Probabilistic Methods in Differential Equations

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Asymptotics and Limit Theorems
for the Linearized Boltzmann Equation

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We consider the initial value problem for the linearized Boltzmann equation

\[
\frac{\partial p}{\partial t} + \xi \cdot \text{grad} \ p = \frac{1}{\varepsilon} Qp, \quad \lim_{t \to 0} p = f,
\]

where the initial data \( f = f(x, \xi) \) and the solution \( p = p(t, x, \xi) \), \( t > 0 \), \( x \in \mathbb{R}^3 \), \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \). \( Q \) is the linearized collision operator corresponding to a spherically symmetric intermolecular potential, and \( \varepsilon > 0 \) is a parameter which represents the mean free path. Corresponding to the conservation of number, momentum, and energy in an individual collision, \( Q \) has a five-dimensional nullspace spanned by \( \mathbf{1} \), \( \xi_1 \) \( (i = 1, 2, 3) \), and \( |\xi|^2 \).

We are interested in the asymptotic behavior of the solution of (1) as \( \varepsilon \to 0 \). This is formally treated at the physical level of rigor by the Chapman-Enskog-Hilbert procedure [9; pp. 254-262]. Given a nice \( f \), we define

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\[ f_0(x) = (f(x, \cdot), 1); f_4(x) = (f(x, \cdot), \xi_1), i = 1, 2, 3; \]

(2) \[ f_4(x) = (f(x, \cdot), \frac{|\xi|^2 - \beta}{\sqrt{\lambda}}); \]

\[ n_0(t, x) = (p_\epsilon(t, x, \cdot), 1); n_1(t, x) = (p_\epsilon(t, x, \cdot), \xi_1), i = 1, 2, 3; \]

\[ n_4(t, x) = (p_\epsilon(t, x, \cdot), \frac{|\xi|^2 - \beta}{\sqrt{\lambda}}); \]

where \( p_\epsilon \) is the solution of (1) with initial data \( f \) and where \((\ , \ )\) denotes the inner product in the Hilbert space \( \mathcal{H}_0 = L^2(\mathbb{R}^3; (2\pi)^{-3/2} \exp(-\frac{1}{2}|\xi|^2) d\xi) \). Provided \( f \) satisfies certain algebraic compatibility conditions (the Hilbert relations), one can show that up to an error \( O(\epsilon) \) the \( n_\perp \) satisfy the following system of partial differential equations (linear Euler equations):

\[ \frac{\partial n_0}{\partial t} = - \text{div} \vec{n}, \]

(4) \[ \frac{\partial \vec{n}}{\partial t} = - \text{grad} n_0 - \frac{\beta}{\sqrt{\lambda}} \text{grad} n_4, \]

\[ \frac{\partial n_4}{\partial t} = - \frac{\beta}{\sqrt{\lambda}} \text{div} \vec{n}, \]

\[ n_\perp(0^+, x) = f_4(x), \]

where \( \vec{n} = (n_1, n_2, n_3) \). Further, if one writes \( p_\epsilon(t/\epsilon, x, \cdot) \) for \( p_\epsilon(t, x, \cdot) \) in (2), then up to an error \( O(\epsilon) \) the resulting \( n_\perp(t/\epsilon, x) \) agree with the solutions (evaluated at time \( t/\epsilon \)) of the following system of partial differential equations (linear Navier-Stokes
equations):
\[ \frac{\partial n_0}{\partial t} = - \text{div} \, \dot{n}, \]
\[ (4) \quad \frac{\partial \dot{n}_i}{\partial t} = - \nabla n_0 - \sqrt{\frac{e}{\eta}} \nabla n_i + e\eta(\Delta \dot{n} + \frac{1}{2} \nabla \text{div} \, \dot{n}), \]
\[ \frac{\partial n_i}{\partial t} = - \sqrt{\frac{e}{\eta}} \text{div} \, \dot{n} + e\lambda \Delta n_i, \]
\[ n_1(0^+, x) = f_1(x). \]

In (4), \( \varepsilon > 0, n_1 = n_1^\varepsilon(t, x), \) \( i = 0, \ldots, 4, \) and \( \eta > 0 \) and \( \lambda > 0 \) are physical constants. Notice that the Euler equations are obtained from (4) by setting \( \varepsilon = 0. \)

In a series of basic papers [7, 8, 10], Grad has studied the existence of the solution of (1) and has sought to make precise the formal results obtained by the Chapman-Enskog-Hilbert procedure. He begins with the decomposition, valid for a class of so-called cut-off hard potentials:

\[ (5) \quad Q = \nu - K. \]

Here \( \nu \) is the operator of multiplication by the collision frequency \( \nu(\xi), \) a strictly positive function of \( |\xi|, \) and \( K \) is a compact operator on \( \mathcal{H}. \) Using (5), Grad writes (1) as an integral equation and then derives \textit{a priori} estimates in the Hilbert space \( \mathcal{H} = L^2(R^6; (2\pi)^{-3/2} \exp(-\frac{1}{2}|\xi|^2) d\xi dx). \) Concerning the asymptotic behavior of \( p_\xi, \) let \( A \) and \( B \) denote, respectively, the first-order and second-order spatial partial derivatives in (4). Defining
\[ p_\varepsilon = T_\varepsilon(t)f, \exp(t(A + \varepsilon B))f = n_0^\varepsilon + \sum_{i=1}^{3} n_i^\varepsilon s_i + n_4^\varepsilon \frac{|s|^2 - 3}{6}, \]

