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Asymptotic Behavior of the Magnetization Near Critical and Tricritical Points via Ginzburg–Landau Polynomials

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Abstract The purpose of this paper is to prove connections among the asymptotic behavior of the magnetization, the structure of the phase transitions, and a class of polynomials that we call the Ginzburg–Landau polynomials. The model under study is a mean-field version of a lattice spin model due to Blume and Capel. It is defined by a probability distribution that depends on the parameters β and K, which represent, respectively, the inverse temperature and the interaction strength. Our main focus is on the asymptotic behavior of the magnetization $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) that converge to a second-order point or to the tricritical point of the model and that lie inside various subsets of the phase-coexistence region. The main result states that as (β_n, K_n) converges to one of these points (β, K), $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^\gamma \to 0$. In this formula γ is a positive constant, and $\bar{x}$ is the unique positive, global minimum point of a certain polynomial g. We call g the Ginzburg–Landau polynomial because of its close connection with the Ginzburg–Landau phenomenology of critical phenomena. For each sequence the structure of the set of global minimum points of the associated Ginzburg–Landau polynomial mirrors the structure of the set of global minimum points of the free-energy functional in the region through which (β_n, K_n) passes and thus reflects the phase-transition structure of the model in that region. This paper makes rigorous the predictions of the Ginzburg–Landau phenomenology of critical phenomena and the tricritical scaling theory for the mean-field Blume–Capel model.

Keywords Ginzburg–Landau phenomenology · Second-order phase transition · First-order phase transition · Tricritical point · Scaling theory · Blume–Capel model

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1 Introduction

In this paper we prove unexpected connections among the asymptotic behavior of the magnetization, the structure of the phase transitions, and a class of polynomials that we call the Ginzburg–Landau polynomials. The investigation is carried out for a mean-field version of an important lattice spin model due to Blume and Capel, to which we refer as the B-C model [2, 5–7]. This mean-field model is equivalent to the B-C model on the complete graph on N vertices. It is certainly one of the simplest models that exhibit the following intricate phase-transition structure: a curve of second-order points; a curve of first-order points; and a tricritical point, which separates the two curves. A generalization of the B-C model is studied in [3].

The main result in the present paper is Theorem 3.2, a general theorem that gives the asymptotic behavior of the magnetization in the mean-field B-C model for suitable sequences. With only changes in notation, the theorem also applies to other mean-field models including the Curie–Weiss model [11] and the Curie–Weiss–Potts model [15].

The mean-field B-C model is defined by a canonical ensemble that we denote by $P_{N,\beta,K}$; N equals the number of spins, β is the inverse temperature, and K is the interaction strength. $P_{N,\beta,K}$ is defined in terms of the Hamiltonian

$$H_{N,K}(\omega) = \sum_{j=1}^{N} \omega_j^2 - \frac{K}{N}\left(\sum_{j=1}^{N} \omega_j\right)^2,$$

in which ω_j represents the spin at site $j \in \{1, 2, \dots, N\}$ and takes values in $\Lambda = \{1, 0, -1\}$. The configuration space for the model is the set Λ^N containing all sequences $\omega = (\omega_1, \omega_2, \dots, \omega_N)$ with each $\omega_j \in \Lambda$.

Before introducing the results in this paper, we summarize the phase-transition structure of the model. For $\beta > 0$ and $K > 0$ we denote by $\mathcal{M}_{\beta,K}$ the set of equilibrium values of the magnetization. $\mathcal{M}_{\beta,K}$ coincides with the set of global minimum points of the free-energy functional $G_{\beta,K}$, which is defined in (2.4). It is known from heuristic arguments and is proved in [14] that there exists a critical inverse temperature $\beta_c = \log 4$ and that for $0 < \beta \le \beta_c$ there exists a quantity $K(\beta)$ and for $\beta > \beta_c$ there exists a quantity $K_1(\beta)$ having the following properties:

1. Fix $0 < \beta \le \beta_c$. Then for $0 < K \le K(\beta)$, $\mathcal{M}_{\beta,K}$ consists of the unique pure phase 0, and for $K > K(\beta)$, $\mathcal{M}_{\beta,K}$ consists of two nonzero values of the magnetization $\pm m(\beta, K)$.
2. For $0 < \beta \le \beta_c$, $\mathcal{M}_{\beta,K}$ undergoes a continuous bifurcation at $K = K(\beta)$, changing continuously from $\{0\}$ for $K \le K(\beta)$ to $\{\pm m(\beta, K)\}$ for $K > K(\beta)$. This continuous bifurcation corresponds to a second-order phase transition.
3. Fix $\beta > \beta_c$. Then for $0 < K < K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of the unique pure phase 0; for $K = K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of 0 and two nonzero values of the magnetization $\pm m(\beta, K_1(\beta))$; and for $K > K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of two nonzero values of the magnetization $\pm m(\beta, K)$.
4. For $\beta > \beta_c$, $\mathcal{M}_{\beta,K}$ undergoes a discontinuous bifurcation at $K = K_1(\beta)$, changing discontinuously from $\{0\}$ for $K < K(\beta)$ to $\{0, \pm m(\beta, K)\}$ for $K = K_1(\beta)$ to $\{\pm m(\beta, K)\}$ for $K > K_1(\beta)$. This discontinuous bifurcation corresponds to a first-order phase transition.

Because of items 2 and 4, we refer to the curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$ as the second-order curve and to the curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$ as the first-order curve. Points on the

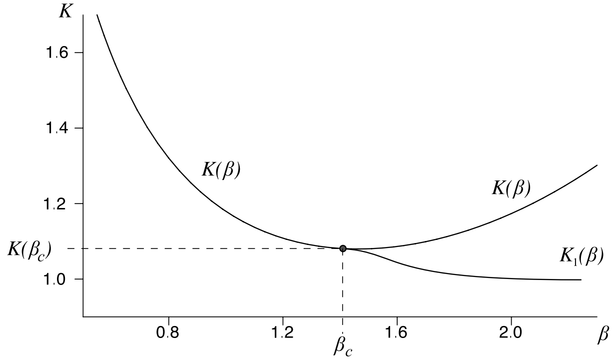

Fig. 1 The sets that describe the phase-transition structure of the mean-field B-C model: the second-order curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$, the first-order curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$, and the tricritical point $(\beta_c, K(\beta_c))$. The phase-coexistence region consists of all (β, K) above the second-order curve, above the tricritical point, on the first-order curve, and above the first-order curve. The extension of the second-order curve to $\beta > \beta_c$ is called the spinodal curve

second-order curve are called second-order points, and points on the first-order curve first-order points. The point $(\beta_c, K(\beta_c)) = (\log 4, 3/2 \log 4)$ separates the second-order curve from the first-order curve and is called the tricritical point. The phase-coexistence region consists of all points in the positive β-K quadrant for which $\mathcal{M}_{\beta,K}$ consists of more than one value. Thus this region consists of all (β, K) above the second-order curve, above the tricritical point, on the first-order curve, and above the first-order curve; i.e., all (β, K) satisfying $0 < \beta \leq \beta_c$ and $K > K(\beta)$ and satisfying $\beta > \beta_c$ and $K \geq K_1(\beta)$. The sets that describe the phase-transition structure of the model are shown in Fig. 1.

We now turn to the main focus of this paper, which is the asymptotic behavior of the magnetization $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) that converge either to a second-order point or to the tricritical point from various subsets of the phase-coexistence region. In the case of second-order points we consider two such sequences in Theorems 4.1 and 4.2, and in the case of the tricritical point we consider four such sequences in Theorems 4.3–4.6. Denoting the second-order point or the tricritical point by (β, K), in each case we prove as a consequence of the general result in Theorem 3.2 that $m(\beta_n, K_n) \to 0$ according to the asymptotic formula

$$m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^{\gamma}; \quad \text{i.e.} \quad \lim_{n\to\infty} |\beta - \beta_n|^{-\gamma} m(\beta_n, K_n) = \bar{x}. \tag{1.1}$$

In this formula γ is a positive constant, and $\bar{x}$ is the unique positive, global minimum point of a certain polynomial g. We call g the Ginzburg–Landau polynomial because of its close connection with the Ginzburg–Landau phenomenology of critical phenomena [16]. Both γ and $\bar{x}$ depend on the sequence (β_n, K_n). The exponent γ and the polynomial g arise via the limit of suitably scaled free-energy functionals; specifically, for appropriate choices of $u \in \mathbb{R}$ and $\gamma > 0$ and uniformly for x in compact subsets of $\mathbb{R}$

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n, K_n}(x/n^{\gamma}) = g(x). \tag{1.2}$$

Possible paths followed by the sequences studied in Theorems 4.1–4.6 are shown in Fig. 2. Two different paths are shown for each of the first three sequences, four different paths for the fourth sequence, and one path for each of the last two sequences. We believe that modulo uninteresting scale changes, these are all the sequences of the form $\beta_n = \beta + b/n^{\alpha}$ and K_n equal to $K(\beta)$ plus a polynomial in $1/n^{\alpha}$, where $(\beta, K(\beta))$ is either a second-order point or the tricritical point and for which $m(\beta_n, K_n) > 0$.

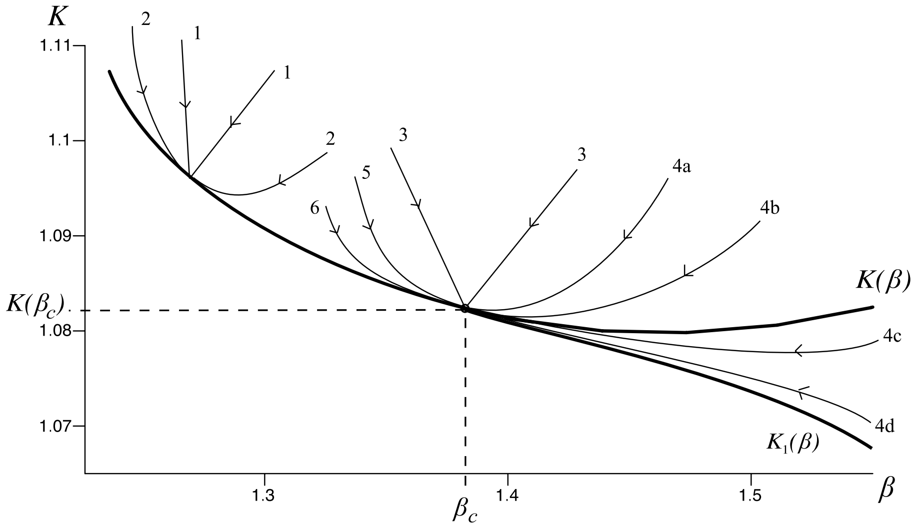

Fig. 2 Possible paths for the six sequences converging to a second-order point and to the tricritical point. The curves labeled *1*, *2*, *3*, *4a–4d*, *5*, and *6* are discussed in the respective Theorems 4.1–4.6. The sequences on the curves labeled *4a–4d* are defined in (4.15) and are discussed in the respective items (i)–(iv) appearing after (4.18)

This paper puts on a rigorous footing the idea, first introduced by Ginzburg and Landau, that low-order polynomial truncations of the free energy functional give correct asymptotic results near continuous phase transitions for mean-field models [16]. The use of sequences (β_n, K_n) that approach second-order points or the tricritical point permits us to establish the validity of truncating the expansion of the free-energy functional at an appropriate low order. The higher-order terms are driven to zero by a power of n and are shown to be asymptotically irrelevant. While the renormalization group methodology also demonstrates the irrelevance of higher order terms in the expansion of the free-energy functional, it does so via a different route that depends on heuristics. By contrast, our approach is rigorous and shows in (1.2) how to obtain the Ginzburg–Landau polynomial as a limit of suitably scaled free-energy functionals. No heuristic approximations or truncations appear.

Let (β_n, K_n) be any particular sequence converging to a second-order point or the tricritical point from the phase-coexistence region and denote the limiting point by (β, K). It is not difficult to obtain an asymptotic formula expressing the rate at which $m(\beta_n, K_n)$ converges to 0. Since $m(\beta_n, K_n)$ is the unique positive minimum point of G_{β_n,K_n}, it solves the equation $G'_{\beta_n,K_n}(m(\beta_n, K_n)) = 0$. As we illustrate in appendix B of [12] in two examples, expanding G'_{β_n,K_n} in a Taylor series of appropriate order, one obtains the formula $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^\gamma$ for some $\bar{x} > 0$ and $\gamma > 0$. However, this method gives only the functional form for $\bar{x}$, not associating it with the model via the Ginzburg–Landau polynomial. This is in contrast to our general result in Theorem 3.2. That result identifies $\bar{x}$ as the unique positive, global minimum point of the Ginzburg–Landau polynomial, using the uniform convergence in (1.2). The proof of that result completely avoids Taylor expansions, making use of the fact that under appropriate conditions, the positive global minimum points of n-dependent minimization problems converge to the positive global minimum point of a limiting minimization problem when such a minimum point is unique.

Another contribution of the present paper is to demonstrate how the structure of the set of global minimum points of the associated Ginzburg–Landau polynomials g mirrors the phase-transition structure of the subsets through which (β_n, K_n) passes. In this way the properties of the Ginzburg–Landau polynomials make rigorous the predictions of the Ginzburg–Landau phenomenology of critical phenomena. Details of this mirroring are given in the discussions leading up to Theorem 4.1 and Theorem 4.4; of all the sequences that we consider, the sequence considered in the latter theorem shows the most varied behavior.

Our work is also closely related to the scaling theory for critical and tricritical points. By choosing sequences that approach second-order points or the tricritical point from various directions and at various rates, we are able to verify a number of predictions of scaling theory. The sequences that approach the tricritical point reveal the subtle geometry of the crossover between critical and tricritical behavior described in Riedel's tricritical scaling theory [20]. In Sect. 5 we will see that a proper application of scaling theory near the tricritical point requires that the scaling parameters be defined in a curvilinear coordinate system, an idea proposed in [20] but, to our knowledge, not previously explored.

Some of the results proved here can be obtained non-rigorously via the methods introduced by Capel and collaborators [5–8, 17–19]. These papers introduce the mean-field B-C model and provide a general framework for studying mean-field models and obtaining the thermodynamic properties of these systems.

