonzero values $\pm m(\beta, K)$.
4. For $\beta > \beta_c$, $\mathcal{M}_{\beta,K}$ undergoes a discontinuous bifurcation at $K = K_1(\beta)$, changing discontinuously from $\{0\}$ for $K < K(\beta)$ to $\{0, \pm m(\beta, K)\}$ for $K = K_1(\beta)$ to $\{\pm m(\beta, K)\}$ for $K > K_1(\beta)$. This discontinuous bifurcation corresponds to a first-order phase transition.

Because of item 2, we refer to the curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$ as the second-order curve and points on this curve as second-order points. Because of item 4, we refer to the curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$ as the first-order curve and points on this curve as first-order points. The point $(\beta_c, K(\beta_c)) = (\log 4, 3/2 \log 4)$, called the tricritical point, separates the second-order curve from the first-order curve. The phase-coexistence region is defined as the set of all points in the positive β-K quadrant for which $\mathcal{M}_{\beta,K}$ consists of more than one value. Therefore the phase-coexistence region consists of all points above the second-order curve, above the tricritical point, on the first-order curve, and above the first-order curve; that is,

$$\big\{(\beta, K)\colon 0 < \beta \le \beta_c, K > K(\beta) \text{ and } \beta > \beta_c, K \ge K_1(\beta)\big\}.$$

Figure 1 exhibits the sets that describe the phase-transition structure of mean-field B-C model.

In order to discuss the contributions of this paper, it is helpful first to explain the main results in [8] and [11]. Those papers focus on positive sequences (β_n, K_n) that lie in the phase-coexistence region for all sufficiently large n, converge to either a second-order point or the tricritical point, and satisfy the hypotheses of Theorem 3.2 in [8]. These sequences are parameterized by $\alpha > 0$ in the sense that the limits

$$b = \lim_{n\to\infty} n^{\alpha}(\beta_n - \beta) \quad \text{and} \quad k = \lim_{n\to\infty} n^{\alpha}\big(K_n - K(\beta)\big)$$

are assumed to exist and are not both 0. Six specific such sequences are introduced in Sect. 4 of that paper. Theorem 3.2 in [8] states that for any $\alpha > 0$, $m(\beta_n, K_n)$ has the asymptotic behavior

$$m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}, \tag{1.1}$$

where $\theta > 0$ and $\bar{x}$ is the positive global minimum point of a certain polynomial $g(x)$ called the Ginzburg-Landau polynomial. This polynomial is defined in terms of the free-energy function $G_{\beta,K}$ in hypothesis (iii)(a) of Theorem 3.1 below.

One of the surprises in our study of the mean-field B-C model is the appearance of the Ginzburg-Landau polynomial in a number of basic results. These include the asymptotic

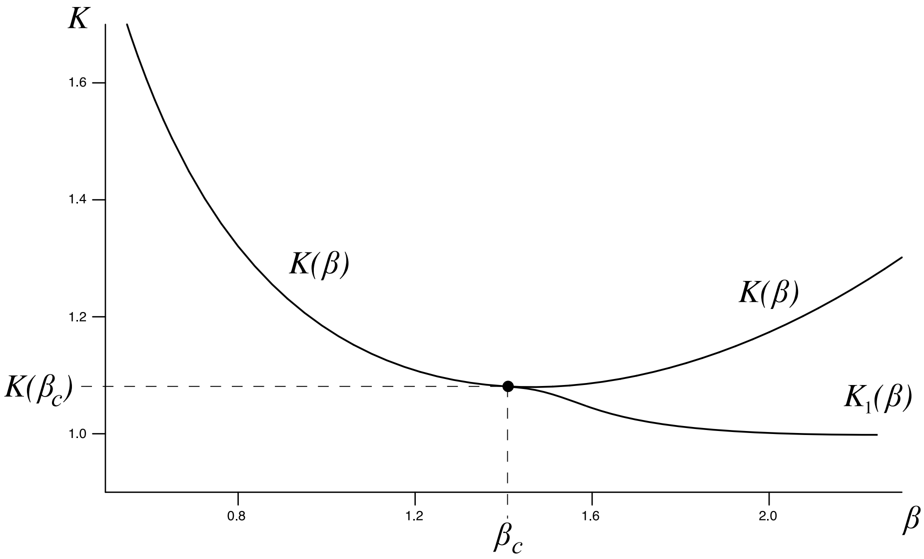

Fig. 1 The sets that describe the phase-transition structure of the mean-field B-C model: the second-order curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$, the first-order curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$, and the tricritical point $(\beta_c, K(\beta_c))$. The phase-coexistence region consists of all (β, K) above the second-order curve, above the tricritical point, on the first-order curve, and above the first-order curve

formula (1.1), the quantity $\bar{z}$ in the asymptotic formula (1.4) and the conditional limit theorem (1.7), the limiting variance in the conditional central limit theorem (1.9), and the rate function in the moderate deviation principle in Theorem 6.2.

A straightforward large-deviation calculation summarized in [11, p. 2120] shows that for fixed (β, K) lying in the phase-coexistence region the spin per site S_N/N has the weak-convergence limit

$$P_{N,\beta,K}\{S_N/N \in dx\} \quad \Longrightarrow \quad \left(\frac{1}{2}\delta_{m(\beta,K)} + \frac{1}{2}\delta_{-m(\beta,K)}\right)(dx). \tag{1.2}$$

This implies that

$$\lim_{N\to\infty} E_{N,\beta,K}\left\{|S_N/N|\right\} = m(\beta, K).$$

Because the thermodynamic magnetization $m(\beta, K)$ is the limit, as the number of spins goes to ∞, of the finite-size magnetization $E_{N,\beta,K}\{|S_N/N|\}$, the thermodynamic magnetization is a physically relevant estimator of the finite-size magnetization, at least when evaluated at fixed (β, K) in the phase-coexistence region.

The main focus of [11] is to determine whether the thermodynamic magnetization is a physically relevant estimator of the finite-size magnetization in a more general sense, namely, when evaluated along positive sequences that lie in the phase-coexistence region for all sufficiently large n, converge to a second-order point or the tricritical point, and satisfy a set of hypotheses including those of Theorem 3.2 in [8]. The criterion for determining whether $m(\beta_n, K_n)$ is a physically relevant estimator is that as $n \to \infty$, $m(\beta_n, K_n)$ is asymptotic to the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\}$, both of which converge to 0. In this formulation we let $N = n$ in the finite-size magnetization; i.e., we let the number of spins N coincide with the index n parametrizing the sequence (β_n, K_n).

As summarized in Theorems 4.1 and 4.2 in [11], the main result is that $m(\beta_n, K_n)$ is a physically relevant estimator when the parameter α controlling the speed at which (β_n, K_n) approaches criticality is below a certain threshold α_0. The value of α_0 depends on the type of the phase transition—first-order, second-order, or tricritical—that influences the sequence, an issue addressed in Section 6 of [11]. For $0 < \alpha < \alpha_0$ this result is summarized by the limit

$$\lim_{n\to\infty} n^{\theta\alpha} \left| E_{n,\beta_n,K_n}\left\{|S_n/n|\right\} - m(\beta_n, K_n)\right| = 0, \tag{1.3}$$

which in combination with (1.1) implies that

$$E_{n,\beta_n,K_n}\left\{|S_n/n|\right\} \sim \bar{x}/n^{\theta\alpha} \sim m(\beta_n, K_n).$$

By contrast, when $\alpha > \alpha_0$, $m(\beta_n, K_n)$ converges to 0 much faster than $E_{n,\beta_n,K_n}\{|S_n/n|\}$. The sequences for which these asymptotic results are valid include the six sequences introduced in [8, §4].

We now turn to the main focus of this paper, which is a refined analysis of the fluctuations of S_n/n around $m(\beta_n, K_n)$ for $0 < \alpha < \alpha_0$. This refined analysis is expressed in three limit theorems and an upper bound. Our first main result involves a quantity that we call the centered, finite-size magnetization. It is defined by $E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$. Let $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. For $0 < \alpha < \alpha_0$ and for a class of sequences that includes the first five sequences introduced in [8, §4] part (a) of Theorem 4.1 gives the following exact rate of convergence to 0 of the centered, finite-size magnetization as $n \to \infty$:

$$E_{n,\beta_n,K_n}\left\{\left||S_n/n| - m(\beta_n, K_n)\right|\right\} \sim \bar{z}/n^{\kappa}. \tag{1.4}$$

In this formula $\bar{z} = (2/[\pi g^{(2)}(\bar{x})])^{1/2}$, where $g^{(2)}(\bar{x})$ denotes the positive second derivative of the Ginzburg-Landau polynomial g evaluated at its unique positive global minimum point $\bar{x}$. For all $0 < \alpha < \alpha_0$, κ is larger than $\theta\alpha$. Thus the rate $\bar{z}/n^{\kappa}$ at which $E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$ converges to 0 is asymptotically faster than the rate $\bar{x}/n^{\theta\alpha}$ at which $E_{n,\beta_n,K_n}\{|S_n/n|\}$ and $m(\beta_n, K_n)$ converge separately to 0.

This asymptotic result generalizes (1.3), which is the conclusion of Theorem 4.1 in [11]. To see this, define $A_n = E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$ and note that

$$\left|E_{n,\beta_n,K_n}\left\{|S_n/n|\right\} - m(\beta_n, K_n)\right| \le A_n. \tag{1.5}$$

Equation (1.4) states that $\lim_{n\to\infty} n^{\kappa} A_n = \bar{z}$. Since $\kappa > \theta\alpha$, this implies that

$$0 = \lim_{n\to\infty} n^{\theta\alpha} A_n \ge \lim_{n\to\infty} n^{\theta\alpha} \left|E_{n,\beta_n,K_n}\left\{|S_n/n|\right\} - m(\beta_n, K_n)\right| = 0.$$

The fact that this second limit equals 0 yields (1.3).

For any $\varepsilon > 0$ and all sufficiently large n, $A_n \le (\bar{z} + \varepsilon)/n^{\kappa}$. Hence (1.5) yields our second main result, which is that for all sufficiently large n

$$\left|E_{n,\beta_n,K_n}\left\{|S_n/n|\right\} - m(\beta_n, K_n)\right| \le (\bar{z} + \varepsilon)/n^{\kappa}. \tag{1.6}$$

It is stated in part (b) of Theorem 4.1.

One of the hypotheses of Theorem 4.1 is that $\theta\alpha_0 < 1/2$. Because of this condition, $\kappa = \frac{1}{2} + \alpha(\theta - 1/(2\alpha_0))$ is a decreasing function of $\alpha \in (0, \alpha_0)$, This property yields interpretations of (1.4) and (1.6) that are relevant to the theory of finite-size scaling. Because (1.4) is an exact asymptotic result, this property of κ shows that the rate at which the centered, finite-size magnetization converges to 0 is slower the closer α is to the threshold value α_0. We now turn to (1.6). The conclusion of Theorem 4.1 in [11] is that for $0 < \alpha < \alpha_0$ the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\}$ and the thermodynamic magnetization are asymptotic to each other. Because (1.6) is an upper bound rather than an exact asymptotic result, the fact that κ is a decreasing function of $\alpha \in (0, \alpha_0)$ suggests, but does not prove, that the rate at which $|E_{n,\beta_n,K_n}\{|S_n/n|\} - m(\beta_n, K_n)|$ converges to 0 is slower the closer α is to the threshold value α_0. In the context of the theory of finite-size scaling, these interpretations of (1.4) and (1.6) are certainly intuitive since α controls the speed at which (β_n, K_n) approaches criticality.

Like the other three main results in this paper, (1.6) is valid only for $0 < \alpha < \alpha_0$. However, it is worthwhile pointing out that if α equals the threshold value α_0, then at least in the rate of decay of the right-hand side, (1.6) gives the correct asymptotic rate at which $|E_{n,\beta_n,K_n}\{|S_n/n|\} - m(\beta_n, K_n)|$ converges to 0 provided a certain constant is nonzero. In order to see this, we refer to Theorem 4.3 in [11], which shows that for $\alpha = \alpha_0$

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim \bar{\zeta}/n^{\theta\alpha_0}, \quad \text{where } \bar{\zeta} = \frac{1}{\int_{\mathbb{R}} \exp[-g(x)]\,dx} \cdot \int_{\mathbb{R}} |x| \exp\big[-g(x)\big]\,dx.$$

Since for $\alpha = \alpha_0$, $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha_0}$ [Thm. 3.1], it follows that when $\bar{\zeta} \neq \bar{x}$,

$$\big|E_{n,\beta_n,K_n}\{|S_n/n|\} - m(\beta_n, K_n)\big| \sim |\bar{\zeta} - \bar{x}|/n^{\theta\alpha_0}.$$

When $\alpha = \alpha_0$, we have $\kappa = \theta\alpha_0$. Hence the right-hand side of the last display has the same rate of decay as the right-hand side of (1.6) provided $\bar{\zeta} \neq \bar{x}$. In general we expect that $\bar{\zeta} \neq \bar{x}$. However, when $\bar{\zeta} = \bar{x}$, the right-hand side of the last display equals 0, and so the last display gives no information.