\((n_i^\varepsilon, i = 0, \ldots, 4,\) solve the Navier-Stokes system \((4)\), Grad proves the following asymptotic results, which are valid for any \(f \in \mathcal{H}\) satisfying a mild growth and smoothness condition:

\[(6) \quad T_\varepsilon(t)f = \exp(tA)f + o(\varepsilon), \text{ as } \varepsilon \to 0,\]

\[(7) \quad T_\varepsilon(t/\varepsilon)f = \exp(t(\varepsilon A + \varepsilon B))f + o(\varepsilon), \text{ as } \varepsilon \to 0.\]

In physical terms, \((6)\) describes the non-viscous fluid approximation at a fixed time \(t > 0\); \((7)\) describes the viscous effects when \(t \to \infty\). Our aim is to show that \((7)\) is only one of a large variety of possible refinements of \((6)\). This is accomplished by the following two results.

**Boltzmann Limit Theorem.** Let \(f \in \mathcal{H}\) be sufficiently regular. Then

\[(8) \quad \exp(-tA/\varepsilon)T_\varepsilon(t/\varepsilon)f = \mathcal{N}(t)f + o(\varepsilon), \text{ as } \varepsilon \to 0,\]

where \(\mathcal{N}(t)\) is a contraction semigroup on \(\mathcal{H}\) whose generator is given by the differential equations

\[ \frac{\partial n_0}{\partial t} = \left(\frac{9 \lambda}{2} + \frac{2 \eta}{5}\right) \Delta n_0 + \sqrt{\frac{\eta}{\varepsilon}} \left(-\frac{6 \lambda}{2} + \frac{2 \eta}{5}\right) \Delta n_4, \]

\[ \frac{\partial n_i}{\partial t} = \eta \Delta n_i + \left(\frac{\lambda}{2} - \frac{\eta}{5}\right) \text{grad div } n, \quad (i = 1, 2, 3), \]

\[ \frac{\partial n_4}{\partial t} = \sqrt{\frac{\eta}{\varepsilon}} \left(-\frac{6 \lambda}{2} + \frac{2 \eta}{5}\right) \Delta n_0 + \left(\frac{11 \lambda}{25} + \frac{4 \eta}{15}\right) \Delta n_4, \]
\[ n_1(0^+, x) = f_1(x); \]

1.e., \( N(t)f = n_0 + \sum_{1}^{3} n_1 t_1^1 + n_4 \frac{|g|^2 - 3}{\sqrt{6}}. \) The semigroup \( N(t), t \geq 0 \) commutes with the Euler semigroup \( \{\exp(tA), t \geq 0\}. \)

In order to make the connection with (7), we also need the following.

**NAVIER-STOKES LIMIT THEOREM.** Let \( f \in \mathcal{H} \) be sufficiently regular. Then

\[ \exp(-tA/\varepsilon)\exp(t/\varepsilon(A + \varepsilon B))f = N(t)f + o(\varepsilon), \text{ as } \varepsilon \to 0. \]  

The proof of (10) proceeds by means of Fourier transformation from the following purely algebraic result.

**MATRIX LIMIT THEOREM.**\(^3\) Let \( A \) be a skew-symmetric \( m \times m \) matrix and \( B \) a real, symmetric, negative semidefinite \( m \times m \) matrix. Let "exp" denote matrix exponentiation. Then

\[ \exp(-tA/\varepsilon)\exp(t/\varepsilon(A + \varepsilon B)) = \exp(t\pi_A B) + o(\varepsilon), \text{ as } \varepsilon \to 0, \]

when \( \pi_A B \) is the orthogonal projection, in the Euclidean space of \( m \times m \) matrices, of \( B \) onto the linear subspace of matrices which commute with \( A \).

In particular, we show that the generator of \( N(t) \) is \( \pi_A B \), the projection of the Navier-Stokes operator \( B \) upon the set of operators which commute with the Euler operator \( A \) (\( B \) and \( A \) do not commute).

\(^3\) T. Kato (preliminary report) has generalized this result to the case of operators on a Banach space.
Using (10) and the commutativity of $\pi_A B$ and $A$, we have

$$\text{(11)} \quad T_\varepsilon(t/\varepsilon)f = \exp\left(\frac{t}{\varepsilon}(A + \varepsilon\pi_A B)\right)f + O(\varepsilon), \text{ as } \varepsilon \to 0.$$  

This is the simplest of an infinite number of alternatives to (7). Indeed, we can show the existence of infinitely many solution operators $\exp(t\tilde{B})$ of parabolic systems of partial differential equations (one needs $\pi_A\tilde{B} = \pi_A B$) such that (11) remains true when $\pi_A B$ is replaced by $\tilde{B}$. This illustrates the asymptotic non-uniqueness of the Navier-Stokes equations. Further, since one of these $\tilde{B}$ is the Navier-Stokes operator $B$, we obtain an independent derivation of Grad's result (7) with, it turns out, weaker assumptions on $f$.

