36 Limit Theorems for Sums of Dependent Random Variables Occurring in Statistical Mechanics

Richard S. Ellis
Department of Mathematics and Statistics
University of Massachusetts Amherst

Collaborators:

Marius Costeniuc
Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22–26
Leipzig, Germany

Jonathan Machta
Department of Physics
University of Massachusetts Amherst

Peter Tak-Hun Otto
Department of Mathematics
Union College
Schenectady, NY 12309

June 2006

Email: rsellis@math.umass.edu

Outline of the Talk

- BEG Spin Model
  - Phase transitions: second order (continuous bifurcation), first order (discontinuous bifurcation), tricritical point
  - Probabilistic limit theorems as a tool for studying the fine structure of the phase transitions
    - 18 scaling limits and 18 moderate deviation principles (MDPs)
- Background on Curie–Weiss Model
  - Continuous, second-order phase transition at β_c
  - Law of large numbers for $\beta \leq \beta_c$ and its breakdown for $\beta > \beta_c$
  - Central limit theorem for $\beta < \beta_c$, its breakdown for $\beta = \beta_c$, conditioned central limit theorem for $\beta > \beta_c$
- Limit Theorems for the BEG Model
  - 1 scaling limit and 1 MDP for non-critical points
  - 4 scaling limits and 4 MDPs for second-order critical points
  - 13 scaling limits and 13 MDPs for tricritical point
  - 1 + 4 + 13 = Chai = 18
  - Limit theorems reveal a geometric structure in a model without geometry.
  - Ideas behind the proofs
- References
  - Scaling Limits for Curie–Weiss Model: RSE and C. M. Newman, *Z. Wahrsch. verw. Geb.* (1979)
  - MDPs for Curie–Weiss Model: P. Eichelsberger and M. Löwe, *Prob. Th. Related Fields* (2004)
  - Phase Transitions in BEG Model: RSE, P. Otto, and H. Touchette, *Ann. Appl. Prob.* (2005)
  - Limit Theorems in BEG Model: M. Costeniuc, RSE, and P. T.-H. Otto, submitted to *Stoch. Proc. Appl.* (2006)

BEG Spin Model

Mean-field approximation to Blume–Capel model (1966) and Blume–Emery–Griffiths model (1971): study He^3-He^4 mixtures and other physical systems

1. n spins $\omega_i \in \Lambda = \{-1, 0, 1\}$
2. Microstates: $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Lambda^n$
3. Total spin: $S_n(\omega) = \sum_{j=1}^n \omega_j$
4. Hamiltonian or energy function ($K > 0$):
$$H_{n,K}(\omega) = \sum_{j=1}^n \omega_j^2 - \frac{K}{n} \sum_{j,k=1}^n \omega_j \omega_k = \sum_{j=1}^n \omega_j^2 - nK \left(\frac{S_n(\omega)}{n}\right)^2$$
5. Prior measure:
$$P_n(\omega) = \prod_{j=1}^n \rho(\omega_j) = \frac{1}{3^n}, \ \rho = \tfrac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$$
6. Canonical ensemble ($\beta > 0$, $K > 0$):
$$P_{n,\beta,K}(\omega) = \frac{1}{Z_n(\beta, K)} \cdot \exp[-\beta H_{n,K}(\omega)] \, P_n(\omega),$$
$$Z_n(\beta, K) = \int_{\Lambda^n} \exp[-\beta H_{n,K}(\omega)] \, P_n(d\omega)$$
7. Alignment effect:
$$\lim_{\beta \to 0^+} P_{n,\beta,K}(\omega) = P_n(\omega) = \tfrac{1}{3^n},$$
$$\lim_{K \to 0^+} P_{n,\beta,K}(\omega) = \prod_{j=1}^n \rho_\beta(\omega)$$
$$\lim_{K \to \infty} P_{n,\beta,K}(\omega) = \tfrac{1}{2}(\delta_{\omega^-}(\omega) + \delta_{\omega^+}(\omega)),$$
where for all j, $\omega_j^- = -1$ and $\omega_j^+ = 1$
8. Phase transition: persistence of alignment effect as $n \to \infty$

