where $\delta^2 g$ is the tensor in (2.20). A sufficient condition for nonnegativity is that the matrices $[\delta^2 j_f]^*$, $[\delta^2 j_\Sigma - (r'T'(TT')^{-1})\delta^2 g]^*$ and $[\delta^2 h]^*$ be positive semidefinite for all $(t,z) \in D$.

III. Concluding Remarks

Some interesting structural features of the necessary conditions, which will also be characteristics of related problems, are worth noting. First, there is a relationship between the boundary conditions on the state (2.4), costate (2.7), and the condition characterizing the boundary controls (2.9). At a given boundary point, there will typically be $n_g(< n_x)$ constrained and $n_x - n_g$ unconstrained combinations of state variables. In a sense (defined in the text), the former n_g combinations of costate variables will be unconstrained, while the remaining $n_x - n_g$ combinations of costate variables will be constrained by the boundary condition (2.7). The boundary controls, furthermore, depend essentially on the boundary values of the n_g unconstrained combinations of costate variables. This relationship explains the apparent indeterminacy which results from a naive formulation of the problem with $n_g = 0$ (no boundary conditions) or $n_g = n_x$ (completely specified boundary values). A second point concerns the second-order conditions (2.28). The second-order perturbations in the state and its boundary value disappear by virtue of the first-order conditions. The term (2.28) involving $\delta^2 g^*$ reflects the role played by the boundary conditions in the second variation.

A few directions in which the present results may be extended are,

1) Consideration of the case where $(\partial g/\partial \mathbf{x})$ is less than full rank.

2) Inclusion of a penalty on the boundary value of the state.

3) In addition, a "Decomposition Theorem" has been derived [5] for the time-invariant case, showing that under certain conditions the optimal control may be decomposed into a steady-state control (which is the solution of a static distributed optimization problem), and a linear regulator (which is the solution of a dynamic "linear-quadratic" distributed optimal control problem).

4) The theory has been applied to quasilinear analytic systems with quasiquadratic criteria; most of the smoothness and well-posedness assumptions (Section II) may be verified in this case [5].

Much future work remains to be done in order to achieve a distributed maximum principle of comparable elegance and generality to that available for lumped systems. The continuity assumptions could be relaxed, for instance, by defining weak solutions of the state and costate equations. The entire problem should ultimately be cast in a distributional framework, and solved in a Banach-space setting. A second major improvement would be the use of global (rather than local) control perturbations to obtain global (rather than local) necessary conditions. Browder and Lions [18] have already made considerable contributions in this area. Finally, there are the generalizations to movable boundaries, control constraints, free terminal time, and fixed terminal state problems.

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An Application of Stochastic Optimal Control Theory to the Optimal Rescheduling of Airplanes

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Abstract—**A model for the air traffic flow between two airports subject to random constraints on the takeoff and landing capacities is set up. For a simple case a dynamic programming algorithm is used to compute the optimal solutions explicitly.**

I. Introduction

Every air traveler is familiar with delays due to random effects such as weather or breakdown of equipment. It is an interesting problem to try to determine optimal rescheduling procedures so as to minimize total passenger inconvenience.

In this paper a simplified model for the traffic flow from one airport to another, subject to random constraints, is set up and studied. We formulate the problem of rescheduling the number of planes taking off and landing so that the air traffic system operates within the random constraints, so that all the scheduled planes make the trip by the terminal time, and so that passenger inconvenience, in terms of total waiting time on the ground and in the air, is minimized.

II. Statement of Problem

To describe the problem that will be considered, let us assume there are given two airports, from one of which aircraft takeoff to land at the other. Time is discretized. There are $n + 1$ intervals of time to complete the transfer of planes between the two airports. Also, j intervals of time are required to fly from the first airport to the second. To start, a desired departure schedule at airport one is given. In general, not all scheduled departures and arrivals can be handled at each time because of the random constraints. What the controller must do is to schedule actual departures from airport one

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to meet the condition that the number of departures at each time is less than or equal to the capacity of airport one at that time to takeoff airplanes.

Similarly, the controller must schedule actual landings at airport two to meet the condition that the number of landings at each time is less than or equal to the capacity of airport two at that time to land airplanes. Both these capacities are modeled as random quantities to take into account such factors as weather and delay due to equipment breakdown. In addition departures and landings must be such that after the $n + 1$ time intervals, all of the scheduled planes will have made the trip.

