where $\delta g$ is the tensor in (2.20). A sufficient condition for non-negativity is that the matrices $[\delta g^*]$, $[\delta g - (r'T(TT')^{-1})\delta g]^*$ and $[\delta g]^*$ be positive semidefinite for all $(t, z) \in D$.

III. CONCLUDING REMARKS

Some interesting structural features of the necessary conditions, which will also be characteristics of related problems, are worth noting. First, there is a relationship between the boundary conditions or the state (2.4), costate (2.7), and the condition characterizing the boundary controls (2.9). At a given boundary point, there will typically be $n_y < n_x$ constrained and $n_x - n_y$ unconstrained combinations of state variables. In a sense (defined in the text), the former $n_y$ combinations of costate variables will be unconstrained, while the remaining $n_x - n_y$ combinations of costate variables will be constrained by the boundary condition (2.7). The boundary controls, furthermore, depend essentially on the boundary values of the $n_y$ unconstrained combinations of costate variables. This relationship explains the apparent indeterminacy which results from a naive formulation of the problem with $n_y = 0$ (no boundary conditions) or $n_x = n_y$ (completely specified boundary values). A second point concerns the second-order conditions (2.28). The second-order perturbations in the state and its boundary value disappear by virtue of the first-order conditions. The term (2.28) involving $\delta g^*$ reflects the role played by the boundary conditions in the second variation.

A few directions in which the present results may be extended are:
1) Consideration of the case where $(\delta g/\delta x)$ is less than full rank.
2) Inclusion of a penalty on the boundary value of the state.
3) In addition, a “Decomposition Theorem” has been derived [5] for the time-invariant case, showing that under certain conditions the optimal control may be decomposed into a steady-state control (which is the solution of a static distributed optimization problem), and a linear regulator (which is the solution of a dynamic “linear-quadratic” distributed optimal control problem).
4) The theory has been applied to quasilinear analytic systems with quasiquadratic criteria; most of the smoothness and well-posedness assumptions (Section II) may be verified in this case [5].

Much future work remains to be done in order to achieve a distributed maximum principle of comparable elegance and generality to that available for lumped systems. The continuity assumptions could be relaxed, for instance, by defining weak solutions of the state and costate equations. The entire problem should ultimately be cast in a distributional framework, and solved in a Banach-space setting. A second major improvement would be the use of global (rather than local) control perturbations to obtain global (rather than local) necessary conditions. Browder and Lions [18] have already made considerable contributions in this area. Finally, there are the generalizations to movable boundaries, control constraints, free terminal time, and fixed terminal state problems.

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An Application of Stochastic Optimal Control Theory to the Optimal Rescheduling of Airplanes

R. S. ELLIS AND R. W. RISHEL

Abstract—A model for the air traffic flow between two airports subject to random constraints on the takeoff and landing capacities is set up. For a simple case a dynamic programming algorithm is used to compute the optimal solutions explicitly.

I. INTRODUCTION

Every air traveler is familiar with delays due to random effects such as weather or breakdowns of equipment. It is an interesting problem to try to determine optimal rescheduling procedures so as to minimize total passenger inconvenience.

In this paper a simplified model for the traffic flow from one airport to another, subject to random constraints, is set up and studied. We formulate the problem of rescheduling the number of planes taking off and landing so that the air traffic system operates within the random constraints, so that all the scheduled planes make the trip by the terminal time, and so that passenger inconvenience, in terms of total waiting time on the ground and in the air, is minimized.

II. STATEMENT OF PROBLEM

To describe the problem that will be considered, let us assume there are given two airports, from one of which aircraft takeoff to land at the other. Time is discretized. There are $n + 1$ intervals of time to complete the transfer of planes between the two airports. Also, $j$ intervals of time are required to fly from the first airport to the second. To start, a desired departure schedule at airport one is given. In general, not all scheduled departures and arrivals can be handled at each time because of the random constraints. What the controller must do is to schedule actual departures from airport one


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R. S. Ellis is with the Department of Mathemetics, Northwestern University, Evanston, Ill. 60201.

R. W. Ris hel is with the Department of Mathematics, University of Kentucky, Lexingon, Ky. 40506.
to meet the condition that the number of departures at each time is less than or equal to the capacity of airport one at that time to takeoff airplanes.