Phase Transitions in BEG Model

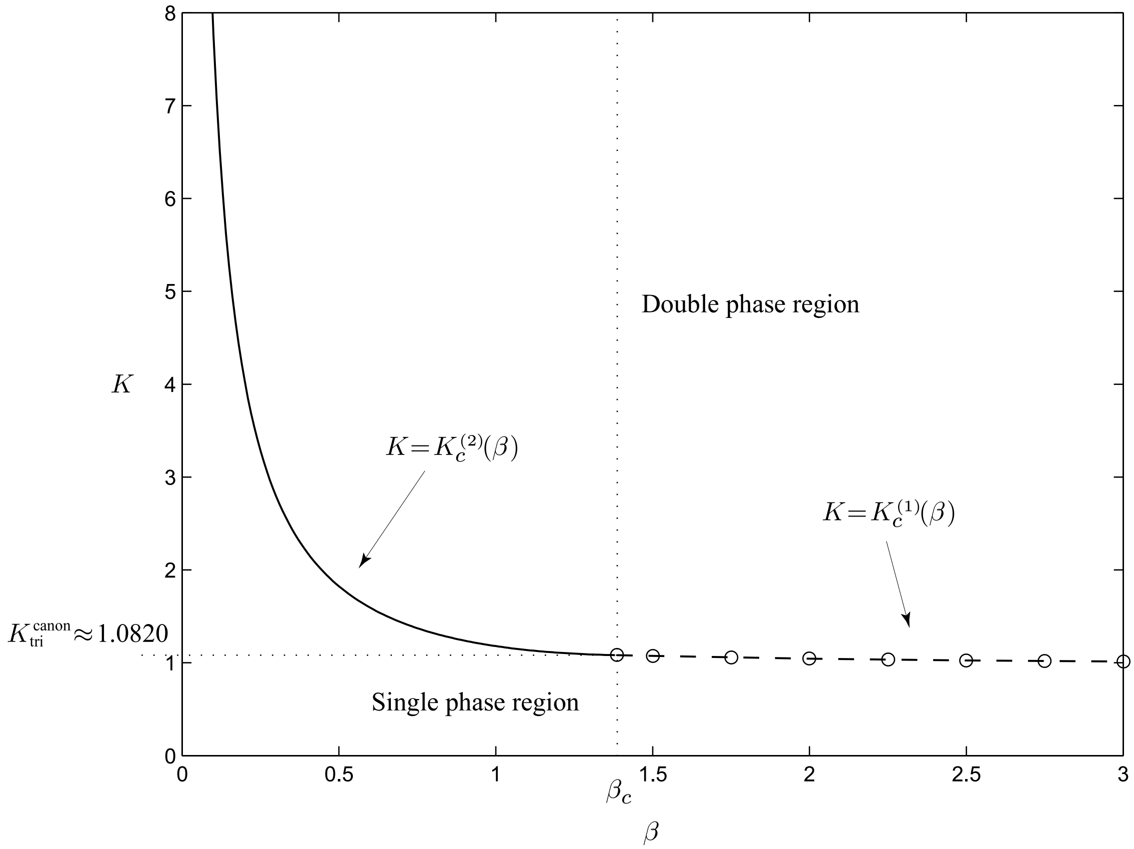

Figure 1: Bifurcation diagram for the BEG model

- Continuous bifurcation (second-order phase transition) for $0 < \beta < \beta_c = \log 4$ as $K \nearrow K_c^{(2)}(\beta)$
- Discontinuous bifurcation (first-order phase transition) for $\beta > \beta_c$ as $K \nearrow K_c^{(1)}(\beta)$
- Tricritical point at $(\beta_c, K_c^{(2)}(\beta_c))$ separates the two phase transitions
- Unique phase for (β, K) under the two phase-transition curves
- Double phase for (β, K) above the two phase-transition curves