The proof of the asymptotic result (1.4) is based on the following new conditional limit stated in part (b) of Theorem 6.1 for $0 < \alpha < \alpha_0$:

$$\lim_{n\to\infty} n^{\kappa} E_{n,\beta_n,K_n}\big\{\big|S_n/n - m(\beta_n, K_n)\big| \,\big|\, S_n/n > \delta m(\beta_n, K_n)\big\} = \bar{z}. \tag{1.7}$$

The conditioning is on the event $\{S_n/n > \delta m(\beta_n, K_n)\}$, where $\delta \in (0, 1)$ is sufficiently close to 1. This conditioning allows us to study the asymptotic behavior of the system in a neighborhood of the pure states having thermodynamic magnetization $m(\beta_n, K_n)$. According to Lemma 6.3

$$\begin{aligned}\lim_{n\to\infty} & P_{n,\beta_n,K_n}\big\{S_n/n > \delta m(\beta_n, K_n)\big\} \\ &= \lim_{n\to\infty} P_{n,\beta_n,K_n}\big\{S_n/n < -\delta m(\beta_n, K_n)\big\} = 1/2.\end{aligned} \tag{1.8}$$

These limits are the analog of the weak convergence limit (1.2), showing that as $n \to \infty$ the mass of the P_{n,β_n,K_n}-distribution of S_n/n concentrates at $\pm m(\beta_n, K_n)$. As we show in Sect. 6, the limits (1.7) and (1.8) and a moderate deviation estimate on the probability $P_{n,\beta_n,K_n}\{\delta m(\beta_n, K_n) \geq S_n/n \geq -\delta m(\beta_n, K_n)\}$ yield

$$\lim_{n\to\infty} n^{\kappa} E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m(\beta_n, K_n)\big|\big\} = \bar{z}.$$

This limit is equivalent to (1.4). In Sect. 7 we outline the proof of the conditional limit (1.7), saving the technical details for [7].

The results (1.4) and (1.6), which are stated in parts (a) and (b) of Theorem 4.1, and the conditional limit (1.7), which is stated in part (b) of Theorem 6.1, are valid for the first five sequences introduced in [8, §4]. Located in the phase-coexistence region for all sufficiently large n, the first two sequences converge to a second-order point, and the last three sequences converge to the tricritical point. Possible paths followed by these sequences are shown in Fig. 2. For each of the five sequences the quantities α_0, θ, and κ appearing in Theorem 4.1 are specified in Table 1.

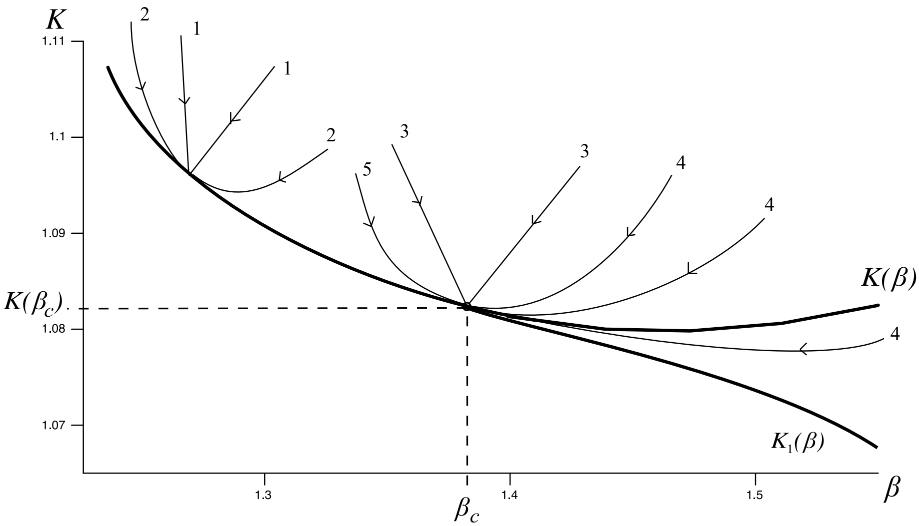

Fig. 2 Possible paths for the five sequences converging to a second-order point and to the tricritical point. In Sect. 5 of this paper and in Appendix A in [7] these sequences are defined and are shown to satisfy the hypotheses of Theorem 4.1 and Theorem 6.1. The sequences labeled 1–5 in this figure correspond to sequences 1a–5a in Table 1 and Table 2

Table 1 The equations where each of the five sequences is defined and the values of α_0, θ, and κ for each sequence

Seq.	Defn.	α_0	θ	κ
1a	(5.5)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}(1-\alpha)$
2a	(5.6)	$\frac{1}{2p}$	$\frac{p}{2}$	$\frac{1}{2}(1-p\alpha)$
3a	(5.7)	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{2}(1-\alpha)$
4a	(5.8)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}(1-2\alpha)$
5a	(5.10)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}(1-2\alpha)$

The conditional limit (1.7) is closely related to our fourth main result, which is stated in part (a) of Theorem 6.1 and is also valid for the first five sequences introduced in [8, §4]. This result is a new conditional central limit theorem for the spin for $0 < \alpha < \alpha_0$. As in (1.7), the conditioning is on the event $\{S_n/n > \delta m(\beta_n, K_n)\}$, where $\delta \in (0,1)$ is sufficiently close to 1. Part (a) of Theorem 6.1 states that when conditioned on $\{S_n/n > \delta m(\beta_n, K_n)\}$, the P_{n,β_n,K_n}-distributions of $n^\kappa(S_n/n - m(\beta_n, K_n))$ converge weakly to a normal random variable $N(0, 1/g^{(2)}(\bar{x}))$ with mean 0 and variance $1/g^{(2)}(\bar{x})$; in symbols,

$$P_{n,\beta_n,K_n}\left\{n^\kappa\left(S_n/n - m(\beta_n, K_n)\right) \in dx \;\middle|\; S_n/n > \delta m(\beta_n, K_n)\right\} \Longrightarrow N\left(0, 1/g^{(2)}(\bar{x})\right). \tag{1.9}$$

Since $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ is less than $\frac{1}{2}$ [Thm. 6.1(c)], the scaling in this result is non-classical. An equivalent formulation is that for any bounded, continuous function f

$$\begin{aligned}
&\lim_{n\to\infty} E_{n,\beta_n,K_n}\big\{f\big(n^{\kappa}\big(S_n/n - m(\beta_n,K_n)\big)\big) \mid S_n/n > \delta m(\beta_n,K_n)\big\} \\
&\quad = \lim_{n\to\infty} E_{n,\beta_n,K_n}\big\{f\big(S_n/n^{1-\kappa} - n^{\kappa} m(\beta_n,K_n)\big) \mid S_n/n > \delta m(\beta_n,K_n)\big\} \\
&\quad = E\big\{f\big(N\big(0, 1/g^{(2)}(\bar{x})\big)\big)\big\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} f(x)\exp\left[-\frac{1}{2}g^{(2)}(\bar{x})x^2\right]dx. \qquad (1.10)
\end{aligned}$$

Through the term $g^{(2)}(\bar{x})$ this conditional central limit theorem and the asymptotic formula (1.4) exhibit a sensitive dependence on the choice of the sequence (β_n, K_n), which lies in the phase coexistence region for all sufficiently large n and converges to a second-order point or the tricritical point. This contrasts sharply with the central limit theorem that is valid for an arbitrary sequence (β_n, K_n) that converges to a point (β, K) in the single-phase region defined by $\{(\beta, K) : 0 < \beta \le \beta_c, 0 < K < K(\beta)\}$. In this situation it is proved in Theorem 5.5 in [6] that

$$P_{n,\beta_n,K_n}\big\{S_n/n^{1/2} \in dx\big\} \quad \Longrightarrow \quad N\big(0, \sigma^2(\beta, K)\big),$$

where the limiting variance $\sigma^2(\beta, K)$ depends only on (β, K) and not on the sequence (β_n, K_n).

Formally, the conditional limit (1.7) follows from the conditional central limit theorem (1.10) if one replaces the bounded, continuous function f by the absolute value function. Then (1.10) would imply

$$\begin{aligned}
&\lim_{n\to\infty} n^{\kappa} E_{n,\beta_n,K_n}\big\{\big|S_n/n - m(\beta_n,K_n)\big| \mid S_n/n > \delta m(\beta_n,K_n)\big\} \\
&\quad = \lim_{n\to\infty} E_{n,\beta_n,K_n}\big\{\big|S_n/n^{1-\kappa} - n^{\kappa} m(\beta_n,K_n)\big| \mid S_n/n > \delta m(\beta_n,K_n)\big\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]\,dx} \cdot \int_{\mathbb{R}} |x|\exp\left[-\frac{1}{2}g^{(2)}(\bar{x})x^2\right]dx = \bar{z}.
\end{aligned}$$

In fact, both the conditional limit (1.7) and the conditional central limit theorem (1.9) are consequences of a weak convergence result stated in (7.10) in the present paper. In Sect. 7 we outline the proof of this weak convergence result. We then show how to derive both (1.7) and (1.9). In so doing, we explain why the constant κ equals $\frac{1}{2}(1-\alpha/\alpha_0) + \theta\alpha$.

The contents of this paper are as follows. In section 2 we define the mean-field B-C model and summarize its phase-transition structure in Theorems 2.1 and 2.2. For a class of sequences (β_n, K_n) lying in the phase-coexistence region for all sufficiently large n and converging either to a second-order point or to the tricritical point, Theorem 3.1 in Sect. 3 describes the asymptotic behavior of $m(\beta_n, K_n) \to 0$ as stated in (1.1). Theorem 3.2 in Sect. 3 states one of the main results of [11], which is that as $n \to 0$, $m(\beta_n, K_n)$ is asymptotic to $E_{n,\beta_n,K_n}\{|S_n/n|\}$ for $0 < \alpha < \alpha_0$, proving that for this range of α the thermodynamic magnetization $m(\beta_n, K_n)$ is a physically relevant estimator of the finite-size magnetization $E_{n,\beta_n,K_n}\{|S_n/n|\}$.

The first two main results in this paper are given in Sect. 4. According to part (a) of Theorem 4.1, for $0 < \alpha < \alpha_0$

$$E_{n,\beta_n,K_n}\big\{\big|S_n/n - m(\beta_n,K_n)\big|\big\} \sim \bar{z}/n^{\kappa},$$

where $\bar{z} = (2/[\pi g^{(2)}(\bar{x})])^{1/2}$ and $\kappa = \frac{1}{2}(1-\alpha/\alpha_0) + \theta\alpha$. Part (b) of this theorem gives the inequality (1.6). Parts (a) and (b) of Theorem 4.1 are applied in Sect. 5 to five specific sequences (β_n, K_n). The first two sequences converge to a second-order point, and the last three sequences converge to the tricritical point. In Sect. 6 part (b) of Theorem 6.1 states the conditional limit (1.7), which is the third main result in this paper. From this conditional limit parts (a) and (b) of Theorem 4.1 are derived. Part (a) of Theorem 6.1 states the conditional central limit theorem (1.9), which is the fourth main result in this paper. In Sect. 7 we motivate the proofs of parts (a) and (b) of Theorem 6.1. We focus on the proof of part (b) because it is somewhat more complicated than the proof of part (a). The technical details of the proof are given in Sects. 7 and 8 of [7]. The analogous but more straightforward proof of part (a) is discussed at the end of Sect. 7 in this paper.

2 Phase-Transition Structure of the Mean-Field B-C Model

For $N \in \mathbb{N}$ the mean-field Blume-Capel model is defined on the complete graph on N vertices $1, 2, \dots, N$. The spin at site $j \in \{1, 2, \dots, N\}$ is denoted by ω_j, a quantity taking values in $\Lambda = \{-1, 0, 1\}$. The Hamiltonian for this model is defined by

$$H_{N,K}(\omega) = \sum_{j=1}^{N} \omega_j^2 - \frac{K}{N}\left(\sum_{j=1}^{N} \omega_j\right)^2,$$

where $K > 0$ is a positive parameter representing the interaction strength and $\omega = (\omega_1, \dots, \omega_N) \in \Lambda^N$. We will refer to this model as the mean-field B-C model.