Similarly, the controller must schedule actual landings at airport two to meet the condition that the number of landings at each time is less than or equal to the capacity of airport two at that time to land airplanes. Both these capacities are modeled as random quantities to take into account such factors as weather and delay due to equipment breakdown. In addition departures and landings must be such that after the n + 1 time intervals, all of the scheduled planes will have made the trip.

Because of factors like gas expenditure, safety, and passenger discomfort, waiting time in the air is considered to be undesirable than waiting time on the ground. Motivated by this, we set takeoff at the first airport in such a way that all of the above discomfort, waiting time in the air is considered to be more undesirable than waiting time on the ground. Let s1(ω) and s2(ω) take on only positive values. Let α, β, and γ denote the maximum and minimum values that s1(ω) and s2(ω), respectively, take on.

It may not be possible to satisfy conditions (4) if the total number of scheduled airplanes is too large. There are worst conditions that s1(ω) = α, s2(ω) = β occur for all times i = 0, 1, . . . , n. In order to complete the transfer of planes between airports before time n with probability one, one would have to be able to do this under the worst conditions. This is possible only if

$$\sum_{k=1}^{n-i} \delta_k \leq \min \{n - i - j, \alpha(n - i - j)\beta\},$$

$$i = 0, \ldots, n - (\gamma + 1)$$

$$\delta_i = 0, \quad i = n - j, \ldots, n.$$  

Let us say that the schedule is feasible if conditions (5) hold. Notice that it is always optimal to land as many planes as possible. This follows because if planes are left in the air when they could be landed, the performance index is only increased. However, we shall continue to consider landings as a control variable, since by doing so, the equations can be written using a unified notation which simplifies the theoretical discussion.

IV. IMPLIED CONSTRAINTS

We shall begin our discussion of the optimization problem by showing that the terminal conditions (4) imply constraints on the values of variables at intermediate times as well as at the initial time and the terminal time. Constraints of this type were noticed in stochastic programming problems by Wets [4, p. 92].

The total number of planes which must leave airport one at and after time i is ti + \sum_{k=i}^{n} \delta_k. Under worst conditions it is possible to takeoff α planes at each time and there are n − i − j times left in which planes can leave airport one and still reach airport two before time n. Thus to assure that all the planes reach airport two before time n with probability one the inequalities

$$t_i + \sum_{k=1}^{n} \delta_k \leq (n - i - j)\alpha, \quad i = 0, \ldots, n - j,$$

$$t_i = 0, \quad i = n - j + 1, \ldots, n,$$

must hold. In order that all these planes can land at airport two with probability one the inequalities

$$t_i + \sum_{k=1}^{n} \delta_k \leq (n - i - j)\beta, \quad i = 0, \ldots, n - j,$$

must also hold. The number of planes which must be landed at and after time i is

$$q_i + \sum_{k=1}^{j} u_{i-k} + t_i + \sum_{k=1}^{n} \delta_k \leq (n - i)\beta$$

To land this number under worst conditions,

$$q_i + \sum_{k=1}^{j} u_{i-k} + t_i + \sum_{k=1}^{n} \delta_k \leq (n - i)\beta$$

must hold. For each m = 1, . . . , j, the number of airplanes which will
arrive at airport two after \( t - m \) is
\[
t_i + \sum_{k=1}^{m} u_{i-k} + \sum_{k=i+1}^{n} \delta_k.
\]

To land this number under worst conditions,
\[
t_i + \sum_{k=1}^{m} u_{i-k} + \sum_{k=i+1}^{n} \delta_k \leq (n - i - m)\rho
\]
(9)

must hold. In addition, summing (2) and (3) and using \( u_i \geq 0, i \geq 0, \)
\( q_0 = \delta_0 = 0 \) gives
\[
q_i + t_i + \sum_{k=1}^{i} u_{i-k} \leq \sum_{k=0}^{i} \delta_k
\]
(10)

Finally the condition \( q_0 = \delta_0 = 0 \) implies that \( q_1 = \cdots = q_f = 0 \)
and \( t_0 - \delta_0 + u_0 = 0 \).