■ Probabilistic Limit Theorems

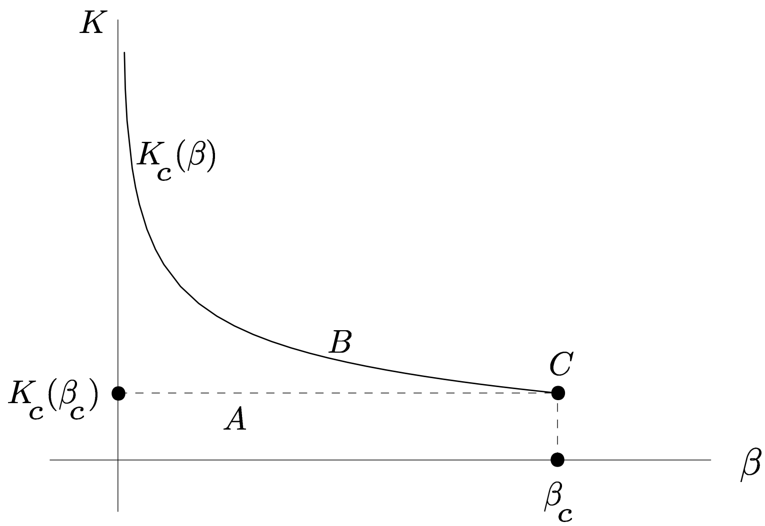

Figure 2: The sets A, B, and C

- Seek $(\beta_n, K_n) \to (\beta, K) \in A \cup B \cup C$, $\gamma \in (0,1)$, $w \in (0,1)$, even polynomials G satisfying $G(x) \to \infty$ as $|x| \to \infty$ and $G(0) = 0$:
  - Scaling limits:
  $$P_{n,\beta_n,K_n}(S_n/n^{1-\gamma} \in dx) \Longrightarrow \exp[-G(x)]\, dx$$
  - Moderate deviation theorems (MDPs):
  $$P_{n,\beta_n,K_n}(S_n/n^{1-\gamma} \in dx) \asymp \exp[-n^w G(x)]\, dx$$
- Scaling limits and MDPs have similar forms.
- Results to be discussed later in the talk
  - For $(\beta, K) \in A$, 1 scaling limit and 1 MDP: deg$(G) = 2$ ($\therefore$ 1 form for G), arbitrary $(\beta_n, K_n) \to (\beta, K)$
  - For $(\beta, K) \in B$, 4 scaling limits and 4 MDPs: deg$(G) = 4$ or 2 ($\therefore$ 3 + 1 = 4 forms for G), special $K_n \to K$, arbitrary $\beta_n \to \beta$
  - For $(\beta, K) \in C$, 13 scaling limits and 13 MDPs: deg$(G) = 6, 4$, or 2 ($\therefore$ 9 + 3 + 1 = 13 forms for G), special $(\beta_n, K_n) \to (\beta, K)$
  - For $(\beta, K) \in B \cup C$, unexpected consequences of MDPs

Curie–Weiss Spin Model

Mean-field approximation to Ising model

1. n spins $\omega_i \in \Lambda = \{-1, 1\}$
2. Microstates: $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Lambda^n$
3. Total spin: $S_n(\omega) = \sum_{j=1}^n \omega_j$
4. Hamiltonian or energy function:

$$H_{n,K}(\omega) = -\frac{1}{2n} \sum_{j,k=1}^{n} \omega_j \omega_k = -\frac{n}{2} \left(\frac{S_n(\omega)}{n} \right)^2$$

5. Prior measure:

$$P_n(\omega) = \prod_{j=1}^{n} \nu(\omega_j) = \frac{1}{2^n},\ \nu = \tfrac{1}{2}(\delta_{-1} + \delta_1)$$

6. Canonical ensemble ($\beta > 0$):

$$P_{n,\beta}(\omega) = \frac{1}{Z_n(\beta)} \cdot \exp[-\beta H_n(\omega)]\, P_n(\omega),$$

$$Z_n(\beta) = \int_{\Lambda^n} \exp[-\beta H_n(\omega)]\, P_n(d\omega)$$

7. Alignment effect:

$$\lim_{\beta \to 0^+} P_{n,\beta,K}(\omega) = P_n(\omega) = \tfrac{1}{2^n},$$

$$\lim_{\beta \to \infty} P_{n,\beta}(\omega) = \tfrac{1}{2}(\delta_{\omega^-}(\omega) + \delta_{\omega^+}(\omega)),$$

where for all j, $\omega_j^- = -1$ and $\omega_j^+ = 1$

8. Phase transition: persistence of alignment effect as $n \to \infty$

■ LDP and Equilibrium Macrostates

$$P_{n,\beta}(\omega) = \frac{1}{Z_n(\beta)} \cdot \exp\left[\frac{n\beta}{2}\left(\frac{S_n(\omega)}{n}\right)^2\right] P_n(\omega)$$

For $t \in \mathbb{R}$ and $x \in [-1,1]$ define

$$c(t) = \log \int_{\{-1,1\}} e^{t\omega}\, \rho(d\omega) = \log\left[\tfrac{1}{2}(e^t + e^{-t})\right]$$

and

$$I(x) = \sup_{t \in \mathbb{R}}\{tx - c(t)\} = \tfrac{1}{2}\left[(1-x)\log(1-x) + (1+x)\log(1+x)\right].$$