Because of factors like gas expenditure, safety, and passenger discomfort, waiting time in the air is considered to be more undesirable than waiting time on the ground. Motivated by this, we set up a performance index equal to the sum of the total waiting time on the ground during the $n + 1$ intervals plus a factor $\alpha > 1$ times the total waiting time in the air during the $n + 1$ intervals.

The control variables of the problem are the number of planes which takeoff and the number which land. These are adjusted by the air traffic controller as functions of the information available to him. Thus, they are functions of the number of planes in the air, those waiting to takeoff, those waiting to land, the current capacities of each airport, and the time.

The optimization problem is to carry out the transfer of aircraft from the first to the second airport in such a way that all of the above conditions are met and in addition the expected value of the performance index is minimized.

We notice that the introduction of the factor $\alpha > 1$ makes the problem nontrivial. Otherwise, if $\alpha = 1$, the optimal strategy would be to takeoff scheduled planes as fast as possible regardless of the number waiting to land at the second airport.

We formally state the problem by introducing the following notation. In these definitions a waiting plane will always be understood to be a plane which has had to wait for at least one time unit.

- δ_i number of planes scheduled to takeoff from airport one at time i.
- t_i number waiting to takeoff from airport one at time i.
- u_i number actually taking off from airport one at time i.
- q_i number waiting to land at airport two at time i.
- l_i number actually landing at airport two at time i.
- $s_i(\omega)$ maximum number which can takeoff from airport one at time i.
- $r_i(\omega)$ maximum number which can land at airport two at time i.

The quantities $s_i(\omega)$ and $r_i(\omega)$ are taken to be random variables of a discrete two-dimensional Markov process with a finite state space $\mathcal{R}$. As a simplifying assumption, we will allow the other variables listed above to have nonnegative real number values.

The number of takeoffs u_i and landings l_i are the variables which are determined at each time by the air traffic controller. They are to be chosen as functions

$$u_i(q_i,t_i,u_{i-1},\cdots,u_{i-j},s_i(\omega),r_i(\omega))$$
$$l_i(q_i,t_i,u_{i-1},\cdots,u_{i-j},s_i(\omega),r_i(\omega))$$

of the variables $(q_i,t_i,u_{i-1},\cdots,u_{i-j},s_i(\omega),r_i(\omega))$ subject to the constraint that they satisfy the random inequalities

$$0 \leqslant u_i \leqslant s_i(\omega), \qquad 0 \leqslant l_i \leqslant r_i(\omega). \tag{1}$$

For convenience in writing certain formulae, variables with negative subscripts will be used. In all cases these will be understood to be identically zero.

The equations of the system are

$$t_i = t_{i-1} + \delta_{i-1} - u_{i-1}, \qquad i = 1,2,\cdots,n \tag{2}$$

$$q_i = q_{i-1} + u_{i-j-1} - l_{i-1}, \qquad i = 1,2,\cdots,n. \tag{3}$$

The subscript $(i - j - 1)$ on u in (3) arises because planes arriving at time $i - 1$ at airport two left airport one at time $i - j - 1$.

As we start fresh at time 0 and want by time n to have transferred all the planes between the two airports, the following conditions (4) must be satisfied with probability one:

$$q_0 = t_0 = q_n = t_n = u_n = u_{n-1} = \cdots = u_{n-j} = 0. \tag{4}$$

III. Comments on the Optimization Problem

Notice that if at each time there were a positive probability that $r_i(\omega)$ or $s_i(\omega)$ equaled 0, it would not be possible to satisfy (4) with probability one. Motivated by this, let us assume that both $r_i(\omega)$ and $s_i(\omega)$ take on only positive values. Let ρ and $\bar{\rho}$, σ and $\bar{\sigma}$ denote the maximum and minimum values that $r_i(\omega)$ and $s_i(\omega)$, respectively, take on.