The proof of (8) depends on a careful spectral analysis of the operator $Q - i(\gamma \cdot \xi)$, where $\gamma \in \mathbb{R}^3$ is a parameter. We prove the existence and differentiability, for $|\gamma|$ sufficiently small, of the hydrodynamical eigenvalues and eigenfunctions $(\alpha(j)(\gamma), e(j)(\gamma); j = 1, \ldots, 5)$, which satisfy $\alpha(j)(0) = 0$, $e(j)(0) \in \text{span} \{1, \xi_1, \xi_2, \xi_3, |\xi|^2\}$. We then prove a contour integral representation

$$\text{(12)} \quad \exp[t(Q - i(\gamma \cdot \xi))]f =$$

$$\quad = \sum_{j=1}^{5} e^{ta(j)(\gamma)} \left< f, e(j)(-\gamma) \right> e(j)(\gamma) +$$

$$\quad (2\pi)^{-1} \int_{\gamma} e^{ta} R(\alpha, \gamma) \frac{(Q - i(\gamma \cdot \xi))^2}{\alpha^2} \, da,$$

where $C$ is a vertical contour in the half plane $\text{Re}\alpha < 0$ and $R(\alpha, \gamma) = (Q - i(\gamma \cdot \xi) - \alpha)^{-1}$. The first term of (12) corresponds
to the Hilbert solution and gives the connection with the Euler and Navier-Stokes equations. The second term is negligible when \( Q/\epsilon \) is written for \( Q \) and \( \epsilon \to 0 \). In case \( \nu(\xi) \sim |\xi|^\alpha \) as \( |\xi| \to \infty \) \((\alpha > 0)\), the contour integral may be replaced by

\[
\int_C e^{\xi a} R(a, \nu) f(a) \, da,
\]

where the contour \( C \) is such that \( \text{Re} a \to -\infty \) \( \text{Im} a \to +\infty \). The existence of the eigenvalues \( a(j)(\nu) \) follows by applying the implicit function theorem to the exact hydrodynamical dispersion law. Previously, exact dispersion laws were obtained [15] only for hard sphere potentials, i.e., \( \nu(\xi) \sim |\xi| \) as \( |\xi| \to \infty \). In this case, the \( a(j)(\nu) \) are analytic functions and can also be obtained from Rellich's perturbation theorem [11,13].

In case \( \nu(\xi) \sim |\xi|^\alpha \) as \( |\xi| \to \infty \), \( 0 \leq \alpha < 1 \), the \( a(j)(\nu) \) will not be analytic around \( \nu = 0 \). Nevertheless, we obtain an asymptotic development

\[
a(j)(\nu) - \sum_{n=1}^{\infty} a_n(j) |\nu|^n, \quad (1 \leq j \leq 5)
\]

where \( a_1(j) \) is imaginary and \( a_2(j) < 0 \). These constants can be computed by formal perturbation theory. They correspond to the adiabatic sound speed and absorption coefficients for low-frequency sound waves [5].

The results (8) and (10) extend known results on finite-state velocity models in one dimension [1,2] to the full three-dimensional linearized Boltzmann equation. These theorems are valid in any number of dimensions. Their proofs and related matters will appear in full detail in [3,4].

We end this paper with several open questions.

1) If one has an external force field \( F(x, \xi) \), then the linearized Boltzmann equation (1) becomes (assuming unit mass)
\[ \frac{\partial p}{\partial t} + \xi \cdot \text{grad}_x p + F \cdot \text{grad}_\xi p = \frac{1}{\varepsilon} Q p, \lim_{t \to 0} p = f. \]

The extensions of our limit theorems to this case have not been worked out. Nelson [14; p. 77] has results for an equation with the same form as (13) but where \( Q \) has a one-dimensional null-space and is not a Boltzmann collision operator.

2) Grad's decomposition (5) of the operator \( Q \) stems from a restriction to a class of intermolecular potentials that are physically unnatural. Pao [16] has shown for a large class of more realistic potentials that \( Q \) is self-adjoint and \((Q + aI)^{-1}\) is compact and negative definite for \( a > 0 \). The limit theorems should hold not only in this case but also for any equation with the same form as (1) provided \( Q \) is self-adjoint, negative-semi-definite, with an isolated, finite dimensional eigenvalue at zero. Our methods, based on the existence of the eigenvalues \( \lambda^n(\nu) \), do not seem to go over.

3) We mentioned the asymptotic nonuniqueness of the Navier-Stokes equations, but question whether these equations have any additional properties which single them out as an asymptotic limit.

4) The statements of our results at least make sense for the nonlinear Boltzmann equation. We feel that a fruitful area of research is the study of nonlinear models. Initial work in this direction has been done by Kurtz [12], who proved the analogue of (8) for the Carleman model. This model, however, has the unsatisfactory feature that its Euler equations are trivial (\( \text{A} = 0 \)). A physically more interesting model has been suggested by Godunov and Sultangazin [6; p. 16].
References