Let P_N be the product measure on Λ^N with identical one-dimensional marginals $\rho = \frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$. Then P_N assigns the probability 3^{-N} to each $\omega \in \Lambda^N$. For inverse temperature $\beta > 0$ and for $K > 0$, the canonical ensemble for the mean-field B-C model is the sequence of probability measures that assign to each subset B of Λ^N the probability

$$\begin{aligned} P_{N,\beta,K}(B) &= \frac{1}{Z_N(\beta, K)} \cdot \int_B \exp[-\beta H_{N,K}]\, dP_N \\ &= \frac{1}{Z_N(\beta, K)} \cdot \sum_{\omega \in B} \exp\bigl[-\beta H_{N,K}(\omega)\bigr] \cdot 3^{-N}, \end{aligned} \tag{2.1}$$

where

$$Z_N(\beta, K) = \int_{\Lambda^N} \exp[-\beta H_{N,K}]\, dP_N = \sum_{\omega \in \Lambda^N} \exp[-\beta H_{N,K(\omega)}] \cdot 3^{-N}.$$

It is useful to rewrite this measure in a different form. Define $S_N(\omega) = \sum_{j=1}^{N} \omega_j$ and let $P_{N,\beta}$ be the product measure on Λ_N with identical one-dimensional marginals

$$\rho_\beta(d\omega_j) = \frac{1}{Z(\beta)} \cdot \exp\bigl(-\beta \omega_j^2\bigr) \rho(d\omega_j),$$

where $Z(\beta) = \int_\Lambda \exp(-\beta \omega_j^2) \rho(d\omega_j) = (1 + 2e^{-\beta})/3$. Define

$$P_{N,\beta}(d\omega) = \prod_{j=1}^{N} \rho_\beta(d\omega_j) = \frac{1}{[Z(\beta)]^N} \prod_{j=1}^{N} \exp\bigl(-\beta \omega_j^2\bigr) \rho(d\omega_j)$$

and

$$\tilde{Z}_N(\beta, K) = \int_{\Lambda^N} \exp\bigl[N\beta K (S_N/N)^2\bigr] dP_{N,\beta} = \frac{Z_N(\beta, K)}{[Z(\beta)]^N}.$$

Then we have

$$P_{N,\beta,K}(d\omega) = \frac{1}{\tilde{Z}_N(\beta, K)} \exp\big[N\beta K\big(S_N(\omega)/N\big)^2\big] P_{N,\beta}(d\omega). \tag{2.2}$$

For $t \in \mathbb{R}$ and $x \in \mathbb{R}$ we also define the cumulant generating function

$$c_\beta(t) = \log \int_\Lambda \exp(t\omega_1)\rho_\beta(d\omega_1) = \log\left[\frac{1 + e^{-\beta}(e^t + e^{-t})}{1 + 2e^{-\beta}}\right] \tag{2.3}$$

and the free-energy function

$$G_{\beta,K}(x) = \beta K x^2 - c_\beta(2\beta K x). \tag{2.4}$$

We denote by $\mathcal{M}_{\beta,K}$ the set of equilibrium macrostates of the mean-field B-C model. As shown in Proposition 3.4 in [12], $\mathcal{M}_{\beta,K}$ can be characterized as the set of global minimum points of $G_{\beta,K}$:

$$\mathcal{M}_{\beta,K} = \big\{x \in [-1, 1]\colon\ x \text{ is the global minimum points of } G_{\beta,K}(x)\big\}.$$

In [12] $\mathcal{M}_{\beta,K}$ is denoted by $\tilde{\mathcal{E}}_{\beta,K}$.

The critical inverse temperature of the mean-field B-C model is $\beta_c = \log 4$. For $0 < \beta \le \beta_c$, the next theorem states that $\mathcal{M}_{\beta,K}$ exhibits a continuous bifurcation as K increases through a value $K(\beta)$. This bifurcation corresponds to a second-order phase transition, and the curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$ is called the second-order curve. The point $(\beta_c, K(\beta_c))$ is called the tricritical point. Theorem 2.1 is proved in Theorem 3.6 in [12], where $K(\beta)$ is denoted by $K_c^{(2)}(\beta)$.

Theorem 2.1 *For $0 < \beta \le \beta_c$, we define*

$$K(\beta) = 1/\big[2\beta c''_\beta(0)\big] = \big(e^\beta + 2\big)/(4\beta).$$

For these values of β, $\mathcal{M}_{\beta,K}$ has the following structure.

(a) *For $0 < K \le K(\beta)$, $\mathcal{M}_{\beta,K} = \{0\}$.*
(b) *For $K > K(\beta)$, there exists $m(\beta, K) > 0$ such that $\mathcal{M}_{\beta,K} = \{\pm m(\beta, K)\}$.*
(c) *$m(\beta, K)$ is a positive, increasing, continuous function for $K > K_c(\beta)$, and as $K \to (K(\beta))^+$, $m(\beta, K) \to 0$. Therefore, $\mathcal{M}_{\beta,K}$ exhibits a continuous bifurcation at $K(\beta)$.*

For $\beta > \beta_c$, the next theorem states that $\mathcal{M}_{\beta,K}$ exhibits a discontinuous bifurcation as K increases through a value $K_1(\beta)$. This bifurcation corresponds to a first-order phase transition, and the curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$ is called the first-order curve. Theorem 2.2 is proved in Theorem 3.8 in [12].

Theorem 2.2 *For $\beta > \beta_c$, $\mathcal{M}_{\beta,K}$ has the following structure in terms of the quantity $K_1(\beta)$, denoted by $K_c^{(1)}(\beta)$ in [12] and defined implicitly for $\beta > \beta_c$ on page 2231 of [12].*

(a) *For $0 < K < K_1(\beta)$, $\mathcal{M}_{\beta,K} = \{0\}$.*
(b) *For $K = K_1(\beta)$ there exists $m(\beta, K_1(\beta)) > 0$ such that $\mathcal{M}_{\beta,K_1(\beta)} = \{0, \pm m(\beta, K_1(\beta))\}$.*
(c) *For $K > K_1(\beta)$ there exists $m(\beta, K) > 0$ such that $\mathcal{M}_{\beta,K} = \{\pm m(\beta, K)\}$.*
(d) *$m(\beta, K)$ is a positive, increasing, continuous function for $K \ge K_1(\beta)$, and as $K \to K_1(\beta)^+$, $m(\beta, K) \to m(\beta, K_1(\beta)) > 0$. Therefore, $\mathcal{M}_{\beta,K}$ exhibits a discontinuous bifurcation at $K_1(\beta)$.*

The positive quantity $m(\beta, K)$ in Theorems 2.1 and 2.2 is called the thermodynamic magnetization. In the next section we describe the asymptotic behavior of the finite-size magnetization for suitable sequences (β_n, K_n) and relate this to the asymptotic behavior of the thermodynamic magnetization $m(\beta_n, K_n)$.

3 Asymptotic Behavior of $E_{n,\beta_n,K_n}\{|S_n/n|\}$

For $\beta > 0$ and $K > 0$ the finite-size magnetization is defined as

$$E_{N,\beta,K}\big\{|S_N/N|\big\} = \int_{\Omega_N} |S_N/N|\, dP_{N,\beta,K},$$

where $P_{N,\beta,K}$ denotes the measure defined in (2.1)–(2.2). In this section we describe the asymptotic behavior of $E_{n,\beta_n,K_n}\{|S_n/n|\}$ for suitable sequences (β_n, K_n) lying in the phase-coexistence region. In this formulation we let $N = n$ in the finite-size magnetization; i.e., we let the number of spins N coincide with the index n parametrizing the sequence (β_n, K_n).

The phase-coexistence region is defined as the set of all points in the positive β-K quadrant for which $\mathcal{M}_{\beta,K}$ consists of more than one value. According to Theorems 2.1 and 2.2, the phase-coexistence region consists of all points above the second-order curve, above the tricritical point, on the first-order curve, and above the first-order curve; that is,

$$\big\{(\beta, K)\colon\ 0 < \beta \le \beta_c,\, K > K(\beta) \text{ and } \beta > \beta_c,\, K \ge K_1(\beta)\big\}.$$

For a class of sequences (β_n, K_n) lying in the phase-coexistence region for all sufficiently large n and converging either to a second-order point or to the tricritical point, Theorem 3.1 describes the asymptotic behavior of the thermodynamic magnetization $m(\beta_n, K_n) \to 0$. The asymptotic behavior is related to the unique positive, global minimum point of the Ginzburg-Landau polynomial, which is defined in hypothesis (iii) of the theorem.

Theorem 3.1 is a special case of the main theorem in [8], Theorem 3.2. In that paper we describe six different sequences that satisfy the hypotheses of Theorem 3.1. The first five of these sequences are revisited in Sect. 5 of this paper, where we show that they satisfy the hypotheses of Theorem 4.1. In that theorem we state our first two main results. These five sequences, labeled 1a–5a, are summarized in Table 2, which appears in Sect. 5.

Theorem 3.1 *Let (β_n, K_n) be a positive sequence that converges either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta_c, K(\beta_c))$. We assume that (β_n, K_n) satisfies the following four hypotheses.*

(i) *(β_n, K_n) lies in the phase-coexistence region for all sufficiently large n.*
(ii) *The sequence (β_n, K_n) is parametrized by $\alpha > 0$. This parameter regulates the speed of approach of (β_n, K_n) to the second-order point or the tricritical point in the following sense*:

$$b = \lim_{n\to\infty} n^\alpha(\beta_n - \beta) \quad \text{and} \quad k = \lim_{n\to\infty} n^\alpha\big(K_n - K(\beta)\big)$$

both exist, and b and k are not both 0; *if $b \neq 0$, then b equals* 1 *or* -1.
(iii) *There exists an even polynomial g of degree* 4 *or* 6 *satisfying $g(x) \to \infty$ as $|x| \to \infty$ together with the following two properties*; *g is called the Ginzburg-Landau polynomial.*
   (a) *There exist $\alpha_0 > 0$ and $\theta > 0$ such that for all $\alpha > 0$*

$$\lim_{n\to\infty} n^{\alpha/\alpha_0} G_{\beta_n,K_n}\big(x/n^{\theta\alpha}\big) = g(x)$$

   uniformly for x in compact subsets of $\mathbb{R}$.
   (b) *There exists $\bar{x} > 0$ such that the set of global minimum points of g equals $\{\pm\bar{x}\}$.*
(iv) *Consider $\alpha_0 > 0$ and $\theta > 0$ in hypothesis* (iii)(a). *There exists a polynomial H satisfying $H(x) \to \infty$ as $|x| \to \infty$ together with the following property*: *for all $\alpha > 0$ there exists $R > 0$ such that for all $n \in \mathbb{N}$ sufficiently large and for all $x \in \mathbb{R}$ satisfying $|x/n^{\theta\alpha}| < R$, $n^{\alpha/\alpha_0} G_{\beta_n,K_n}(x/n^{\theta\alpha}) \ge H(x)$.*

Under hypotheses (i)–(iv), *for any* $\alpha > 0$

$$m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}; \quad i.e., \lim_{n\to\infty} n^{\theta\alpha} m(\beta_n, K_n) = \bar{x}.$$

If $b \neq 0$, *then this becomes* $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^{\theta}$.

Theorem 3.2 restates Theorem 4.1 in [11]. The hypotheses are those of Theorem 3.1 for all $0 < \alpha < \alpha_0$ together with the inequality $0 < \theta\alpha_0 < 1/2$. These hypotheses are satisfied by sequences 1a–5a in Table 2 as well as by a sixth sequence described in Theorem 4.6 in [8]. Part (a) of the next theorem gives the rate at which $E_{n,\beta_n,K_n}\{|S_n/n|\} \to 0$ for $0 < \alpha < \alpha_0$, and part (b) states that for the same values of α, $E_{n,\beta_n,K_n}\{|S_n/n|\} \sim m(\beta_n, K_n)$. Thus Theorem 3.2 shows that the asymptotic behavior of $E_{n,\beta_n,K_n}\{|S_n/n|\}$ coincides with that of $m(\beta_n, K_n)$ for $0 < \alpha < \alpha_0$. Theorem 4.2 in [11] shows that for $\alpha > \alpha_0$, $m(\beta_n, K_n)$ converges to 0 asymptotically faster than $E_{n,\beta_n,K_n}\{|S_n/n|\}$.

Theorem 3.2 *Let* (β_n, K_n) *be a positive sequence parametrized by* $\alpha > 0$ *and converging either to a second-order point* $(\beta, K(\beta))$, $0 < \beta < \beta_c$, *or to the tricritical point* $(\beta_c, K(\beta_c))$. *We assume that* (β_n, K_n) *satisfies the hypotheses of Theorem* 3.1 *for all* $0 < \alpha < \alpha_0$. *We also assume the inequality* $0 < \theta\alpha_0 < 1/2$. *The following conclusions hold.*

(a) *For all* $0 < \alpha < \alpha_0$

$$E_{n,\beta_n,K_n}\big\{|S_n/n|\big\} \sim \bar{x}/n^{\theta\alpha}; \quad i.e., \lim_{n\to\infty} n^{\theta\alpha} E_{n,\beta_n,K_n}\big\{|S_n/n|\big\} = \bar{x}.$$

(b) *For all* $0 < \alpha < \alpha_0$, $E_{n,\beta_n,K_n}\{|S_n/n|\} \sim m(\beta_n, K_n)$.

In Theorem 4.1 in the next section we state our first two main results on the fluctuations of S_n/n around $m(\beta_n, K_n)$ for $0 < \alpha < \alpha_0$. We then explain how Theorem 4.1 generalizes Theorem 3.2.

4 Asymptotic Behavior of $E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$

We denote by E_{n,β_n,K_n} expectation with respect to the measure P_{n,β_n,K_n}. Theorem 4.1 states two of our main results. In part (a) we investigate the asymptotic behavior of the centered, finite-size magnetization $E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$ under the hypotheses of Theorem 3.1 and an additional hypothesis (iii′). We show that this quantity is asymptotic to $\bar{z}/n^{\kappa}$, where $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ and $\bar{z} > 0$ is given explicitly. The rate $\bar{z}/n^{\kappa}$ at which $E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\}$ converges to 0 is asymptotically faster than the rate $\bar{x}/n^{\theta\alpha}$ at which $E_{n,\beta_n,K_n}\{|S_n/n|\}$ and $m(\beta_n, K_n)$ converge to 0 separately. As shown in the introduction, the inequality in part (b) is a simple consequence of part (a), which in turn yields the conclusion of Theorem 3.2 as a corollary. The relevance of parts (a) and (b) to the theory of finite-size scaling is discussed in the introduction after the inequality (1.6). We comment on the hypotheses of Theorem 4.1 at the end of this section.

Part (a) of Theorem 4.1 is proved in Sect. 6. Part (c) of Theorem 4.1 asserts that the hypotheses of this theorem are satisfied by sequences 1a–5a in Table 2. This is discussed in Sect. 5. For each of these sequences the Ginzburg-Landau polynomial has degree 4 or 6.