In order to keep the present paper to a reasonable length, we have omitted a number of routine calculations. Full details can be found in our unpublished, companion paper [12]. We have also omitted the material in section 6 of that paper, in which properties of the Ginzburg–Landau polynomials, mathematical analysis, and numerical calculations are used to determine properties of the first-order curve in a neighborhood of the tricritical point.

We end the introduction by previewing our results on the refined asymptotics of the total spin $S_N = \sum_{j=1}^N \omega_j$, which are the main focus of the sequel to the present paper [13]. When $N = n$—i.e., when the system size N coincides with the index n parametrizing the sequence (β_n, K_n)—these refined asymptotics reveal a fascinating relationship between the asymptotic formulas for $m(\beta_n, K_n)$ obtained here and the finite-size expectation $\langle |S_n/n| \rangle_{n,\beta_n,K_n}$ with respect to P_{n,β_n,K_n}. For a wide class of sequences (β_n, K_n) converging to a second-order point or the tricritical point, including the six sequences considered in the present paper, we prove that there exists a positive constant α_0 depending on the sequence and having the following properties. For $\alpha \in (0, \alpha_0)$ the finite-size expectation $\langle |S_n/n| \rangle_{n,\beta_n,K_n}$ is asymptotic to $m(\beta_n, K_n)$. In this case (β_n, K_n) converges slowly, and the system is in the phase-coexistence regime, where it is effectively infinite. On the other hand, when $\alpha > \alpha_0$, $m(\beta_n, K_n)$ is not related to the finite-size expectation. In this regime, the fluctuations of $|S_n/n|$ as measured by this expectation are much larger than $m(\beta_n, K_n)$, which converges to 0 at a much faster rate. When $\alpha > \alpha_0$, (β_n, K_n) converges quickly, and the system is in the critical regime. The theory of finite-size scaling predicts that for such α, critical singularities are controlled by the size of the system rather than by the distance in parameter space from the phase transition [1, 4, 9, 22].

The contents of the present paper are as follows. In Sect. 2 we summarize the phase-transition structure of the mean-field B-C model. In Sect. 3 we prove our main result (1.1) on the asymptotic behavior of $m(\beta_n, K_n) \to 0$ (Theorem 3.2). In Sect. 4 that result is applied to six different sequences (β_n, K_n), the first two of which converge to a second-order point and the last four of which converge to the tricritical point. In Sect. 5 we relate the results obtained in the preceding section to the scaling theory of critical phenomena. In an appendix we state two results on polynomials of degree 6 needed in the paper.

2 Phase-Transition Structure of the Mean-Field B-C Model

After defining the mean-field B-C model, we introduce a function $G_{\beta,K}$, called the free-energy functional. The global minimum points of this function define the equilibrium values of the magnetization, and the minimum value of this function over $\mathbb{R}$ gives the canonical free energy. We then summarize the phase-transition structure of the model in terms of the behavior of the magnetization. This structure consists of a second-order phase transition, a first-order phase transition, and a tricritical point, which separates the two phase transitions.

The mean-field B-C model is a lattice spin model defined on the complete graph on N vertices $1, 2, \ldots, N$. The spin at site $j \in \{1, 2, \ldots, N\}$ is denoted by ω_j, a quantity taking values in $\Lambda = \{1, 0, -1\}$. The configuration space for the model is the set Λ^N containing all sequences $\omega = (\omega_1, \omega_2, \ldots, \omega_N)$ with each $\omega_j \in \Lambda$. In terms of a positive parameter K representing the interaction strength, the Hamiltonian is defined by

$$H_{N,K}(\omega) = \sum_{j=1}^{N} \omega_j^2 - \frac{K}{N}\left(\sum_{j=1}^{N} \omega_j\right)^2$$

for each $\omega \in \Lambda^N$. Let P_N be the product measure on Λ^N with identical one-dimensional marginals $\rho = \frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$. Thus P_N assigns the probability 3^{-N} to each $\omega \in \Lambda^N$. For $N \in \mathbb{N}$, inverse temperature $\beta > 0$, and $K > 0$, the canonical ensemble for the mean-field B-C model is the sequence of probability measures that assign to each subset B of Λ^N the probability

$$P_{N,\beta,K}(B) = \frac{1}{Z_N(\beta, K)} \cdot \sum_{\omega \in B} \exp[-\beta H_{N,K}(\omega)] \cdot 3^{-N}.$$

In this formula $Z_N(\beta, K) = \sum_{\omega \in \Lambda^N} \exp[-\beta H_{N,K}(\omega)] \cdot 3^{-N}$.

The analysis of the canonical ensemble $P_{N,\beta,K}$ is facilitated by absorbing the noninteracting component of the Hamiltonian into the product measure P_N, obtaining

$$P_{N,\beta,K}(d\omega) = \frac{1}{\tilde{Z}_N(\beta, K)} \cdot \exp\bigl[N\beta K (S_N(\omega)/N)^2\bigr] P_{N,\beta}(d\omega). \tag{2.1}$$

In this formula $S_N(\omega)$ equals the total spin $\sum_{j=1}^{N} \omega_j$, $P_{N,\beta}$ is the product measure on Λ^N with identical one-dimensional marginals

$$\rho_\beta(d\omega_j) = \frac{1}{Z(\beta)} \cdot \exp(-\beta\omega_j^2)\, \rho(d\omega_j), \tag{2.2}$$

$Z(\beta)$ is the normalization equal to $\int_\Lambda \exp(-\beta\omega_j^2)\rho(d\omega_j) = (1 + 2e^{-\beta})/3$, and $\tilde{Z}_N(\beta, K)$ is the normalization equal to $[Z(\beta)]^N / Z_N(\beta, K)$.

In order to summarize the phase-transition structure of the model, we introduce the cumulant generating function c_β of ρ_β, which for $\beta > 0$ and $t \in \mathbb{R}$ is defined by

$$c_\beta(t) = \log \int_\Lambda \exp(t\omega_1)\, \rho_\beta(d\omega_1) = \log\left(\frac{1 + e^{-\beta}(e^t + e^{-t})}{1 + 2e^{-\beta}}\right). \tag{2.3}$$

For $\beta > 0$, $K > 0$, and $x \in \mathbb{R}$ we also define

$$G_{\beta,K}(x) = \beta K x^2 - c_\beta(2\beta K x). \tag{2.4}$$

The large deviation principle in Theorem 3.3 of [14] and the convexity analysis in Proposition 3.4 of [14] shows that if x is not a global minimum point of $G_{\beta,K}$, then any sufficiently small interval containing x has an exponentially small probability with respect to the $P_{N,\beta,K}$-distributions of S_N/N; i.e., x is not observed in the canonical ensemble. Since the global minimum points of $G_{\beta,K}$ lie in $[-1,1]$, we define the set $\mathcal{M}_{\beta,K}$ of equilibrium values of the magnetization by

$$\mathcal{M}_{\beta,K} = \{x \in [-1,1] : x \text{ is a global minimum point of } G_{\beta,K}(x)\}. \tag{2.5}$$

We call $G_{\beta,K}$ the free-energy functional of the mean-field B-C model because the canonical free energy $\varphi(\beta,K)$, defined as $-\lim_{N\to\infty} N^{-1}\log Z_N(\beta,K)$, equals $\min_{x\in\mathbb{R}} G_{\beta,K}(x)$.

In section 2 of [12] the Ginzburg–Landau phenomenology is applied to $G_{\beta,K}$ in order to motivate the phase-transition structure of the model [16]. The next two theorems give the structure of $\mathcal{M}_{\beta,K}$ first for $0 < \beta \le \beta_c = \log 4$ and then for $\beta > \beta_c$. These theorems make rigorous the predictions of the Ginzburg–Landau theory. The first theorem, proved in Theorem 3.6 in [14], describes the continuous bifurcation in $\mathcal{M}_{\beta,K}$ for $0 < \beta \le \beta_c$. This bifurcation corresponds to a second-order phase transition. The quantity $K(\beta)$ is denoted by $K_c^{(2)}(\beta)$ in [14] and by $K_c(\beta)$ in [10].

Theorem 2.1 *For $0 < \beta \le \beta_c$, we define*

$$K(\beta) = 1/[2\beta c_\beta''(0)] = (e^\beta + 2)/(4\beta). \tag{2.6}$$

For these values of β, $\mathcal{M}_{\beta,K}$ has the following structure.

(a) *For $0 < K \le K(\beta)$, $\mathcal{M}_{\beta,K} = \{0\}$.*
(b) *For $K > K(\beta)$, there exists $m(\beta,K) > 0$ such that $\mathcal{M}_{\beta,K} = \{\pm m(\beta,K)\}$.*
(c) *$m(\beta,K)$ is a positive, increasing, continuous function for $K > K(\beta)$, and as $K \to (K(\beta))^+$, $m(\beta,K) \to 0^+$. Therefore, $\mathcal{M}_{\beta,K}$ exhibits a continuous bifurcation at $K(\beta)$.*

The next theorem, proved in Theorem 3.8 in [14], describes the discontinuous bifurcation in $\mathcal{M}_{\beta,K}$ for $\beta > \beta_c$. This bifurcation corresponds to a first-order phase transition. As shown in that theorem, for all $\beta > \beta_c$, $K_1(\beta) < K(\beta)$. The quantity $K_1(\beta)$ is denoted by $K_c^{(1)}(\beta)$ in [14] and by $K_c(\beta)$ in [10].

Theorem 2.2 *For $\beta > \beta_c$, $\mathcal{M}_{\beta,K}$ has the following structure in terms of the quantity $K_1(\beta)$, denoted by $K_c^{(1)}(\beta)$ in [14] and defined implicitly for $\beta > \beta_c$ on page 2231 of [14].*

(a) *For $0 < K < K_1(\beta)$, $\mathcal{M}_{\beta,K} = \{0\}$.*
(b) *For $K = K_1(\beta)$ there exists $m(\beta, K_1(\beta)) > 0$ such that $\mathcal{M}_{\beta,K_1(\beta)} = \{0, \pm m(\beta, K_1(\beta))\}$.*
(c) *For $K > K_1(\beta)$ there exists $m(\beta,K) > 0$ such that $\mathcal{M}_{\beta,K} = \{\pm m(\beta,K)\}$.*
(d) *$m(\beta,K)$ is a positive, increasing, continuous function for $K \ge K_1(\beta)$, and as $K \to K_1(\beta)^+$, $m(\beta,K) \to m(\beta,K_1(\beta)) > 0$. Therefore, $\mathcal{M}_{\beta,K}$ exhibits a discontinuous bifurcation at $K_1(\beta)$.*

In the next section we present a general asymptotic result on the behavior of $m(\beta_n, K_n)$ for sequences (β_n, K_n) converging either to a second-order point or to the tricritical point. In subsequent sections the theorem will be applied to a number of specific sequences.

3 Asymptotic Behavior of $m(\beta_n, K_n)$ in Terms of Ginzburg–Landau Polynomials

Theorem 3.2 is the main result in this paper. It gives the asymptotic behavior of $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) lying in the phase-coexistence region and converging either to a second-order point or to the tricritical point. The asymptotic behavior is expressed in terms of the unique positive, global minimum point of the associated Ginzburg–Landau polynomial.

The phase-coexistence region is defined to be all (β, K) satisfying $0 < \beta \le \beta_c$ and $K > K(\beta)$ and all (β, K) satisfying $\beta > \beta_c$ and $K \ge K_1(\beta)$. Thus for $0 < \beta \le \beta_c$, the phase-coexistence region consists of the region located above the second-order curve and above the tricritical point. For $\beta > \beta_c$, the phase-coexistence region consists of the first-order curve $(\beta, K_1(\beta))$ and the region located above that curve. For all (β, K) in the phase-coexistence region there exists $m(\beta, K) > 0$ such that $\{\pm m(\beta, K)\} \subset \mathcal{M}_{\beta,K}$. This is an equality for all (β, K) in the phase-coexistence region except for $\beta > \beta_c$ and $K = K_1(\beta)$, in which case $\mathcal{M}_{\beta,K} = \{0, \pm m(\beta, K)\}$.

The first theorem in this section shows that for any sequence (β_n, K_n) converging either to a second-order point or to the tricritical point, $m(\beta_n, K_n) \to 0$. This theorem is an essential component in the proof of the asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 3.2.

Theorem 3.1 *Let (β_n, K_n) be an arbitrary positive sequence converging either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. Then* $\lim_{n\to\infty} m(\beta_n, K_n) = 0$.

