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★**Entropy, large deviations, and statistical mechanics.**
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Let $\{\mu_n\}$ be a sequence of probability measures on a Polish space E with Borel field $\mathfrak{B}$, and suppose that, as $n\to\infty$, $\mu_n\to\delta_x$ for some $x\in E$. Loosely speaking, one can say that, as n gets large, the μ_n's "see more and more of E as being like x." Thus, under the μ_n's, x is the "typical" element of E and $y\in E$ which are increasing far from x are seen as increasingly "deviant". In particular, if $\Gamma\in\mathfrak{B}$ and $x\notin\overline{\Gamma}$, then $\mu_n(\Gamma)\to 0$. Now suppose that one has reason to believe that μ_n tends to δ_x so fast that $\mu_n(\Gamma)$ tends to 0 exponentially fast if $x\notin\overline{\Gamma}$, and consider the problem of finding the exponential rate $\lim_{n\to\infty}n^{-1}\log(\mu_n(\Gamma))$ at which $\mu_n(\Gamma)$ goes to 0. In order to get a feeling for what to expect, assume, for the moment, that $\mu_n(dy)=c_n\exp(-nI(y))\lambda(dy)$, where $n^{-1}\log c_n\to 0$ and $I\colon E\to[0,\infty]$ has the properties that $I(x)=0$ and $I(y)>0$ for $y\neq x$. It is then clear, at least when $\lambda(\Gamma)\in(0,\infty)$, that

$$\lim_{n\to\infty} n^{-1}\log(\mu_n(\Gamma))=\lim_{n\to\infty}\log(\|e^{-I}\chi_\Gamma\|_{L^n(\lambda)})=-\inf_{y\in\Gamma}I(y).$$

Of course, one does not want to restrict oneself to μ_n's which are becoming degenerate in such a regular fashion; in particular, the assumption that a reference measure λ exists is unrealistic in most applications. On the other hand, one can hope that the general structure displayed by the preceding example will persist even when λ fails to exist. Thus, one says that $\{\mu_n\}$ satisfies the large deviation principle with rate function I if, for each $\Gamma\in\mathfrak{B}$:

(L.D.) $$\inf_{y\in\operatorname{int}\Gamma}I(y)\le\liminf_{n\to\infty}n^{-1}\log(\mu_n(\Gamma))$$
$$\le\limsup_{n\to\infty}n^{-1}\log(\mu_n(\Gamma))\le-\inf_{y\in\overline{\Gamma}}I(y).$$

Usually, one requires that the rate function $I\colon E \longrightarrow [0,\infty]$ have the property that $\{y\colon I(y) \le l\}$ be a compact subset of E for each $l \ge 0$, in which case Varadhan showed that, for $\Phi \in C(E)$ satisfying reasonable growth conditions:

$$\text{(V)}\ \lim_{n\to\infty} n^{-1} \log\left(\int \exp(n\Phi(y))\mu_n(dy)\right) = \sup\{\Phi(y) - I(y)\colon y \in E\}.$$

What is referred to as the "theory of large deviations" is, in fact, a collection of techniques which have been used to prove, for certain classes of examples, that $\{\mu_n\}$ satisfies the large deviation principle and to describe the associated rate function I. In this book, the author concentrates, for the most part, on the case when $E = \mathbf{R}^N$; and his techniques rest on the fact that the large deviation principle for $\{\mu_n\}$ holds whenever the associated Laplace transforms $\Lambda_n(\xi) = \int e^{\xi\cdot y}\mu_n(dy)$ converge to a sufficiently well-behaved function Λ, in which case the rate function I is given by the Legendre transform $I(y) \equiv \sup\{\xi\cdot y - \Lambda(\xi)\colon \xi \in \mathbf{R}^N\}$. (Note that, by the inversion formula for the Legendre transform, this choice of I guarantees (V) when $\Phi(y) = \xi \cdot y$ for some $\xi \in \mathbf{R}^N$. Thus, the point is that under appropriate conditions one is able to show that (V) for general Φ's follows from (V) for linear Φ's.) Besides the fact that the author's treatment of large deviations is a nice contribution to the literature on the subject, his book has the virtue that it provides a beautifully unified and mathematically appealing account of certain aspects of statistical mechanics. In particular, he carries out the program suggested by O. E. Lanford, III and substantiates, once again, that good mathematics often originates from good physics. Furthermore, he does not make the mistake of assuming that his mathematical audience will be familiar with the physics and has done an admirable job of explaining the necessary physical background. Finally, it is clear that the author's book is the product of many painstaking hours of work; and the reviewer is confident that its readers will benefit from his efforts.

Reviewed by *D. W. Stroock*