V. STATE VARIABLE DESCRIPTION

The discussion of the optimization problem can be notionally and conceptually simplified by adopting a vector state space notation. Let the vector
\[
X_i = (q_i, u_{i-1}, \cdots, u_{i-m})
\]
be defined to be the state of the system at time \( i \). Let the control vector of the system at time \( i \) be defined to be
\[
U_i = (u_i, u_{i-1}, \cdots, u_{i-m-1})
\]

Let the random vector
\[
R_i(\omega) = (s_i(\omega), r_i(\omega))
\]
be defined to be the vector of random constraints on the system. Equations (2) and (3) can be rewritten as
\[
X_i = AX_{i-1} + BU_i + C\delta_{i-1}
\]
(11)

where
\[
A = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 1 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix},
B = \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
\cdots \\
0
\end{bmatrix}, \quad \text{and } C = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\cdots \\
0
\end{bmatrix}
\]

The performance index can be written
\[
E\left\{ \sum_{k=0}^{n} D \cdot X_k \right\}
\]
where \( D = (\alpha_1, 0, \cdots, 0) \).

The implied constraints (6)–(10) define bounded convex sets \( C_i \)

of the state variables. At each time \( i \), the state \( X_i \) must lie in the set \( C_i \). In order that \( X_{i+1} \in C_{i+1} \), the controls \( U_i(X_{i}, R_i(\omega)) \) must be chosen so that
\[
AX_i + BU_i(X_{i}, R_i(\omega)) + C\delta_i \in C_{i+1}.
\]
(12)

We call admissible any sequence of controls \( U_i(X_{i}, R_i(\omega)) \) which are defined on the Cartesian product of the set \( C_i \) determined by the implied constraints and the range \( \Omega \) of the Markov process \( (s_i(\omega)), r_i(\omega)) \) and which satisfy (12) and
\[
0 \leq U_i(X_{i}, R_i(\omega)) \leq R_i(\omega).
\]

If \( X_i \) is the solution of
\[
X_i = AX_{i-1} + BU_{i-1}(X_{i-1}, R_{i-1}(\omega)) + C\delta_{i-1},
\]
then the optimization problem is to choose admissible \( U_i(X_{i}, R_i(\omega)) \)
such that the performance \( E\left\{ \sum_{k=0}^{m} D \cdot X_k \right\} \) is minimized.

VI. DYNAMIC PROGRAMMING

In this section we state the dynamic programming algorithm which
will form the basis for the computation. Although the dynamic programming theorem is a fairly standard one, several features should be pointed out. It is important for computation that the value function need only be defined on \( C_i \times \Omega \) rather than on all of \( B^{i+2} \times \Omega \).

There are counter examples to Theorem 1 when the sets \( C_i \) of implied

constraints are not convex.

For any sequence of admissible controls \( U_i(X, R) \) and for \( i = 1, \cdots, n \) define
\[
V_i^0(X, R) = E\left\{ \sum_{k=0}^{n} D \cdot X_k | X_i = X, R_i(\omega) = R \right\}
\]
(13)

where in (13) the sequence \( X_i \) is generated by (11) using the controls \( U_i(X, R) \). For \( i = 0 \), we shall adopt the convention that
\[
V_0(X, R) = E\left\{ \sum_{k=0}^{n} D \cdot X_k \right\}.
\]
(14)

Theorem 1: For \( i = 0, 1, \cdots, n \), there exist admissible controls \( U_i(X, R) \)

and continuous vector functions \( V_i(X, R) \) defined on \( C_i \times \Omega \) such that
\[
V_{i+1}(AX_i + BU_i(X, R) + C\delta_i, R)
\]
\[
\min_{0 \leq U \leq R} \left\{ \begin{array}{c}
V_i + \sum_{k=0}^{n} D \cdot X_k | X_i = X, R_i(\omega) = R \end{array} \right\}
\]
(15)

\[
V_i(X, R) = D \cdot X
\]
and
\[
V_i(X, R) = V_i^0(X, R)
\]
hold. In addition, for any admissible controls \( U_i(X, R) \),
\[
V_i(X, R) \leq V_i^0(X, R).
\]
(18)

Let \( E_U\{\cdot\} \) denote the conditional expectation of the quantity in brackets when the control \( U \) is used. Notice that (17), (18), and the convention (14) concerning \( V_0(X, R) \) imply that
\[
E_U\left\{ \sum_{k=0}^{n} D \cdot X_k \right\} = V_0(X, R) \leq V_0^0(X, R) = E_U\left\{ \sum_{k=0}^{n} D \cdot X_k \right\}.
\]

This inequality asserts the control \( U_i(X, R) \) is optimal. Thus Theorem 1 implies there is an optimal control and gives in (15) and (16) a procedure for the computation of this control.