Proof Since G_{β_n,K_n} is a real analytic function, $G_{\beta_n,K_n}(m(\beta_n, K_n)) \le 0$, and $G_{\beta_n,K_n}(x) \to \infty$ as $|x| \to \infty$, G_{β_n,K_n} has a largest positive zero, which we denote by x_n. We have the inequality $0 < m(\beta_n, K_n) < x_n$. For any $t \in \mathbb{R}$, $c_\beta(t) \le \log(4e^{|t|}) = \log 4 + |t|$. Because the sequence (β_n, K_n) is bounded and remains a positive distance from the origin and the coordinate axes, there exist numbers $0 < b_1 < b_2 < \infty$ such that $b_1 \le \beta_n \le b_2$ and $b_1 \le K_n \le b_2$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} G_{\beta_n,K_n}(x) &= \beta_n K_n x^2 - c_{\beta_n}(2\beta_n K_n x) \\ &\ge \beta_n K_n x^2 - 2\beta_n K_n |x| - \log 4 \ \ge\ b_1^2(|x| - 1)^2 - b_2^2 - \log 4. \end{aligned}$$

Therefore, if x^* denotes the positive zero of the quadratic $b_1^2(|x| - 1)^2 - b_2^2 - \log 4$, then

$$0 < \sup_{n\in\mathbb{N}} m(\beta_n, K_n) \le \sup_{n\in\mathbb{N}} x_n \le x^*.$$

It follows that $m(\beta_n, K_n)$ is a bounded sequence. Thus given any subsequence $m(\beta_{n_1}, K_{n_1})$, there exists a further subsequence $m(\beta_{n_2}, K_{n_2})$ and $\tilde{x} \in \mathbb{R}$ such that $m(\beta_{n_2}, K_{n_2}) \to \tilde{x}$ as $n_2 \to \infty$. We complete the proof by showing that independently of the subsequence chosen, $\tilde{x} = 0$. To prove this, we use the fact that

$$G_{\beta_{n_2},K_{n_2}}(m(\beta_{n_2}, K_{n_2})) = \inf_{y\in\mathbb{R}} G_{\beta_{n_2},K_{n_2}}(y).$$

Hence for any $y \in \mathbb{R}$, $G_{\beta_{n_2},K_{n_2}}(m(\beta_{n_2}, K_{n_2})) \le G_{\beta_{n_2},K_{n_2}}(y)$. Since $G_{\beta_{n_2},K_{n_2}}(x) \to G_{\beta,K(\beta)}(x)$ uniformly for x in compact subsets of $\mathbb{R}$, it follows that for all $y \in \mathbb{R}$

$$G_{\beta,K(\beta)}(\tilde{x}) = \lim_{n_2\to\infty} G_{\beta_{n_2},K_{n_2}}(m(\beta_{n_2}, K_{n_2})) \le \lim_{n_2\to\infty} G_{\beta_{n_2},K_{n_2}}(y) = G_{\beta,K(\beta)}(y).$$

Therefore $\tilde{x}$ is a minimum point of $G_{\beta,K(\beta)}$. Because $(\beta, K(\beta))$ is either a second-order point or the tricritical point, $\tilde{x}$ must coincide with the unique positive, global minimum point of $G_{\beta,K(\beta)}$ at 0 (Theorem 2.1(a), Theorem 2.2(a)). We have proved that any subsequence $m(\beta_{n_1}, K_{n_1})$ of $m(\beta_n, K_n)$ has a further subsequence $m(\beta_{n_2}, K_{n_2})$ such that $m(\beta_{n_2}, K_{n_2}) \to 0$ as $n_2 \to \infty$. The conclusion is that $\lim_{n\to\infty} m(\beta_n, K_n) = 0$, as claimed. □

In Sect. 4 we consider six different sequences (β_n, K_n) converging either to a second-order point or to the tricritical point. The fact that each of these sequences lies in the phase-coexistence region for all sufficiently large n is the first hypothesis of Theorem 3.2; this property implies that $m(\beta_n, K_n) > 0$ for all sufficiently large n and $m(\beta_n, K_n) \to 0$ (Theorem 3.1). Under three additional hypotheses Theorem 3.2 describes the exact asymptotic behavior of $m(\beta_n, K_n) \to 0$. Examples of sequences for which the hypotheses of the theorem are valid are given in Theorems 4.1 and 4.2 for sequences converging to a second-order point and in Theorems 4.3–4.6 for sequences converging to the tricritical point.

Theorem 3.2 *Let (β_n, K_n) be a positive sequence that converges either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. We assume that (β_n, K_n) satisfies the following four hypotheses*:

(i) *(β_n, K_n) lies in the phase-coexistence region for all sufficiently large n.*

(ii) *The sequence (β_n, K_n) is parametrized by $\alpha > 0$. This parameter regulates the speed of approach of (β_n, K_n) to the second-order point or the tricritical point in the following sense*:

$$b = \lim_{n\to\infty} n^{\alpha}(\beta_n - \beta) \quad \textit{and} \quad k = \lim_{n\to\infty} n^{\alpha}(K_n - K(\beta))$$

both exist, and b and k are not both 0; if $b \neq 0$, then b equals 1 or -1.

(iii) *There exists an even polynomial g of degree 4 or 6 satisfying $g(x) \to \infty$ as $|x| \to \infty$ together with the following two properties*; *g is called the Ginzburg–Landau polynomial.*

   (a) *$\exists \alpha_0 > 0$ and $\exists \theta > 0$ such that $\forall \alpha > 0$, if $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$, then*

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n,K_n}(x/n^{\gamma}) = g(x)$$

   uniformly for x in compact subsets of $\mathbb{R}$.

   (b) *g has a unique positive, global minimum point $\bar{x}$; thus the set of global minimum points of g equals $\{\pm\bar{x}\}$ or $\{0, \pm\bar{x}\}$.*

(iv) *There exists a polynomial H satisfying $H(x) \to \infty$ as $|x| \to \infty$ together with the following property*: *$\forall \alpha > 0$ $\exists R > 0$ such that, if $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$, then $\forall n \in \mathbb{N}$ sufficiently large and $\forall x \in \mathbb{R}$ satisfying $|x/n^{\gamma}| < R$, $n^{1-u} G_{\beta_n,K_n}(x/n^{\gamma}) \geq H(x)$.*

Under hypotheses (i)–(iv), *for any $\alpha > 0$*

$$m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}; \quad \textit{i.e.}, \quad \lim_{n\to\infty} n^{\theta\alpha} m(\beta_n, K_n) = \bar{x}.$$

If $b \neq 0$, then this becomes $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^{\theta}$.

Proof Since $G_{\beta_n,K_n}(0) = 0$ and G_{β_n,K_n} is even, by hypotheses (iii) g is an even polynomial of degree 4 or 6 satisfying $g(0) = 0$. Hence the global minimum points of g are either $\pm\bar{x}$ for some $\bar{x} > 0$ or 0 and $\pm\bar{x}$ for some $\bar{x} > 0$. The proof of the asymptotic relationship $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}$ is much easier in the case where the global minimum points of g are

$\pm\bar{x}$ for some $\bar{x} > 0$. After a number of preliminary steps, we will prove the theorem for such polynomials g. We will then turn to the case where the global minimum points of g are 0 and $\pm\bar{x}$ for some $\bar{x} > 0$.

As in hypothesis (iii)(a), let $\alpha > 0$ be given and define $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$. In order to ease the notation, we write $\bar{m}_n = n^\gamma m(\beta_n, K_n)$ and $G_n(x) = n^{1-u} G_{\beta_n,K_n}(x/n^\gamma)$. For all sufficiently large n, since (β_n, K_n) lies in the phase-coexistence region, we have $m(\beta_n, K_n) > 0$ and

$$G_{\beta_n,K_n}(m(\beta_n, K_n)) = \inf_{y\in\mathbb{R}} G_{\beta_n,K_n}(y).$$

It follows that for all sufficiently large n

$$\begin{aligned} G_n(\bar{m}_n) &= n^{1-u} G_{\beta_n,K_n}(m(\beta_n, K_n)) \\ &= \inf_{y\in\mathbb{R}} [n^{1-u} G_{\beta_n,K_n}(y)] = \inf_{y\in\mathbb{R}} G_n(y); \end{aligned} \tag{3.1}$$

i.e., G_n attains its minimum over $\mathbb{R}$ at $\bar{m}_n > 0$. This fact will be used several times in the proof.

We first prove that the sequence $\{\bar{m}_n, n \in \mathbb{N}\}$ is bounded. If the sequence $\bar{m}_n$ is not bounded, then there exists a subsequence $\bar{m}_{n_1}$ of $\bar{m}_n$ such that $\bar{m}_{n_1} \to \infty$ as $n_1 \to \infty$. Let R be the quantity in hypothesis (iv). Since $m(\beta_{n_1}, K_{n_1}) > 0$ and $m(\beta_{n_1}, K_{n_1}) \to 0$ (Theorem 3.1), we have $0 < \bar{m}_{n_1}/n^\gamma = m(\beta_{n_1}, K_{n_1}) < R$ for all sufficiently large n_1, and so by hypothesis (iv)

$$G_{n_1}(\bar{m}_{n_1}) \ge H(\bar{m}_{n_1}) \to \infty \quad \text{as } n_1 \to \infty.$$

However, this contradicts the inequality

$$G_n(\bar{m}_n) = \inf_{y\in\mathbb{R}} G_n(y) \le G_n(0) = 0,$$

which is valid for all n. This contradiction proves that the sequence $\bar{m}_n$ is bounded.

We now prove that $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\theta\alpha}$ in the case where the global minimum points of g are $\pm\bar{x}$ for some $\bar{x} > 0$. Let $\bar{m}_{n_1}$ be any subsequence of $\bar{m}_n$. Since the sequence $\bar{m}_{n_1}$ is bounded, there exists a further subsequence $\bar{m}_{n_2}$ and $\hat{x} \ge 0$ such that $\bar{m}_{n_2} \to \hat{x}$ as $n_2 \to \infty$. According to (3.1), for any $y \in \mathbb{R}$, $G_{n_2}(\bar{m}_{n_2}) \le G_{n_2}(y)$. Since $G_n(x) \to g(x)$ uniformly for x in compact subsets of $\mathbb{R}$, it follows that

$$g(\hat{x}) = \lim_{n_2\to\infty} G_{n_2}(\bar{m}_{n_2}) \le \lim_{n_2\to\infty} G_{n_2}(y) = g(y).$$

Hence $\hat{x}$ is a nonnegative global minimum point of g. Because g has a unique nonnegative, global minimum point $\bar{x}$, which is positive, $\hat{x}$ coincides with $\bar{x}$. We have proved that any subsequence $\bar{m}_{n_1}$ of $\bar{m}_n$ has a further subsequence $\bar{m}_{n_2}$ such that $\bar{m}_{n_2} \to \bar{x}$ as $n_2 \to \infty$. The conclusion is that $\lim_{n\to\infty} \bar{m}_n = \bar{x}$, which implies that $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\theta\alpha}$.

We now prove that $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\theta\alpha}$ in the case where the global minimum points of g are 0 and $\pm\bar{x}$ for some $\bar{x} > 0$. In this case g is a polynomial of degree 6. There are two subcases to consider: (1) there exists an infinite subsequence n_1 in $\mathbb{N}$ such that the global minimum points of G_{n_1} are $\pm\bar{m}_{n_1}$; (2) there exists an infinite subsequence n_4 in $\mathbb{N}$ such that the global minimum points of G_{n_4} are 0 and $\pm\bar{m}_{n_4}$.

In subcase 1 we will prove that any subsequence n_2 of n_1 has a further subsequence n_3 for which $\bar{m}_{n_3} \to \bar{x}$. This implies that $\bar{m}_{n_1} \to \bar{x}$. In subcase 2 a similar proof shows that any subsequence n_5 of n_4 has a further subsequence n_6 for which $\bar{m}_{n_6} \to \bar{x}$. This implies that

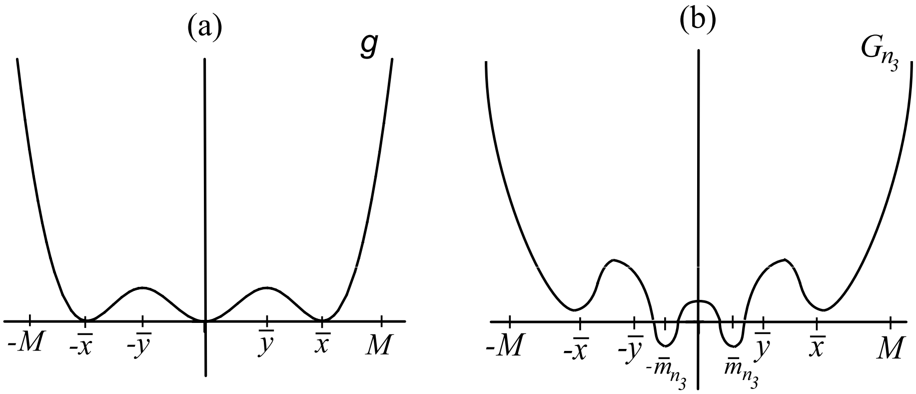

Fig. 3 Proof of Theorem 3.2 in subcase 1. (**a**) Graph of Ginzburg–Landau polynomial g having three global minimum points, (**b**) graph of G_{n_3} showing $\bar{m}_{n_3} \to 0$

$\bar{m}_{n_4} \to \bar{x}$. Now let n_7 be an arbitrary subsequence in $\mathbb{N}$. Then n_7 contains either infinitely many elements of the subsequence n_1 or infinitely many elements of the subsequence n_4. In either case n_7 contains a further subsequence n_8 for which $\bar{m}_{n_8} \to \bar{x}$. The conclusion is that $\bar{m}_n \to \bar{x}$, which yields the desired conclusion, namely, $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\theta\alpha}$.

We focus on subcase 1; subcase 2 is handled similarly. In order to understand the subtlety of the proof, we return to the argument just given in the case where the global minimum points of g are $\pm\bar{x}$ for some $\bar{x} > 0$. Let n_1 be the subsequence in subcase 1 and let n_2 be any further subsequence. Since the sequence $\bar{m}_{n_2}$ is bounded, the same argument shows that there exists a further subsequence n_3 such that $\bar{m}_{n_3} \to \hat{x}$ as $n_2 \to \infty$, where $\hat{x}$ is a nonnegative global minimum point of g. When the global minimum points of g are $\pm\bar{x}$ for some $\bar{x} > 0$, we are able to conclude in fact that $\hat{x}$ equals $\bar{x}$. However, in the present case where the global minimum points of g are 0 and $\pm\bar{x}$ for some $\bar{x} > 0$, it might turn out that $\hat{x}$ equals the global minimum point of g at 0. In this situation we would conclude that $\bar{m}_{n_3} \to 0$, which is not the asymptotic relationship that we want.

As this discussion shows, in subcase 1 it suffices to prove that there exists no subsequence n_3 of n_2 for which $\bar{m}_{n_3} \to 0$ as $n_3 \to \infty$. Under the assumption that there exists such a subsequence, we will reach a contradiction of the fact that $\bar{m}_{n_3}$ is the largest critical point of G_{n_3}. This fact is not directly stated in [14], but it is a straightforward consequence of Lemmas 3.9 and 3.10(a) and Theorem 3.5 in that paper. From these three results it follows that when (β, K) lies in the phase-coexistence region, the positive global minimum point of the function $F_{\beta,K}(z) = G_{\beta,K}(z/2\beta K)$ is also its largest critical point. From the definition of G_n in terms of G_{β_n,K_n}, it then follows that $\bar{m}_n$ is the largest critical point of G_n for all n.