Theorem 4.1 *Let* (β_n, K_n) *be a positive sequence converging either to a second-order point* $(\beta, K(\beta))$, $0 < \beta < \beta_c$, *or to the tricritical point* $(\beta_c, K(\beta_c))$. *We assume that* (β_n, K_n) *satisfies the hypotheses of Theorem* 3.1 *for all* $0 < \alpha < \alpha_0$. *We also assume the following additional hypothesis on the Ginzburg-Landau polynomial* g.

(iii′) *Assume that g has degree* 4. *Then $\theta\alpha_0$ lies in the interval* $[1/4, 1/2)$. *In addition, for all* $0 < \alpha < \alpha_0$ *and for* $j = 2, 3, 4$

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}\big(m(\beta_n, K_n)\big) = g^{(j)}(\bar{x}) > 0.$$

Assume that g has degree 6. *Then $\theta\alpha_0$ lies in the interval* $[1/6, 1/2)$. *In addition, for all* $0 < \alpha < \alpha_0$ *and for* $j = 2, 3, 4, 5, 6$

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}\big(m(\beta_n, K_n)\big) = g^{(j)}(\bar{x}) > 0.$$

For $\alpha \in (0, \alpha_0)$ *we also define* $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. *Then for all* $0 < \alpha < \alpha_0$ *the following conclusions hold.*

(a) *We have the asymptotic behavior*

$$E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m(\beta_n, K_n)\big|\big\} \sim \bar{z}/n^{\kappa},$$

where $\bar{z} = (2/(\pi g^{(2)}(\bar{x})))^{1/2}$; *i.e.*, $\lim_{n\to\infty} n^{\kappa} E_{n,\beta_n,K_n}\{||S_n/n| - m(\beta_n, K_n)|\} = \bar{z}$.

(b) *For any* $\varepsilon > 0$ *and all sufficiently large* n

$$\big|E_{n,\beta_n,K_n}\big\{|S_n/n|\big\} - m(\beta_n, K_n)\big| \le \frac{\bar{z} + \varepsilon}{n^{\kappa}}.$$

(c) *The hypotheses of this theorem are satisfied by sequences* 1a–5a *in Table* 2.

The hypotheses of Theorem 4.1 are those of Theorem 3.1 together with the additional hypothesis (iii′) for all $0 < \alpha < \alpha_0$. The latter hypothesis takes two related forms depending on whether g has degree 4 or degree 6. In this hypothesis, the assumption on $\theta\alpha_0$ yields the inequality $0 < \theta\alpha_0 < 1/2$, which is required by the moderate deviation principle stated in Theorem 6.2. Hypothesis (iii′) also assumes both the form of the asymptotic behavior of certain derivatives of $n^{\alpha/\alpha_0} G_{\beta_n,K_n}$ evaluated at $m(\beta_n, K_n)$ and the positivity of the corresponding derivatives of g evaluated at the positive global minimum point $\bar{x}$. These assumptions are needed in the proof of Lemma 7.5 in [7], a key result needed to prove part (b) of Theorem 6.1, which in turn yields part (a) of Theorem 4.1. In particular, the limits in hypothesis (iii′) of Theorem 4.1 are required to prove both parts (a) and (b) of Lemma 7.5 in [7] while the positivity of $g^{(j)}(\bar{x})$ is needed to prove part (b) of that lemma. The proof of Lemma 7.5 in [7] also requires the fact assumed in hypothesis (iii′) that $\theta\alpha_0$ lies in the interval $[1/4, 1/2)$ or $[1/6, 1/2)$ depending on whether g has degree 4 or degree 6.

In the next section we outline how to verify the hypotheses of Theorem 4.1 for sequences 1a–5a in Table 2.

5 Verification of Hypotheses of Theorem 4.1 for Sequences 1a–5a

Table 2 summarizes five sequences (β_n, K_n) introduced in Sect. 4 of [8]. Depending on the inequalities on the coefficients, sequences 1, 2, 3, and 5 each have two cases labeled a and b, and sequence 4 has three cases labeled a, b, and c. All five sequences 1a–5a lie in the phase-coexistence region for all sufficiently large n as required by hypothesis (i) of Theorem 3.1.

The hypotheses of Theorem 4.1 consist of the hypotheses of Theorem 3.1 for all $0 < \alpha < \alpha_0$ and hypothesis (iii′). Hypothesis (iii′) takes two forms depending on the degree of the Ginzburg-Landau polynomial g. When g has degree 4, $\theta\alpha_0$ is assumed to lie in the interval $[1/4, 1/2)$ and for all $\alpha \in (0, \alpha_0)$ and for $j = 2, 3, 4$

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n,K_n}\big(m(\beta_n, K_n)\big) = g^{(j)}(\bar{x}) > 0. \tag{5.1}$$

Table 2 The equation where each of the 5 sequences is defined and the inequalities on the coefficients guaranteeing that each sequence lies in the phase-coexistence region (Ph-CR) or in the single-phase region (1-PhR). The next-to-last column states the structure of the set $\mathcal{M}_g$ of global minimum points of the Ginzburg-Landau polynomial g for each sequence in terms of a positive number $\bar{x}$ that can be explicitly calculated. The theorems in [8] where this information is verified are also given

Seq.	Defn.	Case	Ineq.	Region	$\mathcal{M}_g$	Thm. in [8]
1	(5.5)	a	$K'(\beta)b - k < 0$	Ph-CR	$\{\pm\bar{x}\}$	Thm. 4.1
		b	$K'(\beta)b - k > 0$	1-PhR	$\{0\}$	
2	(5.6)	a	$(K^{(p)}(\beta) - \ell)b^p < 0$	Ph-CR	$\{\pm\bar{x}\}$	Thm. 4.2
		b	$(K^{(p)}(\beta) - \ell)b^p > 0$	1-PhR	$\{0\}$	
3	(5.7)	a	$K'(\beta_c)b - k < 0$	Ph-CR	$\{\pm\bar{x}\}$	Thm. 4.3
		b	$K'(\beta_c)b - k > 0$	1-PhR	$\{0\}$	
4	(5.8)	a	$\ell > \ell_c, \tilde{\ell} \in \mathbb{R}$	Ph-CR	$\{\pm\bar{x}\}$	Thm. 4.4
		b	$\ell = \ell_c, \tilde{\ell} > K_1'''(\beta_c)$	Ph-CR	$\{0, \pm\bar{x}\}$	
		c	$\ell < \ell_c, \tilde{\ell} \in \mathbb{R}$	1-PhR	$\{0\}$	
5	(5.10)	a	$\ell > K''(\beta_c)$	Ph-CR	$\{\pm\bar{x}\}$	Thm. 4.5
		b	$\ell < K''(\beta_c)$	1-PhR	$\{0\}$	

When g has degree 6, $\theta\alpha_0$ is assumed to lie in the interval $[1/6, 1/2)$ and for all $\alpha \in (0, \alpha_0)$ and for $j = 2, 3, 4, 5, 6$

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n, K_n}\big(m(\beta_n, K_n)\big) = g^{(j)}(\bar{x}) > 0. \tag{5.2}$$

In this section we verify for sequences 1a–5a that when g has degree 4, we have $\theta\alpha_0 \in [1/4, 1/2)$ and $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4$ and that when g has degree 6, we have $\theta\alpha_0 \in [1/6, 1/2)$ and $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4, 5, 6$. The verification of the limits in (5.1) and (5.2) is carried out in Appendix A in [7].

Sequence 6 introduced in Theorem 4.6 in [8] does not satisfy hypothesis (iii′) in Theorem 4.1. In this case g has degree 4, but $\theta\alpha_0$ does not lie in the interval $[1/4, 1/2)$.

The first two sequences converge to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, and the last three sequences converge to the tricritical point $(\beta_c, K(\beta_c))$. For each sequence 1a–5a, the hypotheses of Theorem 3.1 are verified in Theorems 4.1–4.5 in [8]. We follow the same method used in that paper to verify hypothesis (iii′) in Theorem 4.1 for sequences 1a–5a. Hypothesis (iii′) of Theorem 4.1 takes two forms depending on whether the degree of the Ginzburg-Landau polynomial g is 4 or 6. We must verify that $\theta\alpha_0$ lies in a certain interval and that

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n, K_n}\big(m(\beta_n, K_n)\big) = g^{(j)}(\bar{x}) > 0 \tag{5.3}$$

for $j = 2, 3, 4$ when g has degree 4 and for $j = 2, 3, 4, 5, 6$ when g has degree 6. The function $G_{\beta,K}$ is defined in (2.3)–(2.4).

It is straightforward to show that the limit in (5.3) holds for a given j provided the following limit holds uniformly for x in compact subsets of $\mathbb{R}$:

$$\lim_{n\to\infty} n^{\alpha/\alpha_0 - j\theta\alpha} G^{(j)}_{\beta_n, K_n}\big(x/n^{\theta\alpha}\big) = g^{(j)}(x). \tag{5.4}$$

The proof that the uniform convergence in (5.4) implies the limit in (5.3) uses the fact that $n^{\theta\alpha} m(\beta_n, k_n) \to \bar{x}$ [Thm. 3.1]. The uniform convergence in (5.4) can be obtained formally by taking the j-th derivative of the uniform convergence limits in hypothesis (iii)(a) of Theorem 3.1:

$$\lim_{n\to\infty} n^{\alpha/\alpha_0} G_{\beta_n,K_n}\big(x/n^{\theta\alpha}\big) = g(x).$$

The verification of the uniform convergence limits in (5.4), and thus the verification of the limits (5.1) and (5.2) in hypothesis (iii′), depend on asymptotic properties of the Taylor expansions of $G^{(j)}_{\beta_n,K_n}(x/n^{\theta\alpha})$. This analysis closely parallels the proof of Theorem 3.1, which is based on a similar analysis of the Taylor expansions of $G_{\beta_n,K_n}(x/n^{\theta\alpha})$ carried out in [8]. We omit the straightforward but tedious calculations, which can be found in Appendix A in [7].

We now define the five sequences (β_n, K_n) and summarize the verification of the hypotheses of Theorem 4.1 for them.

Sequence 1a

Definition Given $0 < \beta < \beta_c$, $\alpha > 0$, $b \in \{1, 0, -1\}$, and $k \in \mathbb{R}$, $k \neq 0$, sequence 1 is defined by

$$\beta_n = \beta + b/n^\alpha \quad \text{and} \quad K_n = K(\beta) + k/n^\alpha. \tag{5.5}$$

This sequence converges to the second-order point $(\beta, K(\beta))$ along a ray with slope k/b if $b \neq 0$. We assume that $K'(\beta)b - k < 0$. Under this assumption it is proved in Theorem 4.1 in [8] that sequence 1 satisfies the hypotheses of Theorem 3.1 with $\alpha_0 = 1/2$ and $\theta = 1/2$. When $K'(\beta)b - k < 0$, we refer to sequence 1 as sequence 1a.

Hypothesis (iii′) (in Theorem 4.1 for sequence 1a) Since $\alpha_0 = 1/2$ and $\theta = 1/2$, $\theta\alpha_0$ lies in the interval $[1/4, 1/2)$ as required by hypothesis (iii′). The limits in (5.1) for $j = 2, 3, 4$ are proved in Appendix A in [7]. We now prove that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4$ using the formulas for g and $\bar{x}$ in Theorem 4.1 in [8]. Let $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/8 \cdot 4!$. Since $0 < \beta < \beta_c = \log 4$, we have $e^\beta < e^{\beta_c} = 4$, which implies $c_4(\beta) > 0$. Since $K'(\beta)b - k < 0$, these formulas yield

$$g^{(2)}(\bar{x}) = 2\beta\big(K'(\beta)b - k\big) + 3 \cdot 4c_4(\beta)\bar{x}^2 = 4\beta\big(k - K'(\beta)b\big) > 0,$$
$$g^{(3)}(\bar{x}) = 4!c_4(\beta)\bar{x} > 0, \quad \text{and} \quad g^{(4)}(\bar{x}) = 4!c_4(\beta) > 0.$$

Thus under the condition $K'(\beta)b - k < 0$ sequence 1a satisfies all the hypotheses of Theorem 4.1.

Sequence 2a

Definition Given $0 < \beta < \beta_c$, $\alpha > 0$, $b \in \{1, -1\}$, an integer $p \geq 2$, and a real number $\ell \neq K^{(p)}(\beta)$, sequence 2 is defined by

$$\beta_n = \beta + b/n^\alpha \quad \text{and} \quad K_n = K(\beta) + \sum_{j=1}^{p-1} K^{(j)}(\beta)b^j/\big(j!n^{j\alpha}\big) + \ell b^p/\big(p!n^{p\alpha}\big). \tag{5.6}$$

This sequence converges to the second-order point $(\beta, K(\beta))$ along a curve that coincides with the second-order curve to order $n^{-(p-1)\alpha}$. We assume that $(K^{(p)}(\beta) - \ell)b^p < 0$. Under this assumption it is proved in Theorem 4.2 in [8] that sequence 2 satisfies the hypotheses of Theorem 3.1 with $\alpha_0 = 1/(2p)$ and $\theta = p/2$. When $(K^{(p)}(\beta) - \ell)b^p < 0$, we refer to sequence 2 as sequence 2a.