Combinatorics or Cramér's Theorem:

$$P_n\left\{\frac{S_n}{n} \sim x\right\} \asymp e^{-nI(x)}$$

Theorem 1. LDP for $P_{n,\beta}\left\{\frac{S_n}{n} \sim x\right\}$ with rate function $I(x) - \frac{\beta x^2}{2}$:

$$\begin{aligned} P_{n,\beta}\left\{\frac{S_n}{n} \sim x\right\} &= \frac{1}{Z_n(\beta)} \cdot \exp\left(\frac{n\beta x^2}{2}\right) P_n\left\{\frac{S_n}{n} \sim x\right\} \\ &\asymp \frac{1}{Z_n(\beta)} \cdot \exp\left[-n\left(I(x) - \frac{\beta x^2}{2}\right)\right] \end{aligned}$$

For $\beta > 0$ define class of equilibrium macrostates

$$\mathcal{E}_\beta = \left\{x \in [-1,1] : x \text{ minimizes } I(x) - \tfrac{\beta x^2}{2}\right\}.$$

Theorem 2. For any $\beta > 0$ the following conclusions hold.

(a) Let $K \subset [-1,1]$ be any closed set $\ni K \cap \mathcal{E}_\beta = \emptyset$. Then

$$P_{n,\beta}\left\{\frac{S_n}{n} \in K\right\} \to 0 \text{ exponentially fast.}$$

(b) $P_{n,\beta}\left\{\frac{S_n}{n} \in dx\right\}$ concentrates on $\mathcal{E}_\beta$.

Calculating Equilibrium Macrostates

$$x \in \mathcal{E}_\beta \iff x \text{ minimizes } I(x) - \tfrac{\beta x^2}{2} \implies I'(x) = \beta x$$
$$\iff x = c'(\beta x) \impliedby x \text{ minimizes } \tfrac{\beta x^2}{2} - c(\beta x)$$

Theorem 3. For any $\beta > 0$ the following conclusions hold.

(a) x minimizes $I(x) - \frac{\beta x^2}{2} \iff x$ minimizes $\frac{\beta x^2}{2} - c(\beta x)$.

(b) $\mathcal{E}_\beta = \left\{x \in [-1,1] : x \text{ minimizes } \frac{\beta x^2}{2} - c(\beta x)\right\}$.

Minimize $\frac{\beta x^2}{2} - c(\beta x) \iff$ minimize $\frac{w^2}{2\beta} - c(w)$ $[w = \beta x] \implies \frac{w}{\beta} = c'(w)$.

Key property: $c'(w)$ is strictly concave for $w > 0$.

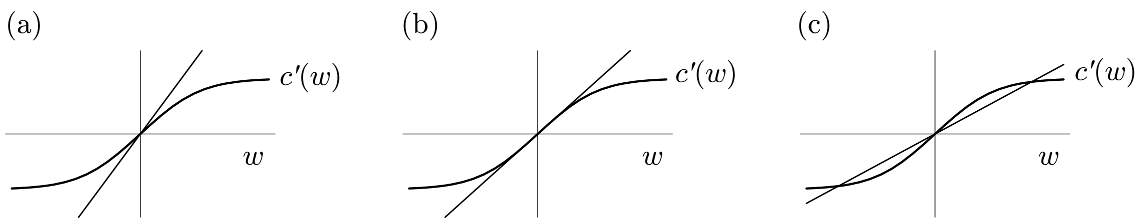

Figure 3: Continuous, second-order phase transition for $\beta = \beta_c = 1$. (a) $\beta < \beta_c$, (b) $\beta = \beta_c$, (c) $\beta > \beta_c$

Theorem 4. Law of large numbers and its breakdown.

(a) For $0 < \beta \leq \beta_c = 1/c''(0) = 1$, $\mathcal{E}_\beta = \{0\}$ and

$$P_{n,\beta}\left\{\frac{S_n}{n} \in dx\right\} \Longrightarrow \delta_0.$$

(b) For $\beta > \beta_c$, there exists $z(\beta) > 0$ such that $\mathcal{E}_\beta = \{\pm z(\beta)\}$ and

$$P_{n,\beta}\left\{\frac{S_n}{n} \in dx\right\} \Longrightarrow \frac{1}{2}(\delta_{z(\beta)} + \delta_{-z(\beta)}).$$

(c) $z(\beta) \to 0$ as $\beta \to (\beta_c)^+$.

(d) $\mathcal{E}_\beta$ has a continuous, second-order phase transition at $\beta = \beta_c$.

Limit Results for CW Model

Theorem 5. Central limit theorem and its breakdown.