It may not be possible to satisfy conditions (4) if the total number of scheduled planes is too large. There are worst conditions that $\sigma_i(\omega) = \sigma$, $r_i(\omega) = \rho$ occur for all times $i = 0,1,\cdots,n$. In order to complete the transfer of planes between airports before time n with probability one, one would have to be able to do this under the worst conditions. This is possible only if

$$\sum_{k=i}^{n-(j+1)} \delta_k \leqslant \min\{(n-i-j)\sigma,(n-i-j)\rho\}, \qquad i = 0,\cdots,n-(j+1)$$
$$\delta_i = 0, \qquad i = n-j,\cdots,n. \tag{5}$$

Let us say that the schedule is *feasible* if conditions (5) hold.

Notice that it is always optimal to land as many planes as possible. This follows because if planes are left in the air when they could be landed, the performance index is only increased. However, we shall continue to consider landings as a control variable, since by doing so, the equations can be written using a unified notation which simplifies the theoretical discussion.

IV. Implied Constraints

We shall begin our discussion of the optimization problem by showing that the terminal conditions (4) imply constraints on the values of variables at intermediate times as well as at the initial time and the terminal time. Constraints of this type were noticed in stochastic programming problems by Wets [4, p. 92].

The total number of planes which must leave airport one at and after time i is $t_i + \sum_{k=i}^{n} \delta_k$. Under worst conditions it is possible to takeoff σ planes at each time and there are $n - i - j$ times left in which planes can leave airport one and still reach airport two before time n. Thus to assure that all the planes reach airport two before time n with probability one the inequalities

$$t_i + \sum_{k=i}^{n} \delta_k \leqslant (n-i-j)\sigma, \qquad i = 0,1,\cdots,n-j, \tag{6}$$
$$t_i = 0, \qquad i = n-j+1,\cdots,n,$$

must hold. In order that all these planes can land at airport two with probability one the inequalities

$$t_i + \sum_{k=i}^{n} \delta_k \leqslant (n-i-j)\rho, \qquad i = 0,1,\cdots,n-j, \tag{7}$$

must also hold. The number of planes which must be landed at and after time i is

$$q_i + \sum_{k=1}^{j} u_{i-k} + t_i + \sum_{k=1}^{n} \delta_k.$$

To land this number under worst conditions,

$$q_i + \sum_{k=1}^{j} u_{i-k} + t_i + \sum_{k=1}^{n} \delta_k \leqslant (n-i)\rho \tag{8}$$

must hold. For each $m = 1,\cdots,j$, the number of airplanes which will arrive at airport two after $i - m$ is

$$t_i + \sum_{k=1}^{m} u_{i-k} + \sum_{k=i}^{n} \delta_k.$$

To land this number under worst conditions,

$$t_i + \sum_{k=1}^{m} u_{i-k} + \sum_{k=i}^{n} \delta_k \leqslant (n - i - m)\rho \tag{9}$$

must hold. In addition, summing (2) and (3) and using $u_i \geqslant 0, l_i \geqslant 0$, $q_0 = t_0 = 0$ gives

$$q_i + t_i + \sum_{k=1}^{j} u_{i-k} \leqslant \sum_{k=0}^{i-1} \delta_k. \tag{10}$$

Finally the condition $q_0 = t_0 = 0$ implies that $q_1 = \cdots = q_j = 0$ and $t_1 - \delta_0 + u_0 = 0$.

V. State Variable Description

The discussion of the optimization problem can be notationally and conceptually simplified by adopting a vector state space notation. Let the vector

$$X_i = (q_i, t_i, u_{i-1}, \cdots, u_{i-j})$$

be defined to be the state of the system at time i. Let the control vector of the system at time i be defined to be

$$U_i = (u_i, l_i).$$

Let the random vector

$$R_i(\omega) = (s_i(\omega), r_i(\omega))$$

be defined to be the vector of random constraints on the system. Equations (2) and (3) can be rewritten as

$$X_i = AX_{i-1} + BU_{i-1} + C\delta_{i-1} \tag{11}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 & 1 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

The performance index can be written

$$E\left\{\sum_{k=0}^{n} D \cdot X_k\right\}$$

where $D = (\alpha, 1, 0, \cdots, 0)$.