Hypothesis (iii′) (in Theorem 4.1 for sequence 2a) Since $\alpha_0 = 1/(2p)$ and $\theta = p/2$, $\theta\alpha_0$ lies in the interval $[1/4, 1/2)$ as required by hypothesis (iii′). The limits in (5.1) for $j = 2, 3, 4$ are proved in Appendix A in [7]. We now prove that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4$ using the formulas for g and $\bar{x}$ in Theorem 4.2 in [8]. Let $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/8 \cdot 4!$. Since $0 < \beta < \beta_c = \log 4$, we have $e^\beta < e^{\beta_c} = 4$, which implies $c_4(\beta) > 0$. Since $(K^{(p)}(\beta) - \ell)b^p < 0$, these formulas yield

$$g^{(2)}(\bar{x}) = \frac{2}{p!}\beta\big(K^{(p)}(\beta) - \ell\big)b^p + 3 \cdot 4c_4(\beta)\bar{x}^2 = \frac{4}{p!}\beta\big(\ell - K^{(p)}(\beta)\big)b^p > 0,$$
$$g^{(3)}(\bar{x}) = 4!c_4(\beta)\bar{x} > 0, \quad \text{and} \quad g^{(4)}(\bar{x}) = 4!c_4(\beta) > 0.$$

Thus under the condition $(K^{(p)}(\beta) - \ell)b^p < 0$ sequence 2a satisfies all the hypotheses of Theorem 4.1.

Sequence 3a

Definition Given $\alpha > 0$, $b \in \{1, 0, -1\}$, and $k \in \mathbb{R}$, $k \neq 0$, sequence 3 is defined by

$$\beta_n = \beta_c + b/n^\alpha \quad \text{and} \quad K_n = K(\beta_c) + k/n^\alpha. \tag{5.7}$$

This sequence converges to the tricritical point $(\beta_c, K(\beta_c))$ along a ray with slope k/b if $b \neq 0$. We assume that $K'(\beta_c)b - k < 0$. Under this assumption it is proved in Theorem 4.3 in [8] that sequence 3 satisfies the hypotheses of Theorem 3.1 with $\alpha_0 = 2/3$ and $\theta = 1/4$. When $K'(\beta_c)b - k < 0$, we refer to sequence 3 as sequence 3a.

Hypothesis (iii′) (in Theorem 4.1 for sequence 3a) Since $\alpha_0 = 2/3$ and $\theta = 1/4$, $\theta\alpha_0$ lies in the interval $[1/6, 1/2)$ as required by hypothesis (iii′). The limits in (5.2) for $j = 2, 3, 4, 5, 6$ are proved in Appendix A in [7]. We now prove that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4, 5, 6$ using the formulas for g and $\bar{x}$ in Theorem 4.3 in [8]. Let $c_6 = 9/40$. Since $K'(\beta_c)b - k < 0$, these formulas yield

$$g^{(2)}(\bar{x}) = 2\beta_c\big(K'(\beta_c)b - k\big) + 5 \cdot 6c_6\bar{x}^4 = 8\beta_c\big(k - K'(\beta_c)b\big) > 0,$$
$$g^{(3)}(\bar{x}) = 4 \cdot 5 \cdot 6c_6\bar{x}^3 > 0, \qquad g^{(4)}(\bar{x}) = 3 \cdot 4 \cdot 5 \cdot 6c_6\bar{x}^2 > 0,$$
$$g^{(5)}(\bar{x}) = 6!c_6\bar{x} > 0, \quad \text{and} \quad g^{(6)}(\bar{x}) = 6!c_6 > 0.$$

Thus under the condition $K'(\beta_c)b - k < 0$ sequence 3a satisfies all the hypotheses of Theorem 4.1.

Sequence 4a

Definition Given $\alpha > 0$, a curvature parameter $\ell \in \mathbb{R}$, and another parameter $\tilde{\ell} \in \mathbb{R}$, sequence 4 is defined by

$$\beta_n = \beta_c + 1/n^\alpha \quad \text{and} \quad K_n = K(\beta_c) + K'(\beta_c)/n^\alpha + \ell/\big(2n^\alpha\big) + \tilde{\ell}/\big(6n^{3\alpha}\big). \tag{5.8}$$

This sequence converges from the right to the tricritical point $(\beta_c, K(\beta_c))$ along the curve $(\beta, \tilde{K}(\beta))$, where for $\beta > \beta_c$

$$\tilde{K}(\beta) = K(\beta_c) + K'(\beta_c)(\beta - \beta_c) + \ell(\beta - \beta_c)^2/2 + \tilde{\ell}(\beta - \beta_c)^3/6.$$

The first-order curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$ is shown in Fig. 1 in the introduction. In order to determine a condition on the coefficients guaranteeing that sequence 4 satisfies the hypotheses of Theorem 3.1, we must study $K_1(\beta)$ more closely.

Since $\lim_{\beta\to\beta_c^+} K_1(\beta) = K(\beta_c)$ [12, Sects. 3.1, 3.3], by continuity we extend the definition of $K_1(\beta)$ from $\beta > \beta_c$ to $\beta = \beta_c$ by define $K_1(\beta_c) = K(\beta_c)$. In addition we must assume other properties of K_1 that are stated in Conjectures 1 and 2 on page 119 of [8]. As a preliminary to stating these conjectures, we assume that the first three right-hand derivatives of $K_1(\beta)$ exist at β_c and denote them by $K_1'(\beta_c)$, $K_1''(\beta_c)$, and $K_1'''(\beta_c)$. We also define $\ell_c = K''(\beta_c) - 5/(4\beta_c)$. Conjectures 1 and 2 state the following: (1) $K_1'(\beta_c) = K'(\beta_c)$ and (2) $K_1''(\beta_c) = \ell_c < 0 < K''(\beta_c)$. These conjectures are discussed in detail in Sect. 5 of [9] and are supported by properties of the Ginzburg-Landau polynomials and numerical calculations.

We assume that $\ell > \ell_c$, which by conjecture 1 equals $K_1''(\beta_c)$. Under this assumption it is proved in Theorem 4.4 in [8] that sequence 4 satisfies the hypotheses of Theorem 3.1 with $\alpha_0 = 1/3$ and $\theta = 1/2$. When $\ell > \ell_c$, we refer to sequence 4 as sequence 4a.

Hypothesis (iii′) (in Theorem 4.1 for sequence 4a) Since $\alpha_0 = 1/3$ and $\theta = 1/2$, $\theta\alpha_0$ lies in the interval $[1/6, 1/2)$ as required by hypothesis (iii′). The limits in (5.2) for $j = 2, 3, 4, 5, 6$ are proved in Appendix A in [7]. Define

$$y = \left(1 + \frac{3}{5}\beta_c\big(\ell - K''(\beta_c)\big)\right)^{1/2}. \tag{5.9}$$

Since $\ell > \ell_c = K''(\beta_c) - 5/(4\beta_c)$, we have $y > 1/2$. We now prove that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4, 5, 6$ using the formulas for g and $\bar{x}$ in Theorem 4.4 in [8]. Let $c_4 = 3/16$ and $c_6 = 9/40$. These formulas yield

$$\begin{aligned}
g^{(2)}(\bar{x}) &= \beta_c\big(K''(\beta_c) - \ell\big) - 3\cdot 4\cdot 4c_4\bar{x}^2 + 5\cdot 6c_6\bar{x}^4 = \frac{20}{3}y^2 + \frac{20}{3}y > 0,\\
g^{(3)}(\bar{x}) &= -4!\cdot 4c_4\bar{x} + 4\cdot 5\cdot 6c_6\bar{x}^3 = 9\bar{x}\left(\frac{4}{3} + \frac{10}{3}y\right) > 0,\\
g^{(4)}(\bar{x}) &= -4!\cdot 4c_4 + 3\cdot 4\cdot 5\cdot 6c_6\bar{x}^2 = -18 + 90(1+y) > 0,\\
g^{(5)}(\bar{x}) &= 6!c_6\bar{x} > 0, \quad \text{and} \quad g^{(6)}(\bar{x}) = 6!c_6 > 0.
\end{aligned}$$

Thus under the condition $\ell > \ell_c = K_1''(\beta_c)$ sequence 4a satisfies all the hypotheses of Theorem 4.1.

Sequence 5a

Definition Given $\alpha > 0$ and a real number $\ell \neq K''(\beta_c)$, sequence 5 is defined by

$$\beta_n = \beta_c - 1/n^{\alpha} \quad \text{and} \quad K_n = K(\beta_c) - K'(\beta_c)/n^{\alpha} + \ell/\big(2n^{2\alpha}\big). \tag{5.10}$$

This sequence converges to the tricritical point $(\beta_c, K(\beta_c))$ from the left along the curve that coincide with the second-order curve to order 2 in powers of $\beta - \beta_c$. We assume that $\ell > K''(\beta_c)$. Under this assumption it is proved in Theorem 4.5 in [8] that sequence 5 satisfies the hypotheses of Theorem 3.1 with $\alpha_0 = 1/3$ and $\theta = 1/2$. When $\ell > K''(\beta_c)$, we refer to sequence 5 as sequence 5a.

Hypothesis (iii′) (in Theorem 4.1 for sequence 5a) Since $\alpha_0 = 1/3$ and $\theta = 1/2$, $\theta\alpha_0$ lies in the interval $[1/6, 1/2)$ as required by hypothesis (iii′). The limits in (5.2) for $j = 2, 3, 4, 5, 6$ are proved in Appendix A in [7]. Define y as in (5.9). Since $\ell > K''(\beta_c)$, we have $y > 1$. We now prove that $g^{(j)}(\bar{x}) > 0$ for $j = 2, 3, 4, 5, 6$ using the formulas for g and $\bar{x}$ in Theorem 4.5 in [8]. Let $c_4 = 3/16$ and $c_6 = 9/40$. These formulas yield

$$g^{(2)}(\bar{x}) = \beta_c\big(K''(\beta_c) - \ell\big) + 3 \cdot 4 \cdot 4c_4\bar{x}^2 + 5 \cdot 6c_6\bar{x}^4 = \frac{20}{3}y(y-1) > 0,$$
$$g^{(3)}(\bar{x}) = 4! \cdot 4c_4\bar{x} + 4 \cdot 5 \cdot 6c_6\bar{x}^3 > 0, \quad g^{(4)}(\bar{x}) = 4! \cdot 4c_4 + 3 \cdot 4 \cdot 5 \cdot 6c_6 \cdot \bar{x}^2 > 0,$$
$$g^{(5)}(\bar{x}) = 6!c_6\bar{x} > 0, \quad \text{and} \quad g^{(6)}(\bar{x}) = 6!c_6 > 0.$$

Thus under the condition $\ell > K''(\beta_c)$ sequence 5a satisfies all the hypotheses of Theorem 4.1.

We have completed the discussion of the verification of the hypotheses of Theorem 4.1 for sequences 1a–5a in Table 2. This is the content of part (c) of Theorem 4.1. Part (a) of that theorem is proved in the next section.

6 Proof of Part (a) of Theorem 4.1

Theorem 6.1, a new theorem stated in this section, has two parts. Under the same hypotheses as Theorem 4.1, part (a) of Theorem 6.1 states a conditional central limit theorem: conditioned on the event $\{S_n/n > \delta m(\beta_n, K_n)\}$ for $\delta \in (0,1)$ sufficiently close to 1, the P_{n,β_n,K_n}-distributions of $n^\kappa(S_n/n - m(\beta_n, K_n))$ converge weakly to an $N(0, 1/g^{(2)}(\bar{x}))$-random variable with mean 0 and variance $1/g^{(2)}(\bar{x})$. Under the same hypotheses as Theorem 4.1, part (b) of Theorem 6.1 states the related conditional limit

$$\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\big\{\big|S_n/n - m(\beta_n, K_n)\big| \,\big|\, S_n/n > \delta m(\beta_n, K_n)\big\}$$
$$= E\big\{\big|N\big(0, 1/g^{(2)}(\bar{x})\big)\big|\big\} = \big(2/\big(\pi g^{(2)}(\bar{x})\big)\big)^{1/2} = \bar{z}.$$

We now comment on the proof of part (a) of Theorem 4.1 from part (b) of Theorem 6.1. In Lemma 6.3, we show that the moderate deviation principle in Theorem 6.2 and the asymptotic behavior of $m(\beta_n, K_n)$ in Theorem 3.1 imply that the event $\{S_n/n > \delta m(\beta_n, K_n)\}$ and the symmetric event $\{S_n/n < -\delta m(\beta_n, K_n)\}$ have large probability and that the event $\{\delta m(\beta_n, K_n) > S_n/n > -\delta m(\beta_n, K_n)\}$ has an exponentially small probability. As we show at the end of this section, combining part (b) of Theorem 6.1 with Lemma 6.3 and using symmetry yield

$$\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m(\beta_n, K_n)\big|\big\} = \bar{z}.$$

This is part (a) of Theorem 4.1.

The proofs of parts (a) and (b) of Theorem 6.1 are long and technical. In Sect. 7 we outline first the proof of part (b) of Theorem 6.1 and then the proof of part (a). As we will see, the latter follows the pattern of proof of part (b) but is more straightforward. Part (b) of Theorem 6.1 is proved in Sects. 8a, 8b, and 8c in [7], using a number of preparatory lemmas in Sect. 7 of [7]. The weak convergence result proved in Lemma 7.7 in [7] is the seed that yields both the conditional central limit theorem in part (a) of Theorem 6.1 and the conditional limit in part (b) of Theorem 6.1.