Since the global minimum points of g are 0 and $\pm\bar{x}$ for some $\bar{x} > 0$, there exists $\bar{y} \in (0, \bar{x})$ such that g attains its maximum on the interval $[0, \bar{x}]$ at $\bar{y}$ and attains its maximum on the interval $[-\bar{x}, 0]$ at $-\bar{y}$. In addition, $g(\pm\bar{y}) > 0 = g(0) = g(\pm\bar{x})$. The graph of g is shown in graph (a) in Fig. 3. The graph of G_{n_3} under the assumption that $\bar{m}_{n_3} \to 0$ is shown in graph (b) in Fig. 3. Referring to these graphs should help the reader follow the proof.

By hypothesis (iii)(a), as $n_3 \to \infty$, $G_{n_3}(z) \to g(z)$ uniformly on compact subsets of $\mathbb{R}$. Thus for all sufficiently large n_3 and each choice of sign

$$G_{n_3}(\pm\bar{y}) \geq 2g(\bar{y})/3 > 0, \qquad G_{n_3}(\pm\bar{x}) \leq g(\bar{y})/3. \tag{3.2}$$

By definition of subcase 1 the global minimum points of G_{n_3} are $\pm\bar{m}_{n_3}$, and by assumption $\bar{m}_{n_3} \to 0$ as $n_3 \to \infty$. For all sufficiently large n_3, the inequality $G_{n_3}(\bar{m}_{n_3}) < G_{n_3}(0) = 0$ and the two inequalities in (3.2) imply that there exists $\bar{y}_{n_3} \in (\bar{m}_{n_3}, \bar{x})$ such that G_{n_3} attains its maximum on the interval $[\bar{m}_{n_3}, \bar{x}]$ at $\bar{y}_{n_3}$ and attains its maximum on the interval $[-\bar{x}, -\bar{m}_{n_3}]$ at $-\bar{y}_{n_3}$. Therefore, $\bar{y}_{n_3}$ is a critical point of G_{n_3} greater than $\bar{m}_{n_3}$. This contradicts the fact that $\bar{m}_{n_3}$ is the largest critical point of G_{n_3}. The proof of subcase 1 is complete.

Subcase 2 is handled similarly. Let n_4 be the subsequence in subcase 2 and let n_5 be any further subsequence. If there exists a subsequence n_6 of n_5 for which $\bar{m}_{n_6} \to 0$ as $n_6 \to \infty$, then for all sufficiently large n_6, there exists $\bar{y}_{n_6} \in (\bar{m}_{n_6}, \bar{x})$ such that G_{n_6} attains its maximum on the interval $[\bar{m}_{n_6}, \bar{x}]$ at $\bar{y}_{n_6}$ and attains its maximum on the interval $[-\bar{x}, -\bar{m}_{n_6}]$ at $-\bar{y}_{n_6}$. Again this contradicts the fact that $\bar{m}_{n_6}$ is the largest critical point of G_{n_6}. This completes the proof of the theorem. □

In the next section we apply Theorem 3.2 to determine the asymptotic behavior of $m(\beta_n, K_n)$ for six sequences (β_n, K_n) converging from the phase-coexistence region to a second-order point or to the tricritical point.

4 Asymptotic Behavior of $m(\beta_n, K_n)$ for Six Sequences

In this section we derive the asymptotic behavior of the magnetization $m(\beta_n, K_n)$ for six sequences (β_n, K_n). The first two sequences converge to a second-order point, and the last four sequences converge to the tricritical point.

By definition, when (β_n, K_n) lies in the phase-coexistence region, $m(\beta_n, K_n)$ is the unique positive, global minimum point of the free-energy functional G_{β_n,K_n}. For each of the sequences considered in this section the asymptotic behavior of $m(\beta_n, K_n)$ is expressed in terms of the unique positive, global minimum point $\bar{x}$ of the limit of a suitable scaled version of G_{β_n,K_n}. This limit is a polynomial called the Ginzburg–Landau polynomial. As we will see, properties of this polynomial reflect the phase-transition structure of the mean-field B-C model in the region through which the associated sequence (β_n, K_n) passes. This makes rigorous the predictions of the Ginzburg–Landau phenomenology of critical phenomena discussed in more detail in section 2 of [12].

We first derive in Theorems 4.1 and 4.2 the asymptotic behavior of $m(\beta_n, K_n)$ for two sequences (β_n, K_n) converging to a second-order point $(\beta, K(\beta))$ from the phase-coexistence region lying above the second-order curve. This asymptotic behavior is derived from the general result in Theorem 3.2. Full details of all the calculations for these two sequences are available in section 3 of [12].

For $0 < \beta < \beta_c$ let (β_n, K_n) be an arbitrary positive sequence converging to a second-order point $(\beta, K(\beta))$. According to hypothesis (iii) of Theorem 3.2, we seek numbers $u \in \mathbb{R}$ and $\gamma > 0$ and a suitable polynomial g such that $n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) \to g(x)$ uniformly on compact subsets of $\mathbb{R}$. In order to carry this out, we consider the Taylor expansion for $nG_{\beta_n,K_n}(x/n^\gamma)$ to order 4 with an error term. Because $G_{\beta_n,K_n}(0) = 0$, G_{β_n,K_n} is an even function, and $G^{(5)}_{\beta,K}(y)$ is uniformly bounded on compact subsets of $[0,\infty) \times [0,\infty) \times \mathbb{R}$, Taylor's Theorem implies that for all $n \in \mathbb{N}$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$nG_{\beta_n,K_n}(x/n^\gamma) = \frac{1}{n^{2\gamma-1}} \frac{G^{(2)}_{\beta_n,K_n}(0)}{2!} x^2 + \frac{1}{n^{4\gamma-1}} \frac{G^{(4)}_{\beta_n,K_n}(0)}{4!} x^4 + O\left(\frac{1}{n^{5\gamma-1}}\right) x^5. \tag{4.1}$$

In this formula the big-oh term is uniform for $x \in (-Rn^\gamma, Rn^\gamma)$.

Define $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/(8 \cdot 4!)$, which is positive since $0 < \beta < \beta_c = \log 4$, and let ε_n denote a sequence that converges to 0 and that represents the various error terms arising in the following calculation. If we substitute into the last display the formulas for $G^{(2)}_{\beta_n,K_n}(0)$ and $G^{(4)}_{\beta_n,K_n}(0)$ and use the convergence $(\beta_n, K_n) \to (\beta, K(\beta))$ and the continuity of $K(\cdot)$, then the last display implies that for all $n \in \mathbb{N}$, any $u \in \mathbb{R}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1+u}} \beta(K(\beta_n) - K_n)(1+\varepsilon_n)x^2 \\ &\quad + \frac{1}{n^{4\gamma-1+u}} c_4(\beta)(1+\varepsilon_n)x^4 + \mathrm{O}\left(\frac{1}{n^{5\gamma-1}}\right)x^5, \end{aligned} \tag{4.2}$$

where the big-oh term is uniform for $x \in (-Rn^\gamma, Rn^\gamma)$.

The two different asymptotic behaviors of $m(\beta_n, K_n)$ to be considered first in this section each depends on the choice of the sequence (β_n, K_n) converging to the second-order point $(\beta, K(\beta))$. Each choice controls, in a different way, the rate at which $(K(\beta_n) - K_n)$ in the quadratic term in (4.2) converges to 0.

Fix $0 < \beta < \beta_c$. For the first choice of sequence we take $\alpha > 0$, $b \in \{1, 0, -1\}$, and a real number $k \neq K'(\beta)b$ and define

$$\beta_n = \beta + b/n^\alpha \quad \text{and} \quad K_n = K(\beta) + k/n^\alpha. \tag{4.3}$$

Since $K(\beta_n) = K(\beta + b/n^\alpha) = K(\beta) + K'(\beta)b/n^\alpha + \mathrm{O}(1/n^{2\alpha})$, it follows from (4.2) that for all $n \in \mathbb{N}$, any $u \in \mathbb{R}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} &n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) \\ &= \frac{1}{n^{2\gamma+\alpha-1+u}} \beta(K'(\beta)b - k)(1+\varepsilon_n)x^2 \\ &\quad + \frac{1}{n^{4\gamma-1+u}} c_4(\beta)(1+\varepsilon_n)x^4 + \mathrm{O}\left(\frac{1}{n^{2\gamma+2\alpha-1+u}}\right)x^2 + \mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right)x^5. \end{aligned} \tag{4.4}$$

We now impose the condition that the powers of n appearing in the first two terms of the last display equal 0; i.e., $2\gamma + \alpha - 1 + u = 0 = 4\gamma - 1 + u$. These two equalities are equivalent to $\gamma = \alpha/2$ and $u = 1 - 4\gamma = 1 - 2\alpha$; in the notation of hypothesis (iii)(a) of Theorem 3.2 $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$, where $\alpha_0 = 1/2$ and $\theta = 1/2$. With this choice of γ and u the powers of n appearing in the last two terms in (4.4) are positive. It follows that as $n \to \infty$, we have uniformly for x in compact subsets of $\mathbb{R}$

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) = g(x) = \beta(K'(\beta)b - k)x^2 + c_4(\beta)x^4. \tag{4.5}$$

We now further assume that $K'(\beta)b - k < 0$. This inequality implies that (β_n, K_n) converges to $(\beta, K(\beta))$ along a ray lying above the tangent line to $(\beta, K(\beta))$ (see path 1 in Fig. 2). Thus, in accordance with hypothesis (i) of Theorem 3.2 (β_n, K_n) lies in the phase-coexistence region above the second-order curve for all sufficiently large n. Since b and k are not both 0, hypothesis (ii) of that theorem is also valid. As required by hypothesis (iii) of Theorem 3.2, the Ginzburg–Landau polynomial g is an even polynomial of degree 4, and since $K'(\beta)b - k < 0$ and $c_4(\beta) > 0$, $g(x) \to \infty$ as $|x| \to \infty$ and g has a unique positive, global minimum point. Substituting $\gamma = \alpha/2$ and $u = 1 - 2\alpha$ into (4.4), one proves the lower bound in hypothesis (iv) of Theorem 3.2 with $H(x) = -2\beta|K'(\beta)b - k|x^2 + \frac{1}{2}c_4(\beta)x^4$.

For all n for which (β_n, K_n) lies in the phase-coexistence region, the global minimum points of the free energy functional G_{β_n,K_n} are $\pm m(\beta_n, K_n)$ (Theorem 2.1(b)), a property reflected in the fact that the global minimum points of the Ginzburg–Landau polynomial g are also a pair of symmetric nonzero points. Alternatively, if $K'(\beta)b - k > 0$, then (β_n, K_n) lies in the single-phase region under the second-order curve for all sufficiently large n, G_{β_n,K_n} has a unique global minimum point at 0 (Theorem 2.1(a)), and g has a unique global minimum point at 0. In this way properties of g reflect the phase-transition structure of the model in the region through which (β_n, K_n) passes.

In the next theorem we describe the asymptotic behavior of $m(\beta_n, K_n)$ for the sequence (β_n, K_n) defined in (4.3) when $K'(\beta)b - k < 0$. Since $\theta = 1/2$, Theorem 3.2 implies that $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha} = \bar{x}/n^{\alpha/2} \to 0$, where $\bar{x}$ denotes the unique positive, global minimum point of g. The next theorem, the proof of which has just been sketched, corresponds to Theorem 3.1 in [12], where full details are given.

Theorem 4.1 *For $\beta \in (0, \beta_c)$, $\alpha > 0$, $b \in \{1, 0, -1\}$, and a real number $k \neq K'(\beta)b$, define*

$$\beta_n = \beta + b/n^\alpha \quad \text{and} \quad K_n = K(\beta) + k/n^\alpha$$

as well as $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/(8 \cdot 4!)$. Then (β_n, K_n) converges to the second-order point $(\beta, K(\beta))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $u = 1 - 2\alpha$, and $\gamma = \alpha/2$*

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) = g(x) = \beta(K'(\beta)b - k)x^2 + c_4(\beta)x^4$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg–Landau polynomial g has nonzero global minimum points if and only if $K'(\beta)b - k < 0$. If this inequality holds, then the global minimum points of g are $\pm\bar{x}$, where*

$$\bar{x} = \big(\beta(k - K'(\beta)b)/[2c_4(\beta)]\big)^{1/2} \tag{4.6}$$

(c) *Assume that $k > K'(\beta)b$. Then for any $\alpha > 0$, $m(\beta_n, K_n) \to 0$ and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2}; \quad \textit{i.e.}, \ \lim_{n\to\infty} n^{\alpha/2} m(\beta_n, K_n) = \bar{x}.$$

If $b \neq 0$, then this becomes $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^{1/2}$.