The implied constraints (6)–(10) define bounded convex sets C_i of the state variables. At each time i, the state X_i must lie in the set C_i. In order that $X_{i+1} \in C_{i+1}$, the controls $U_i(X_i, R_i(\omega))$ must be chosen so that

$$AX_i + BU_i(X_i, R_i(\omega)) + C\delta_i \in C_{i+1}. \tag{12}$$

We call *admissible* any sequence of controls $U_i(X_i, R_i(\omega))$ which are defined on the Cartesian product of the set C_i determined by the implied constraints and the range $\mathcal{R}$ of the Markov process $(s_i(\omega)), r_i(\omega))$ and which satisfy (12) and

$$0 \leqslant U_i(X_i, R_i(\omega)) \leqslant R_i(\omega).$$

If X_i is the solution of

$$X_i = AX_{i-1} + BU_{i-1}(X_{i-1}, R_{i-1}(\omega)) + C\delta_{i-1},$$

then the optimization problem is to choose admissible $U_i(X_i, R_i(\omega))$ such that the performance $E\left\{\sum_{k=0}^{n} D \cdot X_k\right\}$ is minimized.

VI. Dynamic Programming

In this section we state the dynamic programming algorithm which will form the basis for the computation. Although the dynamic programming theorem is a fairly standard one, several features should be pointed out. It is important for computation that the value function need only be defined on $C_i \times \mathcal{R}$ rather than on all of $E^{j+2} \times \mathcal{R}$. There are counter examples to Theorem 1 when the sets C_i of implied constraints are not convex.

For any sequence of admissible controls $U_i(X,R)$ and for $i = 1, \cdots, n$, define

$$V_U^i(X,R) = E\left\{\sum_{j=i}^{n} D \cdot X_j \middle| X_i = X, R_{i-1}(\omega) = R\right\} \tag{13}$$

where in (13) the sequence X_j is generated by (11) using the controls $U_j(X,R)$. For $i = 0$, we shall adopt the convention that

$$V_U(X,R) = E\left\{\sum_{j=0}^{n} D \cdot X_j\right\}. \tag{14}$$

Theorem 1: For $i = 0, 1, \cdots, n$, there exist admissible controls $\bar{U}_i(X,R)$ and continuous vector functions $V_i(X,R)$ defined on $C_i \times \mathcal{R}$ such that

$$V_{i+1}(AX + B\bar{U}_i(X,R) + C\delta_i, R) = \min_{\substack{U: AX+BU+C\delta_i \in C_{i+1} \\ 0 \leqslant U \leqslant R}} V_{i+1}(AX+BU+C\delta_i, R), \tag{15}$$

$$V_i(X,R) = D \cdot X + E\left\{ \min_{\substack{U: AX+BU+C\delta_i \in C_{i+1} \\ 0 \leqslant U \leqslant R_i(\omega)}} \cdot V_{i+1}(AX+BU+C\delta_i, R^i(\omega)) [R_{i-1}(\omega) = R\right\}. \tag{16}$$

and

$$V_i(X,R) = V_{\bar{U}}^i(X,R) \tag{17}$$

hold. In addition, for any admissible controls $U_i(X,R)$,

$$V_i(X,R) \leqslant V_U^i(X,R). \tag{18}$$

Let $E_U\{\cdot\}$ denote the conditional expectation of the quantity in brackets when the control U is used. Notice that (17), (18), and the convention (14) concerning $V_U{}^0(X,R)$ imply that

$$E_{\bar{U}}\left\{\sum_{i=0}^{n} D \cdot X_i\right\} = V_{\bar{U}}^0(X,R) \leqslant V_U^0(X,R) = E_U\left\{\sum_{i=0}^{n} D \cdot X_i\right\}.$$

This inequality asserts the control $\bar{U}_i(X,R)$ is optimal. Thus Theorem 1 implies there is an optimal control and gives in (15) and (16) a procedure for the computation of this control.

VII. Example

Optimal controls were computed for the following example. Time is discretized into five units. The time of flight between airports is one unit. Four planes at airport one are scheduled to takeoff for airport two at time zero and none at the subsequent times. The weather at airport two is such that the capacities at each time at airport two to land planes are independent random variables which take on the value 1 with probability P and 2 with probability $1 - P$. The constants α and p satisfy $P\alpha \geqslant 1$, $P(\alpha + 1) \leqslant 2$.

The optimal control for this example is: Take off two planes at time zero. Take off one plane at time one. At time two take off one plane if no planes are waiting to land at airport two or if the weather is at its best. If there are planes waiting to land and if the weather is bad at airport two at time two, do not take off any planes. If there is a plane remaining at time three, take it off.