The hypotheses of Theorem 6.1 coincide with the hypotheses of Theorem 4.1. Part (c) of Theorem 6.1 states that for $\alpha \in (0, \alpha_0)$, $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ lies in the interval $(\theta\alpha_0, 1/2)$. This fact is needed in the proofs of Lemmas 7.2, 7.5, and 8.4 in [7]. The proof that $\kappa \in (\theta\alpha_0, 1/2)$ is elementary. By hypothesis (iii′) of Theorem 4.1, we have $\theta\alpha_0 < 1/2$, which gives $\theta < 1/(2\alpha_0)$. Therefore

$$\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha = \frac{1}{2} + \alpha\big(\theta - 1/(2\alpha_0)\big) < 1/2.$$

Since $0 < \alpha < \alpha_0$ and $\theta < 1/(2\alpha_0)$, we have $\kappa > \frac{1}{2} + \alpha_0(\theta - 1/(2\alpha_0)) = \theta\alpha_0$. This completes the proof of part (c) of Theorem 6.1.

Concerning part (d) of Theorem 6.1, the hypotheses of this theorem coincide with the hypotheses of Theorem 4.1. Thus, as shown in Sect. 5 of this paper and in Appendix A in [7], these hypotheses are satisfied by sequences 1a–5a in Table 2.

Theorem 6.1 *Let (β_n, K_n) be a positive sequence converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta_c, K(\beta_c))$. We assume that for all $0 < \alpha < \alpha_0$, (β_n, K_n) satisfies the hypotheses of Theorem 4.1, which coincide with the hypotheses of Theorem 3.1 together with hypothesis (iii′). For $\alpha \in (0, \alpha_0)$ we define $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. Then for any $0 < \alpha < \alpha_0$ there exists $\Delta \in (0, 1)$ such that for any $\delta \in (\Delta, 1)$ the following conclusions hold.*

(a) *When conditioned on the event $\{S_n/n > \delta m(\beta_n, K_n)\}$, the P_{n,β_n,K_n}-distributions of $n^\kappa(S_n/n - m(\beta_n, K_n))$ converge weakly to a normal random variable $N(0, 1/g^{(2)}(\bar{x}))$ with mean 0 and variance $1/g^{(2)}(\bar{x})$; in symbols,*

$$P_{n,\beta_n,K_n}\big\{n^\kappa\big(S_n/n - m(\beta_n, K_n)\big) \in dx \,\big|\, S_n/n > \delta m(\beta_n, K_n)\big\} \quad\Longrightarrow\quad N\big(0, 1/g^{(2)}(\bar{x})\big).$$

(b) *We have the conditional limit*

$$\begin{aligned}&\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\big\{\big|S_n/n - m(\beta_n, K_n)\big| \,\big|\, S_n/n > \delta m(\beta_n, K_n)\big\}\\ &\quad= \lim_{n\to\infty} E_{n,\beta_n,K_n}\big\{\big|S_n/n^{1-\kappa} - n^\kappa m(\beta_n, K_n)\big| \,\big|\, S_n/n > \delta m(\beta_n, K_n)\big\} = \bar{z},\end{aligned}$$

where

$$\begin{aligned}\bar{z} &= E\big\{\big|N\big(0, 1/g^{(2)}(\bar{x})\big)\big|\big\}\\ &= \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} |x| \exp\left[-\frac{1}{2}g^{(2)}(\bar{x})x^2\right] dx = \left(\frac{2}{\pi g^{(2)}(\bar{x})}\right)^{1/2}.\end{aligned}$$

(c) *For $\alpha \in (0, \alpha_0)$, $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$ lies in the interval $(\theta\alpha_0, 1/2)$.*
(d) *The hypotheses of this theorem are satisfied by sequences 1a–5a in Table 2.*

In part (a) of Theorem 6.2 we state a moderate deviation principle (MDP) for the mean-field B-C model. This MDP will be used to prove Lemma 6.3, which in turn will be used to prove part (a) of Theorem 4.1 from part (b) of Theorem 6.1. The rate function in the MDP is the continuous function $\Gamma(x) = g(x) - \inf_{y\in\mathbb{R}} g(y)$, where g is the associated Ginzburg-Landau polynomial. Γ satisfies $\Gamma(x) \to \infty$ as $|x| \to \infty$. For A a subset of $\mathbb{R}$ define $\Gamma(A) = \inf_{x\in A} \Gamma(x)$.

Theorem 6.2 *Let (β_n, K_n) be a positive sequence converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta_c, K(\beta_c))$. We assume that (β_n, K_n) satisfies the hypotheses of Theorem 4.1 for all $0 < \alpha < \alpha_0$. The following conclusions hold.*

(a) *For all $0 < \alpha < \alpha_0$, $S_n/n^{1-\theta\alpha}$ satisfies the MDP with respect to P_{n,β_n,K_n} with exponential speed $n^{1-\alpha/\alpha_0}$ and rate function $\Gamma(x) = g(x) - \inf_{y\in\mathbb{R}} g(y)$; i.e., for any closed set F in $\mathbb{R}$*

$$\limsup_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} \log P_{n,\beta_n,K_n}\big\{S_n/n^{1-\theta\alpha} \in F\big\} \le -\Gamma(F)$$

and for any open set G in $\mathbb{R}$

$$\liminf_{n\to\infty} \frac{1}{n^{1-\alpha/\alpha_0}} \log P_{n,\beta_n,K_n}\left\{S_n/n^{1-\theta\alpha} \in G\right\} \geq -\Gamma(G).$$

(b) *The hypotheses of this theorem are satisfied by sequences* 1a–5a *in Table* 2.

The MDP in part (a) of Theorem 6.2 is proved like the MDP in part (a) of Theorem 8.1 in [6] with only changes in notation. Because of the importance of part (a) of Theorem 6.2, the proof is given in Appendix B of [7]. Concerning part (b) of Theorem 6.2, the hypotheses of this theorem coincide with the hypotheses of Theorem 4.1. Thus, as shown in Sect. 5 of this paper and in Appendix A in [7], these hypotheses are satisfied by sequences 1a–5a in Table 2.

After proving the next lemma, we use it to derive part (a) of Theorem 4.1 from part (b) of Theorem 6.1.

Lemma 6.3 *We assume that (β_n, K_n) satisfies the hypotheses of Theorem 4.1 for all $0 < \alpha < \alpha_0$. Then for any $0 < \alpha < \alpha_0$ and any $\delta \in (0,1)$ there exists $c > 0$ such that for all sufficiently large n*

$$P_{n,\beta_n,K_n}\left\{\delta m(\beta_n,K_n) \geq S_n/n \geq -\delta m(\beta_n,K_n)\right\} \leq \exp\left[-c n^{1-\alpha/\alpha_0}\right] \to 0 \quad \textit{as } n\to\infty.$$

In addition,

$$\lim_{n\to\infty} P_{n,\beta_n,K_n}\left\{S_n/n > \delta m(\beta_n,K_n)\right\} = \lim_{n\to\infty} P_{n,\beta_n,K_n}\left\{S_n/n < -\delta m(\beta_n,K_n)\right\} = 1/2.$$

Proof By hypothesis (iii)(b) of Theorem 3.1 the global minimum points of g are $\pm\bar{x}$, and by Theorem 3.1, $n^{\theta\alpha} m(\beta_n,K_n) \to \bar{x}$ as $n\to\infty$. Thus we can choose $\varepsilon > 0$ satisfying $(1+\varepsilon)\delta < 1$ such that $n^{\theta\alpha} m(\beta_n,K_n) \leq (1+\varepsilon)\bar{x}$ for large n. Let F be the closed set $[-(1+\varepsilon)\delta\bar{x},\ (1+\varepsilon)\delta\bar{x}]$. Since $(1+\varepsilon)\delta\bar{x} < \bar{x}$ and $-(1+\varepsilon)\delta\bar{x} > -\bar{x}$, we have

$$\inf_{y\in F} g(y) > \inf_{z\in\mathbb{R}} g(z) = g(\bar{x}),$$

which implies

$$\Gamma(F) = \inf_{y\in F}\left\{g(y) - \inf_{z\in\mathbb{R}} g(z)\right\} > 0.$$

We write $m_n = m(\beta_n,K_n)$. The moderate deviation upper bound in part (a) of Theorem 6.2 implies that for all sufficiently large n

$$\begin{aligned}
&P_{n,\beta_n,K_n}\{\delta m_n \geq S_n/n \geq -\delta m_n\} \\
&\quad = P_{n,\beta_n,K_n}\left\{\delta n^{\theta\alpha} m_n \geq S_n/n^{1-\theta\alpha} \geq -\delta n^{\theta\alpha} m_n\right\} \\
&\quad \leq P_{n,\beta_n,K_n}\left\{(1+\varepsilon)\delta\bar{x} \geq S_n/n^{1-\theta\alpha} \geq -(1+\varepsilon)\delta\bar{x}\right\} \\
&\quad \leq \exp\left[-n^{1-\alpha/\alpha_0}\Gamma(F)/2\right] \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

This yields the first assertion in the lemma.

To prove the second assertion, we write

$$\begin{aligned}
1 &= P_{n,\beta_n,K_n}\{S_n/n \in \mathbb{R}\} \\
&= P_{n,\beta_n,K_n}\{\delta m_n \geq S_n/n \geq -\delta m_n\} \\
&\quad + P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\} \\
&\quad + P_{n,\beta_n,K_n}\{S_n/n < -\delta m_n\}.
\end{aligned}$$

Symmetry and the first assertion imply that

$$\lim_{n\to\infty} P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\} = \lim_{n\to\infty} P_{n,\beta_n,K_n}\{S_n/n < -\delta m_n\} = 1/2.$$

This completes the proof of the lemma. □

Now we are ready to prove part (a) of Theorem 4.1.

Proof of part (a) of Theorem 4.1 from part (b) of Theorem 6.1 and Lemma 6.3 We write $m_n = m(\beta_n, K_n)$. Define

$$p^+_{n,\delta} = P_{n,\beta_n,K_n}\{S_n/n > \delta m_n\}, \qquad p^-_{n,\delta} = P_{n,\beta_n,K_n}\{S_n/n < -\delta m_n\},$$
$$q_{n,\delta} = P_{n,\beta_n,K_n}\{\delta m_n \ge S_n/n \ge -\delta m_n\}.$$

Since by symmetry $p^+_{n,\delta} = p^-_{n,\delta}$ and

$$\begin{aligned} &E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m_n\big| \;\big|\; S_n/n < -\delta m_n\big\} \\ &\quad = E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m_n\big| \;\big|\; S_n/n > \delta m_n\big\} \\ &\quad = E_{n,\beta_n,K_n}\big\{|S_n/n - m_n| \;\big|\; S_n/n > \delta m_n\big\}, \end{aligned}$$

we have

$$\begin{aligned} E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m_n\big|\big\} &= E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m_n\big| \;\big|\; S_n/n > \delta m_n\big\} \cdot p^+_{n,\delta} \\ &\quad + E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m_n\big| \;\big|\; S_n/n < -\delta m_n\big\} \cdot p^-_{n,\delta} \\ &\quad + E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m_n\big| \;\big|\; \delta m_n \ge S_n/n \ge -\delta m_n\big\} \cdot q_{n,\delta} \\ &= 2 \cdot E_{n,\beta_n,K_n}\big\{\big|S_n/n - m_n\big| \;\big|\; S_n/n > \delta m_n\big\} \cdot p^+_{n,\delta} \\ &\quad + E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m_n\big| \;\big|\; \delta m_n \ge S_n/n \ge -\delta m_n\big\} \cdot q_{n,\delta}. \end{aligned}$$

By part (b) of Theorem 6.1 and Lemma 6.3

$$\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\big\{|S_n/n - m_n| \;\big|\; S_n/n > \delta m_n\big\} \cdot p^+_{n,\delta} = \frac{1}{2}\bar{z}.$$

Since $|S_n/n| \le 1$ and $0 \le m_n \le 1$, Lemma 6.3 implies that there exists $c > 0$ such that for all sufficiently large n

$$\begin{aligned} &n^\kappa E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m_n\big| \;\big|\; \delta m_n \ge S_n/n \ge -\delta m_n\big\} \cdot q_{n,\delta} \\ &\quad \le 2n^\kappa q_{n,\delta} \le 2n^\kappa \exp\big[-cn^{1-\alpha/\alpha_0}\big] \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\big\{\big||S_n/n| - m_n\big|\big\} = \frac{1}{2}\bar{z} + \frac{1}{2}\bar{z} = \bar{z}.$$

Part (a) of Theorem 4.1 is proved. □

In the next section we outline the proof of Theorem 6.1.

7 Outline of the Proof of Theorem 6.1

In this section we motivate the proof of parts (a) and (b) of Theorem 6.1. We focus on the proof of part (b) because it is somewhat more complicated than the proof of part (a). The numerous and subtle technical details of the proof are given in Sects. 7 and 8 in [7]. The analogous but more straightforward proof of part (a) is discussed at the end of this section.

Let (β_n, K_n) be a positive sequence. Throughout this section we work with $0 < \alpha < \alpha_0$ and denote $m(\beta_n, K_n)$ by m_n. Let W_n be a sequence of normal random variables with mean 0 and variance $(2\beta_n K_n)^{-1}$ defined on a probability space $(\Omega, \mathcal{F}, Q)$. We denote by $\tilde{E}_{n,\beta_n,K_n}$ expectation with respect to the product measure $P_{n,\beta_n,K_n} \times Q$; P_{n,β_n,K_n} is defined in (2.1)–(2.2).