We now consider the second choice of the sequence (β_n, K_n) converging to a second-order point $(\beta, K(\beta))$ corresponding to $0 < \beta < \beta_c$. Given $\alpha > 0$, $b \in \{1, -1\}$, an integer $p \geq 2$, and a real number $\ell \neq K^{(p)}(\beta)$, we define

$$\beta_n = \beta + b/n^\alpha \quad \text{and} \quad K_n = K(\beta) + \sum_{j=1}^{p-1} K^{(j)}(\beta)b^j/(j!n^{j\alpha}) + \ell b^p/(p!n^{p\alpha}). \tag{4.7}$$

This sequence converges to the second-order point along a curve that coincides with the second-order curve to order $n^{-(p-1)\alpha}$ (see path 2 in Fig. 2). Since

$$K(\beta_n) - K_n = (K^{(p)}(\beta) - \ell)b^p/(p!n^{p\alpha}) + \mathrm{O}(1/n^{(p+1)\alpha}),$$

the expansion (4.2) takes the form

$$n^{1-u}G_{\beta_n,K_n}(x/n^\gamma)$$
$$= \frac{1}{n^{2\gamma+p\alpha-1+u}}\frac{1}{p!}\beta(K^{(p)}(\beta)-\ell)b^p(1+\varepsilon_n)x^2$$
$$+\frac{1}{n^{4\gamma-1+u}}c_4(\beta)(1+\varepsilon_n)x^4+\mathrm{O}\left(\frac{1}{n^{2\gamma+(p+1)\alpha-1+u}}\right)x^2+\mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right)x^5. \quad (4.8)$$

This formula is valid for all $n\in\mathbb{N}$, any $u\in\mathbb{R}$, any $\gamma>0$, any $R>0$, and all $x\in\mathbb{R}$ satisfying $|x/n^\gamma|<R$, and $\varepsilon_n\to 0$. We now impose the condition that the powers of n in the first two terms of the last display equal 0; i.e., $2\gamma+p\alpha-1+u=0=4\gamma-1+u$. These two equalities are equivalent to $\gamma=p\alpha/2$ and $u=1-4\gamma=1-2p\alpha$; in the notation of hypothesis (iii)(a) of Theorem 3.2 $u=1-\alpha/\alpha_0$ and $\gamma=\theta\alpha$, where $\alpha_0=1/(2p)$ and $\theta=p/2$. With this choice of γ and u, the powers of n in the last two terms in (4.8) are positive. It follows that as $n\to\infty$, we have uniformly for x in compact subsets of $\mathbb{R}$

$$\lim_{n\to\infty} n^{1-u}G_{\beta_n,K_n}(x/n^\gamma)=g(x)=\frac{1}{p!}\beta(K^{(p)}(\beta)-\ell)b^px^2+c_4(\beta)x^4.$$

We now further assume that $(K^{(p)}(\beta)-\ell)b^p<0$. This inequality implies that $K_n>K(\beta_n)$ for all sufficiently large n. Hence in accordance with hypothesis (i) of Theorem 3.2, (β_n,K_n) lies in the phase-coexistence region above the second-order curve for all sufficiently large n. Since $b\neq 0$, hypothesis (ii) of that theorem is also valid. As required by hypothesis (iii) of Theorem 3.2, the Ginzburg–Landau polynomial g is an even polynomial of degree 4, and since $(K^{(p)}(\beta)-\ell)b^p<0$ and $c_4(\beta)>0$, $g(x)\to\infty$ as $|x|\to\infty$ and g has a unique positive, global minimum point. As in the case of sequence (4.3), properties of g reflect the phase-transition structure of the model in the region through which (β_n,K_n) passes. Substituting $\gamma=p\alpha/2$ and $u=1-2p\alpha$ into (4.8), one verifies hypothesis (iv) of Theorem 3.2 with $H(x)=-\frac{2}{p!}\beta|K^{(p)}(\beta)b-\ell|x^2+\frac{1}{2}c_4(\beta)x^4$. This completes the verification of the hypotheses of Theorem 3.2.

In the next theorem we describe the asymptotic behavior of $m(\beta_n,K_n)$ for the sequence (β_n,K_n) defined in (4.7) when $(K^{(p)}(\beta)-\ell)b^p<0$. Since $\theta=p/2$, Theorem 3.2 implies that $m(\beta_n,K_n)\sim\bar{x}/n^{\theta\alpha}=\bar{x}/n^{p\alpha/2}\to 0$, where $\bar{x}$ denotes the unique positive, global minimum point of g. The next theorem, the proof of which has just been sketched, corresponds to Theorem 3.2 in [12], where full details are given.

Theorem 4.2 *For* $\beta\in(0,\beta_c)$, $\alpha>0$, $b\in\{1,-1\}$, *an integer* $p\geq 2$, *and a real number* $\ell\neq K^{(p)}(\beta)$, *define*

$$\beta_n=\beta+b/n^\alpha \quad \text{and} \quad K_n=K(\beta)+\sum_{j=1}^{p-1}K^{(j)}(\beta)b^j/(j!n^{j\alpha})+\ell b^p/(p!n^{p\alpha})$$

as well as $c_4(\beta)=(e^\beta+2)^2(4-e^\beta)/(8\cdot 4!)$. *Then* (β_n,K_n) *converges to the second-order point* $(\beta,K(\beta))$. *The following conclusions hold.*

(a) *For any* $\alpha>0$, $u=1-2p\alpha$, *and* $\gamma=p\alpha/2$

$$\lim_{n\to\infty} n^{1-u}G_{\beta_n,K_n}(x/n^\gamma)=g(x)=\frac{1}{p!}\beta(K^{(p)}(\beta)-\ell)b^px^2+c_4(\beta)x^4$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg–Landau polynomial g has nonzero global minimum points if and only if* $(K^{(p)}(\beta)-\ell)b^p<0$. *If this inequality holds, then the global minimum points of g are* $\pm\bar{x}$, *where*

$$\bar{x}=\big(\beta(\ell-K^{(p)}(\beta))b^p/[2c_4(\beta)p!]\big)^{1/2}. \tag{4.9}$$

(c) *Assume that* $(K^{(p)}(\beta)-\ell)b^p<0$. *Then for any* $\alpha>0$, $m(\beta_n,K_n)\to 0$ *and has the asymptotic behavior*

$$m(\beta_n,K_n)\sim \bar{x}/n^{p\alpha/2}=\bar{x}|\beta-\beta_n|^{p/2};\quad \textit{i.e.,}\quad \lim_{n\to\infty} n^{p\alpha/2}m(\beta_n,K_n)=\bar{x}.$$

We next derive in Theorems 4.3–4.6 the asymptotic behavior of $m(\beta_n,K_n)$ for four sequences (β_n,K_n) converging to the tricritical point $(\beta_c,K(\beta_c))$ from various subsets of the phase-coexistence region. A number of new phenomena arise in this case that are not observed in the cases considered earlier in this section. Full details of all the calculations for these four sequences are available in section 5 of [12]. As in the case of the two sequences considered earlier in this section, properties of the Ginzburg–Landau polynomials for these four new sequences reflect the phase-transition structure of the mean-field B-C model in the region through which the associated sequence (β_n,K_n) passes. This again makes rigorous the predictions of the Ginzburg–Landau phenomenology of critical phenomena discussed in section 2 of [12].

Let (β_n,K_n) be an arbitrary positive sequence converging to the tricritical point $(\beta_c,K(\beta_c))=(\log 4, 3/2\log 4)$. According to hypothesis (iii) of Theorem 3.2, we seek numbers $u\in\mathbb{R}$ and $\gamma\in\mathbb{R}$ and a suitable polynomial g such that $n^{1-u}G_{\beta_n,K_n}(x/n^\gamma)\to g(x)$ uniformly on compact subsets of $\mathbb{R}$. In order to carry this out, we consider the Taylor expansion of $nG_{\beta_n,K_n}(x/n^\gamma)$ to order 6 with an error term. This expansion takes the form

$$\begin{aligned} nG_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1}}\frac{G^{(2)}_{\beta_n,K_n}(0)}{2!}x^2+\frac{1}{n^{4\gamma-1}}\frac{G^{(4)}_{\beta_n,K_n}(0)}{4!}x^4\\ &\quad+\frac{1}{n^{6\gamma-1}}\frac{G^{(6)}_{\beta_n,K_n}(0)}{6!}x^6+\mathrm{O}\left(\frac{1}{n^{7\gamma-1}}\right)x^7, \end{aligned} \tag{4.10}$$

which is valid for all $n\in\mathbb{N}$, any $R>0$, and all $x\in\mathbb{R}$ satisfying $|x/n^\gamma|<R$. The big-oh term is uniform for $x\in(-Rn^\gamma,Rn^\gamma)$.

Define $c_4=3/16$ and $c_6=9/40$ and let ε_n denote a sequence that converges to 0 and that represents the various error terms arising in the following calculation. If we substitute into the last display the formulas for $G^{(2)}_{\beta_n,K_n}(0)$, $G^{(4)}_{\beta_n,K_n}(0)$, and $G^{(6)}_{\beta_n,K_n}(0)$ and use the convergence $(\beta_n,K_n)\to(\beta_c,K(\beta_c))$ and the continuity of $K(\cdot)$, the last display implies that for all $n\in\mathbb{N}$, any $u\in\mathbb{R}$, any $\gamma>0$, and all $x\in\mathbb{R}$ satisfying $|x/n^\gamma|<R$

$$\begin{aligned} &n^{1-u}G_{\beta_n,K_n}(x/n^\gamma)\\ &\quad=\frac{1}{n^{2\gamma-1+u}}\beta_c(K(\beta_n)-K_n)(1+\varepsilon_n)x^2+\frac{1}{n^{4\gamma-1+u}}c_4(4-e^{\beta_n})(1+\varepsilon_n)x^4\\ &\qquad+\frac{1}{n^{6\gamma-1+u}}c_6(1+\varepsilon_n)x^6+\mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right)x^7. \end{aligned} \tag{4.11}$$

The four different asymptotic behaviors of $m(\beta_n,K_n)$ to be considered in the remainder of this section each depends on the choice of the sequence (β_n,K_n) converging to the tricritical point $(\beta_c, K(\beta_c))$. Each choice controls, in a different way, the rate at which $(K(\beta_n) - K_n)$ in the quadratic term in (4.11) and the rate at which $(4 - e^{\beta_n})$ in the quartic term converge to 0.

For the first choice of sequence we take $\alpha > 0$, $b \in \{1, 0, -1\}$, and a real number $k \neq K'(\beta_c)b$ and define

$$\beta_n = \beta_c + b/n^\alpha \quad \text{and} \quad K_n = K(\beta_c) + k/n^\alpha. \tag{4.12}$$

Since $K(\beta_n) - K_n = (K'(\beta_c)b - k)/n^\alpha + \mathrm{O}(1/n^{2\alpha})$ and $4 - e^{\beta_n} = -4b/n^\alpha + \mathrm{O}(1/n^{2\alpha})$, it follows from (4.11) that for all $n \in \mathbb{N}$, any $u \in \mathbb{R}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} &n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \\ &\quad = \frac{1}{n^{2\gamma+\alpha-1+u}} \beta_c (K'(\beta_c)b - k)(1+\varepsilon_n)x^2 - \frac{1}{n^{4\gamma+\alpha-1+u}} 4c_4 b(1+\varepsilon_n)x^4 \\ &\qquad + \frac{1}{n^{6\gamma-1+u}} c_6(1+\varepsilon_n)x^6 + \mathrm{O}\left(\frac{1}{n^{2\gamma+2\alpha-1+u}}\right)x^2 \\ &\qquad + \mathrm{O}\left(\frac{1}{n^{4\gamma+2\alpha-1+u}}\right)x^4 + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right)x^7. \end{aligned} \tag{4.13}$$

We now impose the condition that the powers of n appearing in the first and third terms in the last display equal 0; i.e., $2\gamma + \alpha - 1 + u = 0 = 6\gamma - 1 + u$. These two equalities are equivalent to $\gamma = \alpha/4$ and $u = 1 - 6\gamma = 1 - 3\alpha/2$; in the notation of hypothesis (iii)(a) of Theorem 3.2 $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$, where $\alpha_0 = 2/3$ and $\theta = 1/4$. With this choice of γ and u, the powers of n in the second term and the last three terms in (4.13) are positive. It follows that as $n \to \infty$, we have for uniformly for x in compact subsets of $\mathbb{R}$

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) = g(x) = \beta_c(K'(\beta)b - k)x^2 + c_6 x^6.$$

We now further assume that $k > K'(\beta_c)b$. This inequality implies that (β_n, K_n) converges to $(\beta_c, K(\beta_c))$ along a ray lying above the tangent line to $(\beta_c, K(\beta_c))$ (see path 3 in Fig. 2). Hence in accordance with hypothesis (i) of Theorem 3.2, (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n; (β_n, K_n) is above the spinodal curve if $b = 1$ (see Fig. 1), above the second-order curve if $b = -1$, and above the tricritical point if $b = 0$. Since b and k are not both 0, hypothesis (ii) of Theorem 3.2 is also valid. As required by hypothesis (iii) of Theorem 3.2, the Ginzburg–Landau polynomial g is an even polynomial of degree 6, and since $K'(\beta_c)b - k < 0$ and $c_6 > 0$, $g(x) \to \infty$ as $|x| \to \infty$ and g has a unique positive, global minimum point. As in the case of the sequence in (4.3), properties of g reflect the phase-transition structure of the model in the region through which (β_n, K_n) passes. Substituting $\gamma = \alpha/4$ and $u = 1 - 3\alpha/2$ into (4.13), one verifies hypothesis (iv) of Theorem 3.2 with $H(x) = -2\beta_c|K'(\beta_c)b - k|x^2 - 8c_4x^4 + \frac{1}{2}c_6x^6$. This completes the verification of the hypotheses of Theorem 3.2.

In the next theorem we describe the asymptotic behavior of $m(\beta_n, K_n)$ for the sequence (β_n, K_n) defined in (4.12) when $K'(\beta_c)b - k < 0$. Since $\theta = 1/4$, Theorem 3.2 implies that $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha} = \bar{x}/n^{\alpha/4} \to 0$, where $\bar{x}$ denotes the unique positive, global minimum point of g. The next theorem, the proof of which has just been sketched, corresponds to Theorem 5.1 in [12], where full details are given.

Theorem 4.3 *For $\alpha > 0$, $b \in \{1, 0, -1\}$, and a real number $k \neq K'(\beta_c)b$, define*

$$\beta_n = \beta_c + b/n^{\alpha} \quad \textit{and} \quad K_n = K(\beta_c) + k/n^{\alpha}$$

as well as $c_6 = 9/40$. Then (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $u = 1 - 3\alpha/2$, and $\gamma = \alpha/4$*

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n, K_n}(x/n^{\gamma}) = g(x) = \beta_c(K'(\beta_c)b - k)x^2 + c_6 x^6$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg–Landau polynomial g has nonzero global minimum points if and only if $K'(\beta_c)b - k < 0$. If this inequality holds, then the global minimum points of g are $\pm\bar{x}$, where*

$$\bar{x} = \big(\beta_c(k - K'(\beta_c)b)/[3c_6]\big)^{1/4}. \tag{4.14}$$

(c) *Assume that $k > K'(\beta_c)b$. Then for any $\alpha > 0$, $m(\beta_n, K_n) \to 0$ and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/4}; \quad \textit{i.e.,} \quad \lim_{n\to\infty} n^{\alpha/4} m(\beta_n, K_n) = \bar{x}.$$

When $b \neq 0$, this becomes $m(\beta_n, K_n) \sim \bar{x}|\beta_c - \beta_n|^{1/4}$.

