

CONCAVITY OF MAGNETIZATION FOR A CLASS OF EVEN FERROMAGNETS¹

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1. **Introduction.** Let E be the set of even probability measures which satisfy $\int \exp(kx^2)\rho(dx) < \infty$ for all $k \geq 0$ sufficiently small. Given an integer $N \geq 1$, real numbers $h \geq 0$ and $J_{ij} \geq 0$, $1 \leq i < j \leq N$, and measures $\rho_i \in E$, $1 \leq i \leq N$, we define [11, p. 273] real-valued random variables X_i , $1 \leq i \leq N$, with the joint distribution

$$(1) \quad \tau_h(dx_1, \dots, dx_N) = \frac{\exp(\sum_{1 \leq i < j \leq N} J_{ij} x_i x_j + h \sum_{1 \leq i \leq N} x_i) \rho_1(dx_1) \cdots \rho_N(dx_N)}{Z(h)}.$$

$Z(h)$, the partition function, is given by the formula

$$(2) \quad Z(h) = \int_{\mathbb{R}^N} \cdots \int \exp\left(\sum J_{ij} x_i x_j + h \sum x_i\right) \rho_1(dx_1) \cdots \rho_N(dx_N).$$

The J_{ij} are assumed to be so small that the integral in (2) converges for all $h \geq 0$. The inequalities we discuss are to hold for all $h \geq 0$ and all $J_{ij} \geq 0$ subject only to this restriction. The choice of ρ_i as the Bernoulli measure $b(dx) = \frac{1}{2}(\delta(x-1) + \delta(x+1))$ gives the classical Ising model.

We define the average magnetization per site, $m(h)$, by the formula

$$(3) \quad m(h) = \frac{1}{N} \frac{d}{dh} \ln Z(h) = \frac{1}{N} \sum_{i=1}^N E\{X_i\}$$

and consider inequalities on $m(h)$ and its derivatives. While the inequalities $m(h) \geq 0$, $dm(h)/dh \geq 0$ hold for any $\rho_i \in E$ [7, pp. 76–77], the concavity of $m(h)$, i.e.

$$(4) \quad d^2 m(h)/dh^2 \leq 0,$$

requires that further restrictions be placed on the ρ_i . Essentially, (4) is known to hold only in the Ising case and in models which can be built out of Ising models in a suitable way [4], [6]. Measures for which (4) fails are known [6].

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The usual approach to (4) is first to prove the stronger (GHS) inequalities

[5]

$$(5) \quad \frac{\partial^3}{\partial h_i \partial h_j \partial h_k} \ln Z(h_1, \dots, h_N) \leq 0, \quad \text{all } 1 \leq i, j, k \leq N, h_i \geq 0,$$

where

$$Z(h_1, \dots, h_N) = \int \dots \int_{R^N} \exp\left(\sum J_{ij} x_i x_j + \sum h_i x_i\right) \rho_1(dx_1) \dots \rho_N(dx_N).$$

Instead, we shall prove (4) directly for many new measures using a technique which reduces consideration to the case $N = 1$. Afterwards, we shall return to (5).

We state two implications of these inequalities. The first shows that the requirement that the ρ_i in (1) have Gaussian falloff is only an apparent restriction.

THEOREM 1. *Let ρ be an even probability measure satisfying $\int \exp(kx)\rho(dx) < \infty$ for all $k \geq 0$. Assume that (4) holds for $N = 1$ (set $\rho_1 = \rho$). Then ρ is in E .*

The next theorem (known for fourth degree polynomial V [3], [10]) on the spectrum of certain differential operators is a striking consequence of (5).

THEOREM 2. *Let $V(x)$ be an entire function with the expansion*

$$(6) \quad V(x) = \sum_{k=1}^{\infty} a_k x^{2k}, \quad a_k \geq 0 \text{ for } k \geq 2, \quad a_1 \text{ real } (a_1 > 0 \text{ if all } a_k = 0).$$

Let E_1, E_2, E_3 , be the three smallest eigenvalues of the differential operator $-\frac{1}{2}d^2/dx^2 + V(x)$ on $L^2(R^1; dx)$. Then $E_3 - E_2 \geq E_2 - E_1$.

By Theorems 4 and 5 below, we shall see that (5) is satisfied for the measures

$$(7) \quad \rho_i(dx) = c \exp(-V(x))dx, \quad c \text{ a normalization constant,}$$

if V is as in (6). This is the main ingredient needed to prove Theorem 2 [10].

2. The class G_- . Below, we define a subset G_- of measures in E for which we have the following result.

THEOREM 3. *If $\rho_1, \dots, \rho_N \in G_-$, then (4) holds.*

For the proof, we use a closure property of G_- in order to reduce to the case $N = 1$. We call this property the closure of G_- under *ferromagnetic unions*.

(C) *Let Y_1, \dots, Y_N be real-valued random variables with joint distribution τ_0 (see (1)). Let F_0 be the class of all distributions of sums $\sum_{1 \leq i \leq N} r_i Y_i$*