We define $\kappa = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. Part (b) of Theorem 6.1 states that there exists $\Delta \in (0,1)$ such that for any $\delta \in (\Delta, 1)$

$$\begin{aligned}
&\lim_{n\to\infty} n^\kappa E_{n,\beta_n,K_n}\{|S_n/n - m_n| \mid S_n/n > \delta m_n\} \\
&\quad = \lim_{n\to\infty} E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2} g^{(2)}(\bar{x}) x^2] dx} \cdot \int_{\mathbb{R}} |x| \exp\left[-\frac{1}{2} g^{(2)}(\bar{x}) x^2\right] dx = \bar{z}. \qquad (7.1)
\end{aligned}$$

The key idea in proving (7.1) is to show that adding suitably scaled versions of the normal random variables W_n yields a quantity with the following two properties: its limit equals the last line of (7.1) and the second line of (7.1) has the same limit; specifically,

$$\begin{aligned}
&\lim_{n\to\infty} E_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\} \\
&\quad = \lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n + W_n/n^{1/2} > \delta m_n\} \\
&\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2} g^{(2)}(\bar{x}) x^2]\, dx} \cdot \int_{\mathbb{R}} |x| \exp\left[-\frac{1}{2} g^{(2)}(\bar{x}) x^2\right] dx. \qquad (7.2)
\end{aligned}$$

Formula (7.2) is proved in two steps.

Step 1. Prove the second limit in (7.2). The proof is based on the representation formula (7.3) and a Taylor expansion. The verification of the second limit in (7.2) is the heart of the proof of part (b) of Theorem 6.1 and is motivated just below. In the course of this motivation, we explain why κ equals $\frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. The proof of the second limit in (7.2) is given in part (b) of Lemma 8.1 in [7].

Step 2. Prove the first limit in (7.2). This is done in two substeps, which we now explain.

Substep 2a. Define

$$C_n = \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\}$$

and

$$D_n = \tilde{E}_{n,\beta_n,K_n}\{|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n| \mid S_n/n > \delta m_n\}.$$

Substep 2a is to prove that $\lim_{n\to\infty} |C_n - D_n| = 0$. This is certainly plausible since C_n and D_n differ only by $W_n/n^{1/2-\kappa}$ in the argument of the absolute value and since $W_n/n^{1/2-\kappa} \to 0$ with probability 1. Substep 2a is proved in Lemma 8.3 in [7].

Substep 2b. Define

$$F_n = \tilde{E}_{n,\beta_n,K_n}\big\{\big|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n\big| \,\big|\, S_n/n + W_n/n^{1/2} > \delta m_n\big\}.$$

Substep 2b is to prove that

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty} F_n = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]\,dx} \cdot \int_{\mathbb{R}} |x| \exp\left[-\frac{1}{2}g^{(2)}(\bar{x})x^2\right] dx.$$

The limit of F_n as $n \to \infty$ is calculated in Step 1. The first equality in the last display is certainly plausible since D_n and F_n differ only by $W_n/n^{1/2}$ in the conditioned event and $W_n/n^{1/2} \to 0$ with probability 1. The proof of this equality, given in part (b) of Lemma 8.4 in [7], is much more subtle than the proof that $\lim_{n\to\infty} |C_n - D_n| = 0$ in Substep 2a.

Having explained the logic of the proof of part (b) of Theorem 6.1, we now outline the proof of the second limit in (7.2). We start by applying a representation formula that can be proved like Lemma 3.3 in [10], which applies to the Curie-Weiss model, or like Lemma 3.2 in [13], which applies to the Curie-Weiss-Potts model. For any nonnegative measurable function φ, this representation formula states that

$$\begin{aligned}\int_{\Lambda^n\times\Omega} &\varphi\big(S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa}\big)\, d(P_{n,\beta_n,K_n} \times Q)\\ &= \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\kappa)]\,dx} \cdot \int_{\mathbb{R}} \varphi(x) \exp\big[-nG_{\beta_n,K_n}\big(x/n^\kappa\big)\big]\,dx. \qquad (7.3)\end{aligned}$$

In this formula G_{β_n,K_n} is the free-energy function defined in (2.3)–(2.4).

We now adapt this representation to the expected value in the second line of (7.2). For $\bar{\delta} \in (0, 1)$ define

$$A_n(\bar{\delta}) = \big\{S_n/n + W_n/n^{1/2} > \bar{\delta} m_n\big\} = \big\{S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} > \bar{\delta} n^\kappa m_n\big\}.$$

Then the expected value in the second line of (7.2) equals

$$\frac{1}{\tilde{E}_{n,\beta_n,K_n}\{1_{A_n(\bar{\delta})}\}} \cdot \tilde{E}_{n,\beta_n,K_n}\big\{\big|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n\big| \cdot 1_{A_n(\bar{\delta})}\big\}. \qquad (7.4)$$

Let $Z_{n,\kappa} = \int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\kappa)]\,dx$. Since

$$1_{A_n(\bar{\delta})} = 1_{(n^\kappa \bar{\delta} m_n,\infty)}\big(S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa}\big),$$

we can apply (7.3) with $\varphi(x) = |x - n^\kappa m_n| 1_{(n^\kappa \bar{\delta} m_n,\infty)}(x)$ to write the numerator in (7.4) as

$$\begin{aligned}\int_{\Lambda^n\times\Omega} &\big|S_n/n^{1-\kappa} + W_n/n^{1/2-\kappa} - n^\kappa m_n\big| \cdot 1_{A_n(\bar{\delta})}\, d(P_{n,\beta_n,K_n} \times Q)\\ &= \frac{1}{Z_{n,\kappa}} \cdot \int_{\mathbb{R}} \big|x - n^\kappa m_n\big| \cdot 1_{(n^\kappa \bar{\delta} m_n,\infty)}(x) \exp\big[-nG_{\beta_n,K_n}\big(x/n^\kappa\big)\big]\,dx,\\ &= \frac{1}{Z_{n,\kappa}} \cdot \int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty} |x| \exp\big[-nG_{\beta_n,K_n}\big(x/n^\kappa + m_n\big)\big]\,dx.\end{aligned}$$

The denominator in (7.4) can be written in the same form with the absolute value function replaced by the constant function 1. Multiplying the numerator and denominator in (7.4) by the constant $\exp[G_{\beta_n,K_n}(m_n)]$, we see that the conditional expectation in the second line of (7.2) can be written as

$$
\begin{aligned}
&\tilde{E}_{n,\beta_n,K_n}\left\{\left|S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n\right| \,\middle|\, S_n/n+W_n/n^{1/2}>\delta m_n\right\}\\
&\quad=\frac{1}{\int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty}\exp[-nG_{\beta_n,K_n}(x/n^\kappa+m_n)+nG_{\beta_n,K_n}(m_n)]\,dx}\\
&\qquad\cdot\int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty}|x|\exp\bigl[-nG_{\beta_n,K_n}\bigl(x/n^\kappa+m_n\bigr)+nG_{\beta_n,K_n}(m_n)\bigr]\,dx. \qquad (7.5)
\end{aligned}
$$

The conclusion is that Step 1 in the proof of part (b) of Theorem 6.1 is proved by showing that for $h(x)=|x|$ and for $h\equiv 1$

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty}h(x)\exp\bigl[-nG_{\beta_n,K_n}\bigl(x/n^\kappa+m_n\bigr)+nG_{\beta_n,K_n}(m_n)\bigr]\,dx\\
&\quad=\int_{\mathbb{R}}h(x)\exp\left[-\frac{1}{2}g^{(2)}(\bar{x})x^2\right]dx. \qquad (7.6)
\end{aligned}
$$

To facilitate the calculation in (7.6), we reexpress it in terms of measures. For $n\in\mathbb{N}$ let Ψ_n and Ψ denote the measures on $\mathbb{R}$ defined by

$$
\Psi_n(dx)=1_{(-n^\kappa(1-\bar{\delta})m_n,\infty)}(x)\cdot\exp\bigl[-nG_{\beta_n,K_n}\bigl(x/n^\kappa+m_n\bigr)+nG_{\beta_n,K_n}(m_n)\bigr]\,dx
$$

and

$$
\Psi(dx)=\exp\left[-\frac{1}{2}g^{(2)}(\bar{x})x^2\right]dx.
$$

Then the limits in (7.6) for $h(x)=|x|$ and $h(x)\equiv 1$ can be written as

$$
\lim_{n\to\infty}\int_{\mathbb{R}}|x|\,d\Psi_n=\int_{\mathbb{R}}|x|\,d\Psi \quad\text{and}\quad \lim_{n\to\infty}\int_{\mathbb{R}}1\,d\Psi_n=\int_{\mathbb{R}}1d\Psi. \qquad (7.7)
$$

We claim that the sequence Ψ_n converges weakly to Ψ, meaning that for any bounded, continuous function f

$$
\lim_{n\to\infty}\int_{\mathbb{R}}f\,d\Psi_n=\int_{\mathbb{R}}f\,d\Psi. \qquad (7.8)
$$

This will be motivated in the next few paragraphs. Taking $f\equiv 1$ gives the second limit in (7.7). As discussed in Theorem 4 in §II.6 of [14], the standard way of extending (7.8) from bounded, continuous functions f to the unbounded absolute function $|x|$ is to prove the uniform integrability estimate

$$
\lim_{j\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|x|>j\}}|x|\,d\Psi_n=0.
$$

In the present setting there is a condition weaker than uniform integrability that yields the same conclusion. As shown in [11, Prop. 8.3] and [7, Prop. 8.2], if

$$
\lim_{j\to\infty}\limsup_{n\to\infty}\int_{\{|x|>j\}}|x|d\Psi_n=0, \qquad (7.9)
$$

then $\int_{\mathbb{R}}|x|d\Psi_n\to\int_{\mathbb{R}}|x|d\Psi$. This is the first limit in (7.7). The verification of the limit in the last display is carried out in Sect. 8a of [7].

We now motivate the limit

$$
\lim_{n\to\infty}\int_{\mathbb{R}}f\,d\Psi_n=\int_{\mathbb{R}}f\,d\Psi=\int_{\mathbb{R}}f(x)\exp\left[-\frac{1}{2}g^{(2)}(\bar{x})x^2\right]dx \qquad (7.10)
$$

for any bounded, continuous function f. By definition of Ψ_n

$$\int_{\mathbb{R}} f\, d\Psi_n = \int_{\mathbb{R}} f \cdot 1_{(-n^\kappa(1-\bar{\delta})m_n,\infty)}(x) \cdot \exp\big[-nG_{\beta_n,K_n}(x/n^\kappa + m_n) + nG_{\beta_n,K_n}(m_n)\big]\, dx. \tag{7.11}$$

Formally, we motivate (7.10) by interchanging the limit with the integral and using the fact that, since $\bar{\delta} \in (0,1)$, $1_{(-n^\kappa(1-\bar{\delta})m_n,\infty)}(x)$ has the limit 1 for each x. We obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}} f\, d\Psi_n = \int_{\mathbb{R}} f(x)\Big(\lim_{n\to\infty} \exp\big[-nG_{\beta_n,K_n}\big(x/n^\kappa + m_n\big) + nG_{\beta_n,K_n}(m_n)\big]\Big)\, dx. \tag{7.12}$$

We complete the motivation of (7.10) by showing that

$$\lim_{n\to\infty} \big(nG_{\beta_n,K_n}\big(x/n^\kappa + m_n\big) - nG_{\beta_n,K_n}(m_n)\big) = \frac{1}{2} g^{(2)}(\bar{x}) x^2. \tag{7.13}$$

Given the limit in (7.13), the interchange of the limit with the integral in (7.12) must be justified. As shown in Lemma 7.7 in [7], we do this by appealing to two facts: first, the Dominated Convergence Theorem, which follows from Lemma 7.5 in [7], and second, the large deviation estimate given in Lemma 7.6 in [7]. In order to prove Lemma 7.5 in [7], hypothesis (iii′) in Theorem 4.1 is needed. In the statement of the conditional limit theorem in part (b) of Theorem 6.1, there appears a quantity $\Delta \in (0,1)$; the number δ appearing in the conditioned set must satisfy $\delta \in (\Delta, 1)$. The choice of Δ is dictated by the proof of the lower bound in part (b) of Lemma 7.5 in [7].

We complete the outline of the conditional limit theorem in part (b) of Theorem 6.1 by proving the limit (7.13), which is carried out by means of a Taylor expansion. The proof is outlined in the case when the degree of the Ginzburg-Landau polynomial g is 4. The Taylor expansion that yields (7.13) is subtle. Although the limit is quadratic, the expansion must include all terms through order x^5 to prove the convergence. When the degree of g is 6, the proof of (7.13) follows the same pattern except that the corresponding Taylor expansion must include all terms through order x^7 to prove the convergence. For details see the proof of part (a) of Lemma 7.5 in [7].

By Taylor's Theorem, for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\begin{aligned} &nG_{\beta_n,K_n}\big(x/n^\kappa + m_n\big) - nG_{\beta_n,K_n}(m_n) \\ &\quad = \sum_{j=1}^{4} \frac{1}{n^{j\kappa-1}} \cdot \frac{G^{(j)}_{\beta_n,K_n}(m_n)}{j!} x^j + \frac{1}{n^{5\kappa-1}} \cdot \frac{G^{(5)}_{\beta_n,K_n}(m_n + \tau x/n^\kappa)}{5!} x^5, \end{aligned}$$

where τ is a number in $[0,1]$. Since m_n is the unique, positive, global minimum point of G_{β_n,K_n}, we have $G^{(1)}_{\beta_n,K_n}(m_n) = 0$. The sequence (β_n, K_n) is positive and bounded, and the argument of $G^{(5)}_{\beta_n,K_n}$ is bounded uniformly in n. Thus, the term in the last display involving $G^{(5)}_{\beta_n,K_n}$ is bounded uniformly in n. We summarize the last display by writing

$$\begin{aligned} &nG_{\beta_n,K_n}\big(x/n^\kappa + m_n\big) - nG_{\beta_n,K_n}(m_n) \\ &\quad = \sum_{j=1}^{4} \frac{1}{n^{j\kappa-1}} \cdot \frac{G^{(j)}_{\beta_n,K_n}(m_n)}{j!} x^j + \mathrm{O}\Big(\frac{1}{n^{5\kappa-1}}\Big) x^5. \end{aligned}$$

Let ε_n denote a sequence that converges to 0 and that represents the various error terms arising in the proof. The same notation ε_n will be used to represent different error terms.