VIII. Conclusion

An optimal rescheduling problem for traffic between two airports was modeled. To analyze it, a dynamic programming computational algorithm was given and in a simple case used to obtain the optimal solution.

An important step in attacking the problem was to show for the n-stage process how the initial and terminal conditions implied inequality constraints on intermediate values of the state and control variables of the problem. These inequalities restrict the domain over which the control and the value functions must be defined, and they also restrict the values which the controls may take on. This reformulation allowed the optimal control law to be found by the dynamic programming algorithm.

The computation for the example was very complex and suggests that carrying out similar computations for more realistic, more complex systems will be difficult. In more general problems, in which a network of airports is considered, the minimization step becomes a high dimensional nonlinear programming problem. The development of efficient computer algorithms for carrying out the computation in more general cases appears to be an interesting area of research.

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Technical Notes and Correspondence

Computation of Regions of Transient Stability of Multimachine Power Systems

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Abstract—**A major difficulty in applying Lyapunov theory to the problem of specifying transient stability regions of n-machine power systems is computational complexity, which increases markedly with n. This note outlines a method, requiring only a nominal amount of computation, to determine such regions.**

I. Introduction

The application of Lyapunov theory to the study of transient stability of multimachine power systems, initiated by power engineers in 1966 [1], [2], has continued to the present time. A recent paper by Willems in these Transactions [3] presents some of the more significant advances and provides an extensive bibliography.

The chief attraction of the Lyapunov method is its potential for reducing the computation time associated with investigating the transient stability of an n-machine interconnection. However, since the number and complexity of the computations increases rapidly with n, the potential for savings in overall computation may not be realized. In this note, we present a technique providing significant savings in computation; however, somewhat more conservative regions are obtained.

II. Mathematical Model

The starting point for the analysis is the swing equation model of a multimachine power system. For a detailed development of the model with the usual simplifying assumptions see [3]. For an n-machine interconnection, the dynamics are expressed in terms of the state vector $(\boldsymbol{\alpha},\boldsymbol{\omega})$ by [4].

$$\dot{\alpha} = T\omega \qquad (1)$$

$$\dot{\omega} = -M^{-1} D\,\omega - M^{-1} T^t[f(\alpha) - f(\alpha^0)]$$

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where

$$T = \begin{bmatrix} & \cdot & -1 \\ & \cdot & \\ & \cdot & -1 \\ I_{n-1} & \cdot & \vdots \\ & \cdot & \\ & \cdot & -1 \end{bmatrix}$$

$M = \operatorname{diag}\{M_i, i = 1,2,\cdots,n\}$, $M_i > 0$, $D = \operatorname{diag}\{D_i, i = 1,2,\cdots,n\}$, $D_i \geq 0$, $f(\alpha) = \operatorname{col}\{f_i(\alpha)\colon i = 1,2,\cdots,n-1\}$ with

$$f_i(\alpha) = \sum_{\substack{j=1\\ j\neq i}}^{n-1} b_{ij}\sin(\alpha_i - \alpha_j) + b_{in}\sin\alpha_i, \qquad i = 1,2,\cdots,n-1 \qquad (2)$$

$b_{ij} = b_{ji} \geq 0$, and $\omega \in \mathbb{R}^n$ contains the velocity components. Here $\boldsymbol{\alpha}$ is the $n-1$ vector of intermachine angles obtained by taking the difference of the power angle of ith machine and that of the nth machine which has been *arbitrarily* chosen as the reference machine.

III. Stability

To study the stability of the equilibrium point $(\boldsymbol{\alpha}^0,\mathbf{0})$ of (1), we pick the "total energy" as a Lyapunov function V:

$$V(\alpha,\omega) = \tfrac{1}{2}\langle\omega, M\omega\rangle + W(\alpha) \qquad (3)$$

where

$$W(\alpha) \triangleq \int_{\alpha^0}^{\alpha} \langle [f(\xi) - f(\alpha^0)], d\xi\rangle. \qquad (4)$$

Since $H(\alpha) \triangleq (\partial f/\partial\alpha)(\alpha)$ is symmetric, the integral is path independent and well-defined.

V vanishes at the equilibrium point $(\boldsymbol{\alpha}^0,\mathbf{0})$ and along trajectories