We now consider the second choice of sequence (β_n, K_n) converging to the tricritical point. Given $\alpha > 0$, $\ell \in \mathbb{R}$, and $\tilde{\ell} \in \mathbb{R}$ we define

$$\beta_n = \beta_c + 1/n^{\alpha} \quad \text{and} \quad K_n = K(\beta_c) + K'(\beta_c)/n^{\alpha} + \ell/(2n^{2\alpha}) + \tilde{\ell}/(6n^{3\alpha}). \tag{4.15}$$

Since

$$K(\beta_n) - K_n = (K''(\beta_c) - \ell)/(2n^{2\alpha}) + (K'''(\beta_c) - \tilde{\ell})/(6n^{3\alpha}) + \mathrm{O}(1/n^{4\alpha})$$

and $4 - e^{\beta_n} = -4/n^{\alpha} + \mathrm{O}(1/n^{2\alpha})$, it follows from (4.11) that for all $n \in \mathbb{N}$, any $u > 0$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^{\gamma}| < R$

$$\begin{aligned} &G_n(x) \\ &\quad = \frac{1}{n^{2\gamma+2\alpha-1+u}} \frac{1}{2}\beta_c(K''(\beta_c) - \ell)(1+\varepsilon_n)x^2 - \frac{1}{n^{4\gamma+\alpha-1+u}} 4c_4(1+\varepsilon_n)x^4 \\ &\qquad + \frac{1}{n^{6\gamma-1+u}} c_6(1+\varepsilon_n)x^6 + \mathrm{O}\left(\frac{1}{n^{2\gamma+3\alpha-1+u}}\right)x^2 + \mathrm{O}\left(\frac{1}{n^{2\gamma+4\alpha-1+u}}\right)x^2 \\ &\qquad + \mathrm{O}\left(\frac{1}{n^{4\gamma+2\alpha-1+u}}\right)x^4 + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right)x^7. \end{aligned} \tag{4.16}$$

We now impose the condition that the powers of n appearing in the first three terms in the last display equal 0; i.e., $2\gamma + 2\alpha - 1 + u = 0 = 4\gamma + \alpha - 1 + u = 6\gamma - 1 + u$. These three equalities are equivalent to $\gamma = \alpha/2$ and $u = 1 - 6\gamma = 1 - 3\alpha$; in the notation of hypothesis (iii) of Theorem 3.2 $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$, where $\alpha_0 = 1/3$ and $\theta = 1/2$. With this choice of γ and u, the powers of n in the last four terms in (4.16) are positive. It follows that as $n \to \infty$, we have uniformly for x in compact subsets of $\mathbb{R}$

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) = g_\ell(x) = \frac{1}{2}\beta_c(K''(\beta_c) - \ell)x^2 - 4c_4x^4 + c_6x^6. \tag{4.17}$$

We write the Ginzburg–Landau polynomial as g_ℓ in order to emphasize the dependence on the parameter ℓ; g_ℓ does not depend on the choice of $\tilde{\ell}$ in (4.15).

For $\ell \in \mathbb{R}$ and suitable $\tilde{\ell}$ we now describe the region in which (β_n, K_n) lies for sufficiently large n. The discussion depends in part on the validity of Conjectures 1 and 2 stated below. These conjectures are discussed in detail in section 5 of [12] and are supported by properties of the Ginzburg–Landau polynomials and numerical calculations. The conjectures involve the behavior, in a neighborhood of the tricritical point, of the first-order curve defined by $K_1(\beta)$ for $\beta > \beta_c$. Since $\lim_{\beta\to\beta_c^+} K_1(\beta) = K(\beta_c)$ [14, Sects. 3.1, 3.3], by continuity we extend the definition of $K_1(\beta)$ to $\beta = \beta_c$ by defining $K_1(\beta_c) = K(\beta_c)$. We assume that the first three right-hand derivatives of $K_1(\beta)$ exists at β_c and denote them by $K_1'(\beta_c)$, $K_1''(\beta_c)$, and $K_1'''(\beta_c)$. We also define $\ell_c = K''(\beta_c) - 5/(4\beta_c)$. Conjectures 1 and 2 state the following: (1) $K_1'(\beta_c) = K'(\beta_c)$, (2) $K_1''(\beta_c) = \ell_c < 0 < K''(\beta_c)$.

Since $\beta_n - \beta_c = 1/n^\alpha$, (β_n, K_n) converges to $(\beta_c, K(\beta_c))$ along the curve $(\beta, \tilde{K}(\beta))$, where for $\beta > \beta_c$

$$\tilde{K}(\beta) = K(\beta_c) + K'(\beta_c)(\beta - \beta_c) + \ell(\beta - \beta_c)^2/2 + \tilde{\ell}(\beta - \beta_c)^3/6. \tag{4.18}$$

Hence we have the following picture. Possible paths for the sequences in items i–iv are the respective curves labeled 4a–4d in Fig. 2. The spinodal curve is the extension of the second-order curve to $\beta > \beta_c$.

i. For $\ell > K''(\beta_c)$ and any $\tilde{\ell} \in \mathbb{R}$, (β_n, K_n) lies in the phase-coexistence region located above the spinodal curve for all sufficiently large n,
ii. For $\ell = K''(\beta_c)$ and $\tilde{\ell} > K'''(\beta_c)$, (β_n, K_n) lies in the phase-coexistence region located above the spinodal curve for all sufficiently large n.
iii. We assume Conjectures 1 and 2. Then for $\ell \in (\ell_c, K''(\beta_c))$ and any $\tilde{\ell} \in \mathbb{R}$, (β_n, K_n) lies in the phase-coexistence region located above the first-order curve and below the spinodal curve for all sufficiently large n.
iv. We assume Conjectures 1 and 2. Then for $\ell = \ell_c$ and any $\tilde{\ell} > K_1'''(\beta_c)$, (β_n, K_n) lies in the phase-coexistence region located above the first-order curve and below the spinodal curve for all sufficiently large n. The curve $(\beta, \tilde{K}(\beta))$ defined in (4.18) coincides to order 2 in powers of $\beta - \beta_c$ with the first-order curve.
v. We assume Conjectures 1 and 2. Then for $\ell < \ell_c$ and any $\tilde{\ell} \in \mathbb{R}$, (β_n, K_n) lies in the single-phase region located below the first-order curve for all sufficiently large n.

For $\ell \in \mathbb{R}$ define $\mathcal{M}_{g_\ell}$ to be the set of global minimum points of the Ginzburg–Landau polynomial g_ℓ in (4.17), which are easily determined using Theorem A.1. For $\beta > 0$ and $K > 0$, define $\mathcal{M}_{\beta,K}$ to be the set of global minimum points of $G_{\beta,K}$ as in Sect. 2. We next describe the structure of $\mathcal{M}_{g_\ell}$ and show that it mirrors that of $\mathcal{M}_{\beta_n,K_n}$, which describes the phase-transition structure of the region through which the corresponding sequence (β_n, K_n) passes. These regions are described in items (i)–(v) after (4.18), and the phase-transition structure is given in Theorem 2.2. In terms of $\ell_c = K''(\beta_c) - 5/(4\beta_c)$, Theorem A.1 gives the following picture.

1. For $\ell > \ell_c$, $\mathcal{M}_{g_\ell}$ equals $\{\pm\bar{x}(\ell)\}$, where $\bar{x}(\ell)$ is defined in (4.19). For these values of ℓ and suitable $\tilde{\ell}$ given items (i)–(iii), this behavior of $\mathcal{M}_{g_\ell}$ mirrors the fact that for (β_n, K_n) lying above the first-order curve $\mathcal{M}_{\beta_n,K_n}$ equals $\{\pm m(\beta_n, K_n)\}$.

2. For $\ell = \ell_c$, $\mathcal{M}_{g_\ell}$ equals $\{0, \pm\bar{x}(\ell_c)\}$, where $\bar{x}(\ell_c) = \sqrt{5/3}$. As specified in item (iv), for $\ell = \ell_c$ and $\tilde{\ell} > K_1'''(\beta_c)$ this structure of $\mathcal{M}_{g_\ell}$ mirrors the fact that for (β_n, K_n) lying on the first-order curve $\mathcal{M}_{\beta_n,K_n}$ equals $\{0, \pm m(\beta_n, K_n)\}$.
3. For $\ell < \ell_c$, g_ℓ has a unique global minimum point at 0. As specified in item (v), for $\ell < \ell_c$ and $\tilde{\ell} \in \mathbb{R}$ this structure of $\mathcal{M}_{g_\ell}$ mirrors the fact that for (β_n, K_n) lying below the first-order curve $\mathcal{M}_{\beta_n,K_n}$ equals $\{0\}$.
4. As described in items 1–3, $\mathcal{M}_{g_\ell}$ undergoes a discontinuous bifurcation at $\ell = \ell_c$, mirroring the discontinuous bifurcation undergone by $\mathcal{M}_{\beta,K}$ at $K = K_1(\beta)$ for $\beta > \beta_c$.

In Theorem 4.4 we describe the asymptotic behavior of $m(\beta_n, K_n)$ for the sequence (β_n, K_n) defined in (4.15). For the choice of parameters in items i–iv after (4.18), (β_n, K_n) lies in the phase-coexistence region as required by hypothesis (i) of Theorem 3.2. For the choice of parameters in items iii and iv, Conjectures 1 and 2 in section 5 of [12] are needed. Hypothesis (ii) of Theorem 3.2 is obviously valid. As required by hypothesis (iii) of Theorem 3.2, the Ginzburg–Landau polynomial g_ℓ is an even polynomial of degree 6, $g_\ell(x) \to \infty$ as $|x| \to \infty$, and by Theorem A.1 g_ℓ has a unique positive, global minimum point for $\ell \geq \ell_c$. Substituting $\gamma = \alpha/2$ and $u = 1 - 3\alpha$ into (4.16), one verifies hypothesis (iv) of Theorem 3.2 with $H(x) = -\beta_c|K''(\beta_c) - \ell|x^2 - 8c_4x^4 + \frac{1}{2}c_6x^6$ when $\ell \neq K''(\beta_c)$ and with a slightly modified H when $\ell = K''(\beta_c)$. This completes the verification of the hypotheses of Theorem 3.2. Since $\theta = 1/2$, Theorem 3.2 implies that $m(\beta_n, K_n) \sim \bar{x}(\ell)/n^{\theta\alpha} = \bar{x}(\ell)/n^{\alpha/2} \to 0$, where $\bar{x}(\ell)$ denotes the unique positive, global minimum point of g_ℓ. The next theorem, the proof of which has just been sketched, corresponds to Theorem 5.2 in [12], where full details are given.

Theorem 4.4 *For $\alpha > 0$, $\ell \in \mathbb{R}$, and $\tilde{\ell} \in \mathbb{R}$, define*

$$\beta_n = \beta_c + 1/n^\alpha \quad \textit{and} \quad K_n = K(\beta_c) + K'(\beta_c)/n^\alpha + \ell/(2n^{2\alpha}) + \tilde{\ell}/(6n^{3\alpha})$$

as well as $c_4 = 3/16$ and $c_6 = 9/40$. Then (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $u = 1 - 3\alpha$, and $\gamma = \alpha/2$*

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) = g_\ell(x) = \frac{1}{2}\beta_c(K''(\beta_c) - \ell)x^2 - 4c_4x^4 + c_6x^6$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg–Landau polynomial g_ℓ has nonzero global minimum points if and only if $\ell \geq \ell_c = K''(\beta_c) - 5/(4\beta_c) = (-\beta_c^2 - 8\beta_c + 12)/4\beta_c^3$.*

(i) *Assume that $\ell > \ell_c$. Then the global minimum points of g_ℓ are $\pm\bar{x}(\ell)$, where*

$$\bar{x}(\ell) = \frac{\sqrt{10}}{3}\left(1 + \left(1 - \frac{3\beta_c}{5}(K''(\beta_c) - \ell)\right)^{1/2}\right)^{1/2}. \tag{4.19}$$

(ii) *Assume that $\ell = \ell_c$. Then the global minimum points of the Ginzburg–Landau polynomial $g_\ell(x)$ are 0 and $\pm\bar{x}(\ell_c)$, where $\bar{x}(\ell_c) = (5/3)^{1/2}$.*

(c) *In each of the cases* (i)–(iv) *appearing after* (4.18) *and for any $\alpha > 0$, $m(\beta_n, K_n) \to 0$ and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}(\ell)/n^{\alpha/2} = \bar{x}(\ell)(\beta_n - \beta_c)^{1/2}; \quad \textit{i.e.,} \quad \lim_{n\to\infty} n^{\alpha/2} m(\beta_n, K_n) = \bar{x}(\ell).$$

We now consider the third choice of sequence (β_n, K_n) converging to the tricritical point. Given $\alpha > 0$, an integer $p \geq 2$, and $\ell \in \mathbb{R}$, we define

$$\beta_n = \beta_c - 1/n^\alpha \quad \text{and} \quad K_n = K(\beta_c) + \sum_{j=1}^{p-1} K^{(j)}(\beta_c)(-1)^j/(j!n^{j\alpha}) + \ell(-1)^p/(p!n^{p\alpha}). \tag{4.20}$$

This sequence converges to the tricritical point from the left along a curve that coincides with the second-order curve to order $n^{-(p-1)\alpha}$ (see paths 5 and 6 in Fig. 2). Since

$$K(\beta_n) - K_n = (K^{(p)}(\beta_c) - \ell)(-1)^p/(p!n^{p\alpha}) + \mathrm{O}(1/n^{(p+1)\alpha})$$

and $4 - e^{\beta_n} = 4/n^\alpha + \mathrm{O}(1/n^{2\alpha})$, it follows from (4.11) that for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} &n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \\ &\quad = \frac{1}{n^{2\gamma+p\alpha-1+u}} \frac{1}{p!} \beta_c (K^{(p)}(\beta_c) - \ell)(-1)^p (1+\varepsilon_n) x^2 \\ &\qquad + \frac{1}{n^{4\gamma+\alpha-1+u}} 4c_4 (1+\varepsilon_n) x^4 + \frac{1}{n^{6\gamma-1+u}} c_6 (1+\varepsilon_n) x^6 \\ &\qquad + \mathrm{O}\left(\frac{1}{n^{2\gamma+(p+1)\alpha-1+u}}\right) x^2 + \mathrm{O}\left(\frac{1}{n^{4\gamma+2\alpha-1+u}}\right) x^4 + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x^7. \end{aligned} \tag{4.21}$$

We first consider $p = 2$, which gives rise to a different asymptotic behavior of $m(\beta_n, K_n) \to 0$ from $p \geq 3$. We impose the condition that the three powers of n appearing in the first three terms in (4.21) equal 0; i.e., $2\gamma + 2\alpha - 1 + u = 0 = 4\gamma + \alpha - 1 + u = 6\gamma - 1 + u$. These three equalities are equivalent to $\gamma = \alpha/2$ and $u = 1 - 6\gamma = 1 - 3\alpha$; in the notation of hypothesis (iii)(a) of Theorem 3.2 $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$, where $\alpha_0 = 1/3$ and $\theta = 1/2$. With this choice of γ and u, the powers of n in the last three terms in (4.21) are positive. It follows that as $n \to \infty$, we have uniformly for x in compact subsets of $\mathbb{R}$

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \to g(x) = \frac{1}{2}\beta_c(K''(\beta_c) - \ell)x^2 + 4c_4 x^4 + c_6 x^6. \tag{4.22}$$

We now further assume that $\ell > K''(\beta_c)$. This inequality implies that $K_n > K(\beta_n)$ for all sufficiently large n. Hence in accordance with hypothesis (i) of Theorem 3.2, (β_n, K_n) lies in the phase-coexistence region above the second-order curve for all sufficiently large n. Hypothesis (ii) of Theorem 3.2 is obviously valid. As required by hypothesis (iii) of Theorem 3.2, the Ginzburg–Landau polynomial g is an even polynomial of degree 6, and since $K''(\beta_c) - \ell < 0$, $c_4 > 0$, and $c_6 > 0$, $g(x) \to \infty$ as $|x| \to \infty$ and by Theorem A.2 g has a unique positive, global minimum point at the quantity $\bar{x}$ defined in (4.23). As in previous cases, properties of g reflect the phase-transition structure of the model in the region through which (β_n, K_n) passes. Substituting $\gamma = \alpha/2$ and $u = 1 - 3\alpha$ into (4.21), one verifies hypothesis (iv) of Theorem 3.2 with $H(x) = -\beta_c|K''(\beta_c) - \ell|x^2 + \frac{1}{2}c_6(\beta)x^6$. This completes the verification of the hypotheses of Theorem 3.2.