VII. EXAMPLE

Optional controls were computed for the following example. Time is discretized into five units. The time of flight between airports is one unit. Four planes at airport one are scheduled to takeoff for airport two at time zero and none at the subsequent times. The weather at airport two is such that the capacities at each time at airport two to land planes are independent random variables which take on the value 1 with probability 0.2 and with probability 1 - P. The constants \( \alpha \) and \( \beta \) satisfy \( \alpha P \geq 1, P(\alpha + 1) \leq 2 \).

The optimal control for this example is: Take off two planes at time zero. Take off one plane at time one. At time two take off one plane if no planes are waiting to land at airport two or if the weather is at its best. If there are planes waiting to land and if the weather is bad at airport two at time two, do not take off any planes. If there is a plane remaining at time three, take it off.

VIII. CONCLUSION

An optimal rescheduling problem for traffic between two airports was modeled. To analyze it, a dynamic programming computational
Technical Notes and Correspondence

Computation of Regions of Transient Stability of Multimachine Power Systems

ARTHUR R. BERGEN AND GEORGE GROSS

Abstract—A major difficulty in applying Lyapunov theory to the problem of specifying transient stability regions of n-machine power systems is computational complexity, which increases markedly with n. This note outlines a method, requiring only a nominal amount of computation, to determine such regions.

I. INTRODUCTION

The application of Lyapunov theory to the study of transient stability of multimachine power systems, initiated by power engineers in 1966 [1], [2], has continued to the present time. A recent paper by Willems in these TRANSACTIONS [3] presents some of the more significant advances and provides an extensive bibliography.

The chief attraction of the Lyapunov method is its potential for reducing the computation time associated with investigating the transient stability of an n-machine interconnection. However, since the number and complexity of the computations increases rapidly with n, the potential for savings in overall computation may not be realized. In this note, we present a technique providing significant savings in computation; however, somewhat more conservative regions are obtained.

II. MATHEMATICAL MODEL

The starting point for the analysis is the swing equation model of a multimachine power system. For a detailed development of the model with the usual simplifying assumptions see [3]. For an n-machine interconnection, the dynamics are expressed in terms of the state vector \( (\alpha, \omega) \) by [4].

\[
\dot{\alpha} = T \omega \\
\dot{\omega} = -M^{-1} D \omega - M^{-1} T [f(\alpha) - f(\omega)]
\]

where

\[
T = \begin{bmatrix}
0 & -1 \\
1 & 0 \\
\vdots & \vdots \\
0 & -1
\end{bmatrix}
\]

\[
M = \text{diag} \{M_i, i = 1, 2, \ldots, n\}, M_i > 0, D = \text{diag} \{D_i, i = 1, 2, \ldots, n\}, D_i \geq 0, f(\alpha) = \text{col} \{f_i(\alpha); i = 1, 2, \ldots, n, n - 1\}
\]

\[
f_i(\alpha) = \sum_{j=1}^{n-1} b_{ij} \sin(\alpha_i - \alpha_j) + b_{in} \sin \alpha_n, \quad i = 1, 2, \ldots, n - 1 \tag{2}
\]

\[
b_{ij} = b_{ji} \geq 0, \text{ and } \omega \in \mathbb{R}^n \text{ contains the velocity components. Here } \alpha \text{ is the } n-1 \text{ vector of intermachine angles obtained by taking the difference of the power angle of } i\text{th machine and that of the } n\text{th machine which has been arbitrarily chosen as the reference machine.}

III. STABILITY

To study the stability of the equilibrium point \((\alpha^*, 0)\) of (1), we pick the "total energy" as a Lyapunov function \(V\):

\[
V(\alpha, \omega) = \frac{1}{2} (\omega, M \omega) + W(\alpha) \tag{3}
\]

where

\[
W(\alpha) \triangleq \int_{\alpha^*}^{\alpha} (|f(\xi) - f(\xi^*)|) d\xi. \tag{4}
\]

Since \(H(\alpha) \triangleq \frac{dW(\alpha)}{d\alpha} \) is symmetric, the integral's path independent and well-defined.

\(V\) vanishes at the equilibrium point \((\alpha^*, 0)\) and along trajectories

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The authors are with the Department of Electrical Engineering and Computer Science and the Electronics Research Laboratory, University of California, Berkeley, Calif. 94720.