To simplify the arithmetic, we introduce $u = 1 - \alpha/\alpha_0 > 0$. We now use hypothesis (iii′) of Theorem 4.1, which states that for $j = 2, 3, 4$

$$G^{(j)}_{\beta_n,K_n}(m_n) = \frac{g^{(j)}(\bar{x}) + \varepsilon_n}{n^{\alpha/\alpha_0 - j\theta\alpha}} = \frac{g^{(j)}(\bar{x}) + \varepsilon_n}{n^{1-u-j\theta\alpha}}.$$

Substituting these values, we obtain the following asymptotic formula:

$$\begin{aligned} &nG_{\beta_n,K_n}\big(x/n^\kappa + m_n\big) - nG_{\beta_n,K_n}(m_n) \\ &\quad = \sum_{j=2}^{4} \frac{1}{n^{j\kappa-1}} \cdot \frac{g^{(j)}(\bar{x}) + \varepsilon_n}{n^{1-u-j\theta\alpha} \cdot j!} \cdot x^j + \mathrm{O}\bigg(\frac{1}{n^{5\kappa-1}}\bigg)x^5 \\ &\quad = \sum_{j=2}^{4} \frac{1}{n^{j\kappa-u-j\theta\alpha}} \cdot \frac{g^{(j)}(\bar{x}) + \varepsilon_n}{j!} \cdot x^j + \mathrm{O}\bigg(\frac{1}{n^{5\kappa-1}}\bigg)x^5. \end{aligned}$$

We choose κ so that the power of n is 0 in the term corresponding to $j = 2$. This gives $\kappa = \frac{1}{2}u + \theta\alpha = \frac{1}{2}(1 - \alpha/\alpha_0) + \theta\alpha$. Since for $j = 2, 3, 4$ we have $j\kappa - u - j\theta\alpha = (\frac{j}{2} - 1)u$, the last display takes the form

$$\begin{aligned} &nG_{\beta_n,K_n}\big(x/n^\kappa + m_n\big) - nG_{\beta_n,K_n}(m_n) \\ &\quad = \frac{1}{2!} \cdot \big(g^{(2)}(\bar{x}) + \varepsilon_n\big)x^2 + \frac{1}{n^{u/2}} \cdot \frac{(g^{(3)}(\bar{x}) + \varepsilon_n)}{3!}x^3 \\ &\qquad + \frac{1}{n^u} \cdot \frac{(g^{(4)}(\bar{x}) + \varepsilon_n)}{4!}x^4 + \mathrm{O}\bigg(\frac{1}{n^{5\kappa-1}}\bigg)x^5. \end{aligned} \tag{7.14}$$

By hypothesis (iii′) of Theorem 4.1 and part (c) of Theorem 6.1, we have $1/4 \le \theta\alpha_0 < \kappa < 1/2$. Therefore $5\kappa - 1 > 5\theta\alpha_0 - 1 > 0$. Since $u > 0$ and $\varepsilon_n \to 0$, it follows that for all $x \in \mathbb{R}$

$$\lim_{n\to\infty} \big(nG_{\beta_n,K_n}\big(x/n^\kappa + m_n\big) - nG_{\beta_n,K_n}(m_n)\big) = \frac{1}{2}g^{(2)}(\bar{x})x^2.$$

This completes the proof of (7.13), and thus the outline of the proof of part (b) of Theorem 6.1, when g has degree 4.

We now turn to the proof of the conditional central limit theorem in part (a) of Theorem 6.1. It follows the same pattern of proof as the conditional limit theorem in part (b) of Theorem 6.1, but is more straightforward. As we will see, like part (b) of Theorem 6.1, part (a) of that theorem is a consequence of the weak convergence limit stated in (7.10).

In order to prove the conditional central limit theorem, we must prove that for any bounded, continuous function f

$$\begin{aligned} &\lim_{n\to\infty} E_{n,\beta_n,K_n}\big\{f\big(n^\kappa\big(S_n/n - m(\beta_n, K_n)\big)\big) \bigm| S_n/n > \delta m(\beta_n, K_n)\big\} \\ &\quad = \lim_{n\to\infty} E_{n,\beta_n,K_n}\big\{f\big(S_n/n^{1-\kappa} - n^\kappa m(\beta_n, K_n)\big) \bigm| S_n/n > \delta m(\beta_n, K_n)\big\} \\ &\quad = E\big\{f\big(N\big(0, 1/g^{(2)}(\bar{x})\big)\big)\big\} \\ &\quad = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]dx} \cdot \int_{\mathbb{R}} f(x)\exp\bigg[-\frac{1}{2}g^{(2)}(\bar{x})x^2\bigg]dx. \end{aligned} \tag{7.15}$$

By the theory of weak convergence of measures, it suffices to prove this limit for any bounded, uniformly continuous function f [14, Remark 1, p. 313].

We prove (7.15) by the analogous two steps used to prove the conditional limit theorem in part (b) of Theorem 6.1, replacing the absolute value function by the bounded, uniformly continuous function f. Step 1 is based on the representation formula (7.3), which allows us to write

$$\begin{aligned}
&\tilde{E}_{n,\beta_n,K_n}\{f(S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n) \mid S_n/n+W_n/n^{1/2}>\delta m_n\} \\
&= \frac{1}{\int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty} \exp[-nG_{\beta_n,K_n}(x/n^\kappa+m_n)+nG_{\beta_n,K_n}(m_n)]\,dx} \\
&\quad \cdot \int_{-n^\kappa(1-\bar{\delta})m_n}^{\infty} f(x)\exp[-nG_{\beta_n,K_n}(x/n^\kappa+m_n)+nG_{\beta_n,K_n}(m_n)]\,dx. \qquad (7.16)
\end{aligned}$$

The weak convergence limit (7.10) applied to the given f and to $f \equiv 1$ shows that

$$\begin{aligned}
&\tilde{E}_{n,\beta_n,K_n}\{f(S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n) \mid S_n/n+W_n/n^{1/2}>\delta m_n\} \\
&= \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]\,dx} \cdot \int_{\mathbb{R}} f(x)\exp\left[-\frac{1}{2}g^{(2)}(\bar{x})x^2\right]dx. \qquad (7.17)
\end{aligned}$$

This completes Step 1 in the proof of part (a) of Theorem 6.1. We avoid the uniform integrability-type estimate (7.9), which is necessary for the unbounded absolute value function $|x|$.

Step 2 in the proof of part (a) of Theorem 6.1 is to show that

$$\begin{aligned}
&\lim_{n\to\infty} E_{n,\beta_n,K_n}\{f(n^\kappa(S_n/n-m(\beta_n,K_n))) \mid S_n/n>\delta m(\beta_n,K_n)\} \\
&= \lim_{n\to\infty} \tilde{E}_{n,\beta_n,K_n}\{f(S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n) \mid S_n/n+W_n/n^{1/2}>\delta m_n\}. \qquad (7.18)
\end{aligned}$$

Like the proof of part (b) of Theorem 6.1, Step 2 is done in two substeps.

Define

$$\tilde{C}_n = \tilde{E}_{n,\beta_n,K_n}\{f(S_n/n^{1-\kappa}-n^\kappa m_n) \mid S_n/n>\delta m_n\}$$

and

$$\tilde{D}_n = \tilde{E}_{n,\beta_n,K_n}\{f(S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n) \mid S_n/n>\delta m_n\}.$$

Substep 2a is to prove that $\lim_{n\to\infty}|\tilde{C}_n-\tilde{D}_n|=0$. In order to prove this, we write $|\tilde{C}_n-\tilde{D}_n|$ as a sum of an integral on the set where $|W_n/n^{1/2-\kappa}|\le\xi$, for a suitable choice of $\xi>0$, and an integral on the set where $|W_n/n^{1/2-\kappa}|>\xi$. By choosing $\xi>0$ sufficiently small, the uniform continuity of f guarantees that the first integral can be made arbitrarily small. Because there exists a constant $c>0$ such that for all $n\in\mathbb{N}$, $Q\{|W_n/n^{1/2-\kappa}|>\xi|\}\le\exp[-cn^{1-2\kappa}]$, the second integral can also be made arbitrarily small for all sufficiently large n.

Substep 2b is to prove that

$$\lim_{n\to\infty}\tilde{D}_n = \lim_{n\to\infty}\tilde{F}_n = \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2}g^{(2)}(\bar{x})x^2]\,dx} \cdot \int_{\mathbb{R}} f(x)\exp\left[-\frac{1}{2}g^{(2)}(\bar{x})x^2\right]dx,$$

where

$$\tilde{F}_n = \tilde{E}_{n,\beta_n,K_n}\{f(S_n/n^{1-\kappa}+W_n/n^{1/2-\kappa}-n^\kappa m_n) \mid S_n/n+W_n/n^{1/2}>\delta m_n\}.$$

Thus $\tilde{F}_n$ is obtained from $\tilde{D}_n$ by replacing S_n/n in the conditioned event $\{S_n/n>\delta m_n\}$ by $S_n/n+W_n/n^{1/2}$. The limit of $\tilde{F}_n$ as $n\to\infty$ is calculated in Step 1. We write $f=f^+-f^-$, where $f^+=\max(f,0)$ and $f^-=\max(-f,0)$. Substep 2b is proved by following the pattern of proof of Substep 2b in the proof of part (b) of Theorem 6.1. Specifically, we apply the calculations in the latter proof twice. The first time the absolute value function is replaced by f^+, and the second time the absolute value function is replaced by f^-.

Together, Step 1 and Substeps 2a and 2b show that

$$\begin{aligned}\lim_{n\to\infty} \tilde{C}_n &= \lim_{n\to\infty} \tilde{D}_n = \lim_{n\to\infty} \tilde{F}_n \\ &= \frac{1}{\int_{\mathbb{R}} \exp[-\frac{1}{2} g^{(2)}(\bar{x}) x^2]\, dx} \cdot \int_{\mathbb{R}} f(x) \exp\left[-\frac{1}{2} g^{(2)}(\bar{x}) x^2\right] dx.\end{aligned}$$

This is the limit (7.15). The outline of the proof of part (a) of Theorem 6.1 is now complete.

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References

1. Barber, M.N.: Finite-size scaling. In: Domb, C., Lebowitz, J. (eds.) Phase Transitions and Critical Phenomena, vol. 8, pp. 145–266. Academic Press, London (1983)
2. Blume, M.: Theory of the first-order magnetic phase change in UO_2. Phys. Rev. **141**, 517–524 (1966)
3. Capel, H.W.: On the possibility of first-order phase transitions in Ising systems of triplet ions with zero-field splitting. Physica **32**, 966–988 (1966)
4. Capel, H.W.: On the possibility of first-order phase transitions in Ising systems of triplet ions with zero-field splitting II. Physica **33**, 295–331 (1967)
5. Capel, H.W.: On the possibility of first-order phase transitions in Ising systems of triplet ions with zero-field splitting III. Physica **37**, 423–441 (1967)
6. Costeniuc, M., Ellis, R.S., Otto, P.T.-H.: Multiple critical behavior of probabilistic limit theorems in the neighborhood of a tricritical point. J. Stat. Phys. **127**, 495–552 (2007)
7. Ellis, R.S., Li, J.: Refined asymptotics of the finite-size magnetization via a new conditional limit theorem for the spin. http://arxiv.org/abs/1205.0970 (2012), 78 pages
8. Ellis, R.S., Machta, J., Otto, P.T.: Asymptotic behavior of the magnetization near critical and tricritical points via Ginzburg-Landau polynomials. J. Stat. Phys. **133**, 101–129 (2008)
9. Ellis, R.S., Machta, J., Otto, P.T.: Ginzburg-Landau polynomials and the asymptotic behavior of the magnetization near critical and tricritical points. http://arxiv.org/abs/0803.0178 (2008), 75 pages
10. Ellis, R.S., Newman, C.M.: Limit theorems for sums of dependent random variables occurring in statistical mechanics. Z. Wahrscheinlichkeitstheor. Verw. Geb. **44**, 117–139 (1979)
11. Ellis, R.S., Machta, J., Otto, P.T.: Asymptotic behavior of the finite-size magnetization as a function of the speed of approach to criticality. Ann. Appl. Probab. **20**, 2118–2161 (2010)
12. Ellis, R.S., Otto, P.T., Touchette, H.: Analysis of phase transitions in the mean-field Blume-Emery-Griffiths model. Ann. Appl. Probab. **15**, 2203–2254 (2005)
13. Ellis, R.S., Wang, K.: Limit theorems for the empirical vector of the Curie-Weiss-Potts model. Stoch. Process. Appl. **35**, 59–79 (1990)
14. Shiryaev, A.N.: Probability, 2nd edn. Springer, New York (1995). translated by R.P. Boas