In the next theorem we describe the asymptotic behavior of $m(\beta_n, K_n)$ for the sequence (β_n, K_n) defined in (4.20) when $p = 2$ and $\ell > K''(\beta_c)$. Since $\theta = 1/2$, Theorem 3.2 implies that $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha} = \bar{x}/n^{\alpha/2} \to 0$, where $\bar{x}$ denotes the unique positive, global minimum point of g. The next theorem, the proof of which has just been sketched, corresponds to Theorem 5.3 in [12], where full details are given.

Theorem 4.5 *For $\alpha > 0$ and $\ell \in \mathbb{R}$ define*

$$\beta_n = \beta_c - 1/n^\alpha \quad \textit{and} \quad K_n = K(\beta_c) + K'(\beta_c)/n^\alpha + \ell/2n^{2\alpha}$$

as well as $c_4 = 3/16$ and $c_6 = 9/40$. Then the sequence (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $u = 1 - 3\alpha$, and $\gamma = \alpha/2$*

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) = g(x) = \frac{1}{2}\beta_c(K''(\beta_c) - \ell)x^2 + 4c_4x^4 + c_6x^6$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg–Landau polynomial g has nonzero global minimum points if and only if $\ell > K''(\beta_c)$. If this inequality holds, then the global minimum points of g are $\pm\bar{x}$, where*

$$\bar{x} = \frac{\sqrt{10}}{3}\left(-1 + \left(1 + \frac{3\beta_c}{5}(\ell - K''(\beta_c))\right)^{1/2}\right)^{1/2}. \tag{4.23}$$

(c) *Assume that $\ell > K''(\beta_c)$. Then for any $\alpha > 0$, $m(\beta_n, K_n) \to 0$ and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2} = \bar{x}(\beta_c - \beta_n)^{1/2}; \quad \textit{i.e.}, \quad \lim_{n\to\infty} n^{\alpha/2} m(\beta_n, K_n) = \bar{x}.$$

This theorem completes the analysis for $p = 2$. We now continue with the more complicated analysis for the sequences (β_n, K_n) defined in (4.20) for $p \geq 3$, $\alpha > 0$, and $\ell \neq K^{(p)}(\beta_c)$. We omit the calculation showing that $\gamma = (p-1)\alpha/2$ and $u = 1 - p\alpha - 2\gamma = 1 - (2p-1)\alpha$ is the only combination of γ and u for which the limit of $n^{1-u}G_{\beta_n,K_n}(x/n^\gamma)$ in (4.21) is a polynomial having a positive, global minimum point. In the notation of hypothesis (iii)(a) of Theorem 3.2, $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$, where $\alpha_0 = 1/(2p-1)$ and $\theta = (p-1)/2$. With this choice of γ and u, the powers of n in the last four terms in (4.21) are positive. It follows that as $n \to \infty$, we have uniformly for x in compact subsets of $\mathbb{R}$

$$n^{1-u}G_{\beta_n,K_n}(x/n^\gamma) \to g(x) = \frac{1}{p!}\beta_c(K^{(p)}(\beta_c) - \ell)(-1)^p x^2 + 4c_4x^4. \tag{4.24}$$

We now further assume that $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$. This inequality implies that $K_n > K(\beta_n)$ for all sufficiently large n. Hence in accordance with hypothesis (i) of Theorem 3.2, (β_n, K_n) lies in the phase-coexistence region above the second-order curve for all sufficiently large n. Hypothesis (ii) of Theorem 3.2 is obviously valid. As required by hypothesis (iii) of Theorem 3.2, the Ginzburg–Landau polynomial g is an even polynomial of degree 4, and since $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$, $g(x) \to \infty$ as $|x| \to \infty$ and by g has a unique positive, global minimum point. As in previous cases, properties of g reflect the phase-transition structure of the model in the region through which (β_n, K_n) passes. Substituting $\gamma = (p-1)\alpha/2$ and $u = 1 - (2p-1)\alpha$ into (4.21), one verifies hypothesis (iv) of Theorem 3.2 with $H(x) = -2\frac{1}{p!}\beta_c|K^{(p)}(\beta_c) - \ell|x^2 + 2c_4x^4$. This completes the verification of the hypotheses of Theorem 3.2.

In the next theorem we describe the asymptotic behavior of $m(\beta_n, K_n)$ for the sequence (β_n, K_n) defined in (4.20) when $p \geq 3$ is a positive integer and $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$. Since $\theta = (p-1)/2$, Theorem 3.2 implies that $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha} = \bar{x}/n^{(p-1)\alpha/2} \to 0$, where $\bar{x}$ denotes the unique positive, global minimum point of g. The next theorem, the proof of which has just been sketched, corresponds to Theorem 5.4 in [12], where full details are given.

Theorem 4.6 *For p a positive integer satisfying $p \geq 3$, $\alpha > 0$, and a real number $\ell \neq K^{(p)}(\beta_c)$, define*

$$\beta_n = \beta_c - 1/n^{\alpha} \quad \textit{and} \quad K_n = K(\beta_c) + \sum_{j=1}^{p-1} K^{(j)}(\beta_c)(-1)^j/(j!n^{j\alpha}) + \ell(-1)^p/(p!n^{p\alpha})$$

as well as $c_4 = 3/16$. Then (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $\gamma = (p-1)\alpha/2$, and $u = 1 - (2p-1)\alpha$*

$$\lim_{n \to \infty} n^{1-u} G_{\beta_n, K_n}(x/n^{\gamma}) = g(x) = \frac{1}{p!}\beta_c (K^{(p)}(\beta_c) - \ell)(-1)^p x^2 + 4c_4 x^4$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg–Landau polynomial has nonzero global minimum points if and only if $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$. If this inequality holds, then the global minimum points of g are $\pm\bar{x}$, where*

$$\bar{x} = \left(\beta_c\left(\ell - K^{(p)}(\beta_c)(-1)^p\right)/[8c_4 p!]\right)^{1/2}. \tag{4.25}$$

(c) *Assume that $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$. Then for any $\alpha > 0$, $m(\beta_n, K_n) \to 0$ and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}/n^{(p-1)\alpha/2} = \bar{x}(\beta_n - \beta_c)^{(p-1)/2}; \quad \textit{i.e.,} \quad \lim_{n \to \infty} n^{(p-1)\alpha/2} m(\beta_n, K_n) = \bar{x}.$$

This completes our analysis of the asymptotic behavior of $m(\beta_n, K_n)$ for the six sequences introduced in this section. In the next section we relate this asymptotic behavior to the scaling theory of critical phenomena.

5 Relationship with Scaling Theory of Critical Phenomena

The results on the asymptotic behavior of $m(\beta_n, K_n)$ obtained in Sect. 4 are related to scaling theory for critical and tricritical points [20, 23]. In this section we review scaling theory and show that its predictions for the magnetization are consistent with the results obtained in that section.

Scaling theory is based on the idea that the singular parts of thermodynamic functions near continuous phase transitions are homogeneous functions of the distance to the phase transition. If there is a single parameter controlling the approach to the phase transition, then the content of scaling theory for a single thermodynamic quantity is simply that its singularities are power laws. If there is more than one parameter, as is the case here, then scaling theory has a richer content, especially near the tricritical point where the type of phase transition changes in a small neighborhood.

We are interested in the magnetization m as a function of (β_n, K_n), a sequence converging either to a second-order point $(\beta, K(\beta))$ with $0 < \beta < \beta_c$ or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. In either case, the relevant parameter-space is two dimensional. Given any phase-transition point $(\beta, K(\beta))$ with $0 < \beta \leq \beta_c$, the natural coordinate system

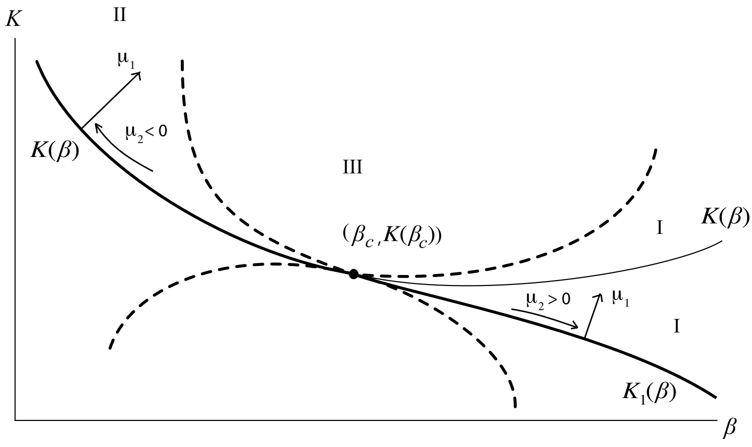

Fig. 4 Curvilinear coordinate system for scaling theory showing the coordinates μ_1 and μ_2; μ_1 is the signed distance from the phase transition line and μ_2 the signed distance from the tricritical point along the phase transition line. Regions I, II, and III are dominated, respectively, by the first-order, second-order, and tricritical phase transition. A similar coordinate system can be defined for any point along the second-order curve

for scaling theory is a curvilinear system (μ_1, μ_2) measuring the signed distances from the phase transition point; μ_1 is the signed distance from the curve of phase transitions and μ_2 the signed distance from the chosen point along the curve of phase transitions. Since we are concerned with the phase-coexistence region, in all our considerations $\mu_1 \geq 0$; however, μ_2 may take either sign. At the tricritical point, $\mu_2 > 0$ and $\mu_1 = 0$ correspond to the first-order line to the right of the tricritical point while $\mu_2 < 0$ and $\mu_1 = 0$ correspond to the second order line to the left of the tricritical point. At a second-order point, for sufficiently small $|\mu_2|$, $(0, \mu_2)$ is also a second-order point. Figure 4 shows this coordinate system for the special case of the tricritical point.

Scaling theory for the magnetization in a two-dimensional parameter space takes the general form

$$m(\mu_1\tau, \mu_2\tau^a) = \tau^b m(\mu_1, \mu_2), \tag{5.1}$$

where τ is an arbitrary scale factor and a and b are exponents to be determined [20]. The exponents a and b are chosen so that the theory is consistent with known exponents for the particular type of phase transition. In our case, a and b depend on whether the phase transition point is a second-order point or the tricritical point.

We first consider the simpler case of a second-order point. Then the neighboring points along the phase-transition curve are also second-order points, and there is no singular dependence on μ_2, implying that $a = 0$. The singular behavior of the magnetization is controlled by $\tilde{\beta}$, the mean-field magnetization exponent for second-order transitions, which has the value $\tilde{\beta} = 1/2$ [23]. Choosing $b = \tilde{\beta} = 1/2$, we obtain from (5.1)

$$m(\mu_1\tau, \mu_2) = \tau^{\tilde{\beta}} m(\mu_1, \mu_2) = \tau^{1/2} m(\mu_1, \mu_2). \tag{5.2}$$

Setting $\tau = 1/\mu_1$ yields

$$m(\mu_1, \mu_2) = \mu_1^{\tilde{\beta}} m(1, \mu_2) = \mu_1^{1/2} f(\mu_2); \tag{5.3}$$

$f(\mu_2)$ is a smooth function of μ_2 that depends on the chosen point $(\beta, K(\beta))$, and the critical amplitude $f(0)$ is presumed to be positive. Equation (5.3) reflects the standard power-law behavior of the magnetization near a critical point.

We now show that (5.3) is consistent with Theorems 4.1 and 4.2. These theorems give the exact asymptotic behavior of $m(\beta_n, K_n)$ for sequences (β_n, K_n) converging to a second-order point. For ease of exposition, we refer to the definitions of the sequences according to the labeling in Fig. 2 in the introduction, calling them sequences of type 1 and type 2, respectively.

