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Asymptotics and Limit Theorems
for the Linearized Boltzmann Equation¹

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We consider the initial value problem for the linearized Boltzmann equation

$$(1) \quad \frac{\partial p}{\partial t} + \xi \cdot \text{grad } p = \frac{1}{\epsilon} Qp, \quad \lim_{t \downarrow 0} p = f,$$

where the initial data $f = f(x, \xi)$ and the solution $p = p_\epsilon(t, x, \xi)$, $t > 0$, $x \in \mathbb{R}^3$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Q is the linearized collision operator corresponding to a spherically symmetric intermolecular potential, and $\epsilon > 0$ is a parameter which represents the mean free path. Corresponding to the conservation of number, momentum, and energy in an individual collision, Q has a five-dimensional nullspace spanned by 1, ξ_i ($i = 1, 2, 3$), and $|\xi|^2$.

We are interested in the asymptotic behavior of the solution of (1) as $\epsilon \downarrow 0$. This is formally treated at the physical level of rigor by the Chapman-Enskog-Hilbert procedure [9; pp. 254-262]. Given a nice f , we define

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$$f_0(x) \equiv \langle f(x, \cdot), 1 \rangle; f_i(x) \equiv \langle f(x, \cdot), \xi_i \rangle, i = 1, 2, 3;$$

$$(2) \quad f_4(x) \equiv \langle f(x, \cdot), \frac{|\xi|^2 - 3}{\sqrt{6}} \rangle;$$

$$n_0(t, x) \equiv \langle p_\epsilon(t, x, \cdot), 1 \rangle; n_i(t, x) \equiv \langle p_\epsilon(t, x, \cdot), \xi_i \rangle, i = 1, 2, 3;$$

$$n_4(t, x) \equiv \langle p_\epsilon(t, x, \cdot), \frac{|\xi|^2 - 3}{\sqrt{6}} \rangle,$$

where p_ϵ is the solution of (1) with initial data f and where $\langle \cdot, \cdot \rangle$ denotes the inner product in the Hilbert space

$\mathcal{H}_0 = L^2(\mathbb{R}^3; (2\pi)^{-3/2} \exp(-\frac{1}{2}|\xi|^2) d\xi)$. Provided f satisfies certain algebraic compatibility conditions (the Hilbert relations), one can show that up to an error $O(\epsilon)$ the n_i satisfy the following system of partial differential equations (linear Euler equations):

$$\frac{\partial n_0}{\partial t} = - \operatorname{div} \vec{n},$$

$$(4) \quad \frac{\partial \vec{n}}{\partial t} = - \operatorname{grad} n_0 - \sqrt{\frac{2}{3}} \operatorname{grad} n_4,$$

$$\frac{\partial n_4}{\partial t} = - \sqrt{\frac{2}{3}} \operatorname{div} \vec{n},$$

$$n_i(0^+, x) = f_i(x),$$

where $\vec{n} = (n_1, n_2, n_3)$. Further, if one writes $p_\epsilon(t/\epsilon, x, \cdot)$ for $p_\epsilon(t, x, \cdot)$ in (2), then up to an error $O(\epsilon)$ the resulting $n_i(\frac{t}{\epsilon}, x)$ agree with the solutions (evaluated at time t/ϵ) of the following system of partial differential equations (linear Navier-Stokes