We first consider the sequence of type 1, which converges to a second-order point $(\beta_0, K(\beta_0))$ along a ray that is above the tangent line to the second-order curve at that point. Defined in (4.3), this sequence takes the form

$$\beta_n = \beta_0 + b/n^\alpha \quad \text{and} \quad K_n = K(\beta_0) + k/n^\alpha, \tag{5.4}$$

where $b \in \{1, 0, -1\}$ and $K'(\beta_0)b - k < 0$. To leading order, the coordinate μ_1 is given by the distance to the tangent to the second-order curve at $(\beta_0, K(\beta_0))$; i.e.,

$$\mu_1 \approx (K - K(\beta_0)) - K'(\beta_0)(\beta - \beta_0). \tag{5.5}$$

Hence we obtain

$$\mu_1 \approx (k - K'(\beta_0)b)/n^\alpha. \tag{5.6}$$

The distance μ_2 is also of order $1/n^\alpha$. However, f is a smooth function of μ_2 that converges to $f(0) > 0$ as $\mu_2 \to 0$. Hence we need only know that $\mu_2 \to 0$ in order to obtain the leading-order behavior of m from (5.3) and (5.6), namely, $m \approx (k - K'(\beta_0)b)^{1/2}/n^{\alpha/2}$. This asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 4.1, correctly predicting both the exponent of n and the dependence on k and b in the prefactor $\bar{x}$ as given in (4.6) with $\beta = \beta_0$.

We next consider the sequence of type 2, which converges to a second-order point $(\beta_0, K(\beta_0))$ along a curve lying in the phase-coexistence region and having the same tangent as the second-order curve at that point. Defined in (4.7), this sequence takes the form

$$\beta_n = \beta_0 + b/n^\alpha \quad \text{and} \quad K_n = K(\beta_0) + \sum_{j=1}^{p-1} K^{(j)}(\beta_0)b^j/(j!n^{j\alpha}) + \ell b^p/(p!n^{p\alpha}), \tag{5.7}$$

where $b \in \{1, -1\}$, $p \geq 2$, and $(K^{(p)}(\beta_0) - \ell)b^p < 0$. In this case it is crucial to recall that the scaling variables comprise a curvilinear coordinate system. In particular, the coordinate μ_1 measures the distance from the second-order curve, not the distance from the tangent to this curve at $(\beta_0, K(\beta_0))$; as a result (5.5) is not sufficient to determine the asymptotic behavior of m. The sequence of type 2 converges to the second-order point along a curve that agrees with the second-order curve to order $p - 1$ in powers of $\beta - \beta_0$. Hence to leading order μ_1 is proportional to the difference between the last term in the definition of K_n and the term of order p in the Taylor expansion of $K(\beta_n)$, namely, $\mu_1 \approx |\ell - K^{(p)}(\beta_0)|/(p!n^{p\alpha})$. Substituting this expression into (5.3) yields

$$m \approx (|\ell - K^{(p)}(\beta_0)|/p!n^{p\alpha})^{\tilde{\beta}} = (|\ell - K^{(p)}(\beta_0)|/p!)^{1/2}/n^{p\alpha/2}.$$

Again, this asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 4.2 and correctly captures the square-root dependence of the prefactor $\bar{x}$ on $|\ell - K^{(p)}(\beta_0)|$ given in (4.9) with $\beta = \beta_0$.

If there is more than one type of phase transition in a neighborhood of a phase-transition point, as is the case near a tricritical point, then scaling theory becomes more complicated [20]. In the case of the tricritical point, the theory involves crossovers between the nearby first-order, second-order, and tricritical phase transitions. Figure 4 shows the region near the tricritical point.

The three regions I, II, and III separated by dotted lines are controlled by the first-order, the second-order, and the tricritical phase transitions, respectively. The mean-field tricritical crossover exponent φ_t determines the boundaries of the regions. In regions I and II we have $|\mu_1| \ll |\mu_2|^{1/\varphi_t}$ while in region III $|\mu_1| \gg |\mu_2|^{1/\varphi_t}$. In region II the magnetization m is controlled by $\tilde{\beta}$, the mean-field magnetization exponent for second-order transitions. In region III the magnetization m is controlled by $\tilde{\beta}_t$, the mean-field magnetization exponent for tricritical transitions, while in region I the magnetization m approaches a constant value as the first-order line is approached. These insights are incorporated in the scaling hypothesis

$$m(\mu_1\tau, \mu_2\tau^{\varphi_t}) = \tau^{\tilde{\beta}_t} m(\mu_1, \mu_2), \tag{5.8}$$

where τ is an arbitrary scale factor [20]. This corresponds to (5.1) with $a = \varphi_t$ and $b = \tilde{\beta}_t$. Setting $\tau = |\mu_2|^{-1/\varphi_t}$ yields the alternate form

$$m(\mu_1, \mu_2) = |\mu_2|^{\tilde{\beta}_t/\varphi_t} m(\mu_1/|\mu_2|^{1/\varphi_t}, 1) = |\mu_2|^{\tilde{\beta}_t/\varphi_t} f_\pm(\mu_1/|\mu_2|^{1/\varphi_t}), \tag{5.9}$$

where f_+ is used on the first-order side of the tricritical point ($\mu_2 > 0$) and f_- is used on the second-order side of the tricritical point ($\mu_2 < 0$). The values of the three relevant mean-field exponents are $\varphi_t = 1/2$, $\tilde{\beta} = 1/2$, and $\tilde{\beta}_t = 1/4$ [21].

We now consider the form taken by the right side of (5.9) in each of the three regions. In region III the arguments of f_+ and of f_- are large. Hence in order to recover the tricritical power-law behavior of m we require that $f_+(x) \approx x^{\tilde{\beta}_t}$ and $f_-(x) \approx x^{\tilde{\beta}_t}$ as $x \to \infty$, yielding

$$m(\mu_1, \mu_2) \approx \mu_1^{\tilde{\beta}_t} = \mu_1^{1/4} \ \ \text{[region III]}. \tag{5.10}$$

In region II with fixed μ_2 we expect that the scaling is the one given in (5.2) for the second-order curve; i.e.,

$$m(\mu_1\tau, \mu_2) = \tau^{\tilde{\beta}} m(\mu_1, \mu_2) = \tau^{1/2} m(\mu_1, \mu_2). \tag{5.11}$$

The requirement that the two scaling assumptions (5.9) and (5.11) are consistent yields an interesting result for the behavior of m in region II for small $|\mu_2|$. The asymptotic behavior of $f_-(x)$ as $x \to 0^+$ must be of the form $x^{\tilde{\beta}}$ in order that second-order scaling is recovered. Thus in region II we find

$$m(\mu_1, \mu_2) \approx \mu_1^{\tilde{\beta}} |\mu_2|^{(\tilde{\beta}_t - \tilde{\beta})/\varphi_t} = \mu_1^{1/2} |\mu_2|^{-1/2} \ \ \text{[region II]}. \tag{5.12}$$

Near the first-order curve in region I, for small positive μ_2 a similar result can be obtained except that $m(\mu_1, \mu_2)$ must converge to a constant as $\mu_1 \to 0^+$. For (5.9) to be consistent with first-order behavior, $f_+(x)$ must also converge to a constant as $x \to 0^+$. Hence along the first-order curve, which is defined by $\mu_1 = 0$ and $\mu_2 > 0$, we have

$$m(0, \mu_2) \approx \mu_2^{\tilde{\beta}_t/\varphi_t} = \mu_2^{1/2} \ \ \text{[region I]}. \tag{5.13}$$

We now show that these results in tricritical scaling theory are consistent with Theorems 4.3–4.6.

We first consider the sequence of type 3, which converges to the tricritical point along a ray that is above the tangent line to the phase-transition curve at the tricritical point. This sequence is defined as in (5.4) with β_0 replaced by β_c, $b \in \{1, 0, -1\}$, and $K'(\beta_c)b - k < 0$. For this sequence (5.6) holds with β_0 replaced by β_c. Thus $\mu_1 \approx (k - K'(\beta_c)b)/n^\alpha$, and μ_2 is of order $1/n^\alpha$. Since this sequence lies in region III, the asymptotic formula (5.10) predicts $m \approx \mu_1^{\tilde{\beta}_t} \approx (k - K'(\beta_c)b)^{1/4}/n^{\alpha/4}$. This asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 4.3 and correctly predicts the 1/4-power dependence on $(k - K'(\beta_c)b)$ given in (4.14).

The sequence of type 4 is defined in (4.15) in terms of real parameters ℓ and $\tilde{\ell}$. For $\ell > \ell_c = K''(\beta_c) - 5/(4\beta_c)$ and appropriate choices of $\tilde{\ell}$, the sequences of type 4a, 4b, and 4c converge to the tricritical point in the crossover region between regions I and III in a neighborhood of the first-order curve (see items i–iii after (4.18)). For these sequences $\mu_2 \approx 1/n^\alpha$ and $\mu_1 \approx (\ell - \ell_c)/n^{2\alpha}$. Hence the scaling expression for the magnetization in (5.9) becomes

$$m \approx n^{-\alpha\tilde{\beta}_t/\varphi_t} f_+(\ell - \ell_c) = f_+(\ell - \ell_c)/n^{\alpha/2}.$$

We note that n does not appear in the argument of f_+ since $1/\varphi_t = 2$ and the powers of n cancel. This asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 4.4.

The sequence of type 4d is defined in (4.15) with $\ell = \ell_c$ and $\tilde{\ell} > K_1'''(\beta_c)$. We conjecture that this sequence converges to the tricritical point along a curve that coincides with the first-order curve to order 2 in powers of $\beta - \beta_c$ and lies in the phase-coexistence region for all sufficiently large n (see item iv after (4.18)). Thus when $\ell = \ell_c$, $\mu_1 \approx 0$ and (5.13) holds. Since $\mu_2 \approx 1/n^\alpha$, we have $m \approx \mu_2^{1/2} \approx 1/n^{\alpha/2}$. This result is consistent with part (c) of Theorem 4.4.

The sequences of type 5 and type 6 approach the tricritical point along a curve that coincides with the second-order curve to order $p - 1$ in powers of $\beta - \beta_c$. These sequences are defined in terms of a parameter ℓ as in (5.7) with β_0 replaced by β_c and $b = -1$; the sequence of type 5 corresponds to the choice $p = 2$ while the sequence of type 6 corresponds to $p \geq 3$. Since $1/\varphi_t = 2$, the dotted line separating regions II and III deviates quadratically from the second-order curve. Thus the sequence of type 5, defined for $\ell > K''(\beta_c)$, lies in the crossover range between region II and region III. The sequence of type 6 lies within region II since it approaches the second-order curve faster than quadratically. For a sequence of type 5 we have $\mu_1 \approx (\ell - K''(\beta_c))/n^{2\alpha}$ and $\mu_2 \approx 1/n^\alpha$. From the general expression (5.9) we obtain

$$m \approx n^{-\alpha\tilde{\beta}_t/\varphi_t} f_-(\ell - K''(\beta_c)) = n^{-\alpha/2} f_-(\ell - K''(\beta_c)).$$

Since $f_-(x) \approx x^{\tilde{\beta}}$, for small x we find that $m \approx (\ell - K''(\beta_c))^{1/2}/n^{\alpha/2}$. This asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 4.5. It captures the correct dependence of the prefactor $\bar{x}$ on $\ell - K''(\beta_c)$ for small $\ell - K''(\beta_c)$ that follows from (4.23).

The sequence of type 6 is defined as in (5.7) with β_0 replaced by β_c, $p \geq 3$, $b = -1$, and $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$. Because this sequence converges to the tricritical point in region II, the scaling expression (5.12) is valid. In this case $\mu_1 \approx |\ell - K^{(p)}(\beta_c)|/n^{p\alpha}$ and $\mu_2 \approx 1/n^\alpha$. Substituting these values into (5.12) yields

$$m \approx (|\ell - K^{(p)}(\beta_c)|/p!)^{1/2}/n^{(p-1)\alpha/2}.$$

Once again this asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 4.6. We note that scaling theory predicts the correct square-root dependence of the prefactor $\bar{x}$ on $|\ell - K^{(p)}(\beta_c)|$ given in (4.25).

This completes the discussion of the relationship between the results obtained in Sect. 4 with scaling theory for critical and tricritical points [23]. We have shown that scaling theory, together with the known mean-field exponents, predicts many of the exact results for $m(\beta_n, K_n)$, capturing both the correct power laws and, in some cases, the dependence on the parameters defining the sequences.

In the Appendix we discuss the structure of the set of global minimum points of polynomials of degree 6 that arise in this paper.

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Appendix: Properties of Polynomials of Degree 6

In this section we present without proof two elementary results on the set of global minimum points of polynomials of degree 6. Theorem A.1 is applied in Theorem 4.4 and Theorem A.2 in Theorem 4.5.

Theorem A.1 *For variable $a_2 \in \mathbb{R}$ and fixed $a_4 > 0$ and $a_6 > 0$, define $g(x) = a_2x^2 - a_4x^4 + a_6x^6$ and $a_c = a_4^2/4a_6$. If $0 \le a_2 \le a_4^2/3a_6$, then also define the positive number*

$$\bar{x}(a_2) = \frac{1}{\sqrt{3a_6}}\left(a_4 + (a_4^2 - 3a_2a_6)^{1/2}\right)^{1/2}. \tag{A.1}$$

The structure of the set of global minimum points of g is as follows.

(a) *If $a_2 > a_c$, then g has a unique global minimum point at* 0.
(b) *If $a_2 = a_c$, then the global minimum points of g are* 0 *and* $\pm\bar{x}(a_c) = \pm(2a_2/a_4)^{1/2}$.
(c) *If $a_2 < a_c$, then the global minimum points of g are $\pm\bar{x}(a_2)$.*
(d) *$\bar{x}(a_2)$ is a positive, decreasing, continuous function for $a_2 < a_c$, and as $a_2 \to (a_c)^-$, $\bar{x}(a_2) \to \bar{x}(a_c)$, the unique positive, global minimum point in part* (b).

The polynomials considered in the next theorem differ from those just considered in the sign of the quartic term.

Theorem A.2 *For variable $a_2 \in \mathbb{R}$ and fixed $a_4 > 0$ and $a_6 > 0$, define $h(x) = a_2x^2 + a_4x^4 + a_6x^6$. The following conclusions hold.*

(a) *If $a_2 \ge 0$, then h has a unique global minimum point at* 0.
(b) *If $a_2 < 0$, then the global minimum points of h are $\pm\bar{x}(a_2)$, where*

$$\bar{x}(a_2) = \frac{1}{\sqrt{3a_6}}\left(-a_4 + (a_4^2 - 3a_2a_6)^{1/2}\right)^{1/2}. \tag{A.2}$$

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