(a) *CLT:* For $0 < \beta < \beta_c = 1$, $\mathcal{E}_\beta = \{0\}$ and

$$P_{n,\beta}\Big\{\frac{S_n}{n^{1/2}} \in dx\Big\} \Longrightarrow \exp[-x^2/(2\sigma^2(\beta))]\,dx, \text{ where } \sigma^2(\beta) = (1-\beta)^{-1}.$$

(b) *Conditioned CLT:* For $\beta > \beta_c$, $\mathcal{E}_\beta = \{\pm z(\beta)\}$. For $\tilde{z} = \pm z(\beta)$ and each choice of sign, there exists $b = b(\beta) > 0$ and $\lambda(\beta) \in (0, \infty)$ such that

$$P_{n,\beta}\Big\{\frac{S_n}{n^{1/2}} \in dx \Big| \frac{S_n}{n} \in [\tilde{z} - b, \tilde{z} + b]\Big\} \Longrightarrow \exp[-x^2/(2\lambda^2(\beta))]\,dx.$$

(c) *Breakdown of CLT due to onset of long-range order:* For $\beta = \beta_c$, $\mathcal{E}_\beta = \{0\}$ and

$$P_{n,\beta_c}\Big\{\frac{S_n}{n^{3/4}} \in dx\Big\} \Longrightarrow \exp\Big[-\tfrac{1}{12}x^4\Big]\,dx.$$

Let $\beta = \beta_c$. If $\gamma \in (0, 1/4)$, then for all $\varepsilon > 0$,

$$\lim_{n\to\infty} P_{n,\beta_c}\Big\{\frac{S_n}{n^{1-\gamma}} \not\in (-\varepsilon, \varepsilon)\Big\} = 0.$$

Theorem 6. MDP for $\beta = \beta_c$ due to Eichelsberger-Löwe (2004):

$$P_{n,\beta_c}\Big\{\frac{S_n}{n^{1-\gamma}} \in dx\Big\} \asymp \exp\Big[-n^{1-4\gamma}\tfrac{1}{12}x^4\Big]\,dx$$

New topic. Scaling limits and MDPs for $\frac{S_n}{n^{1-\gamma}}$, where $\beta_n = \beta_c - \frac{k}{n^\alpha}$.

- For suitable choices of $\gamma \in [\frac{1}{4}, \frac{1}{2})$, $\alpha > 0$, $k \neq 0$, and $v \in (0, 1)$, 4 scaling limits and 4 MDPs:

  $$P_{n,\beta_n}\Big\{\frac{S_n}{n^{1-\gamma}} \in dx\Big\} \Longrightarrow \exp[-G(x)]\,dx,$$
  $$P_{n,\beta_n}\Big\{\frac{S_n}{n^{1-\gamma}} \in dx\Big\} \asymp \exp[-n^v\Gamma(x)]\,dx,$$

  G an even polynomial satisfying $G(0) = 0$, $G(x) \to \infty$, $\deg(G) = 4$ (3 forms) or $\deg(G) = 2$ (1 form); $\Gamma(x) = G(x) - \inf_{y\in\mathbb{R}} G(y)$.

- The scaling limits and the MDPs have similar forms and are proved by a unified method.

- Details to be given for BEG model.

■ BEG Model as a Curie-Weiss Model

1. n spins $\omega_i \in \Lambda = \{-1, 0, 1\}$
2. Microstates: $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Lambda^n$
3. Total spin: $S_n(\omega) = \sum_{j=1}^n \omega_j$
4. Hamiltonian or energy function ($K > 0$):
$$H_{n,K}(\omega) = \sum_{j=1}^n \omega_j^2 - \frac{K}{n} \sum_{j,k=1}^n \omega_j \omega_k = \sum_{j=1}^n \omega_j^2 - nK \left(\frac{S_n(\omega)}{n} \right)^2$$
5. Prior measure:
$$P_n(\omega) = \prod_{j=1}^n \rho(\omega_j) = \frac{1}{3^n}, \rho = \tfrac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$$
6. Canonical ensemble ($\beta > 0$, $K > 0$):
$$P_{n,\beta,K}(\omega) = \frac{1}{Z_n(\beta, K)} \cdot \exp[-\beta H_{n,K}(\omega)]\, P_n(\omega),$$
$$Z_n(\beta, K) = \int_{\Lambda^n} \exp[-\beta H_{n,K}(\omega)]\, P_n(d\omega)$$
7. Absorb noninteracting component of $H_{n,K}$ into P_n:
$$P_{n,\beta,K}(\omega) = \frac{1}{\tilde{Z}_n(\beta, K)} \cdot \exp\left[n\beta K \left(\frac{S_n(\omega}{n} \right)^2 \right] \prod_{j=1}^n \rho_\beta(\omega_j),$$
$$\rho_\beta(\omega_j) = \frac{1}{1 + 2\exp(-\beta)} \cdot \exp(-\beta \omega_j^2)\, \rho(\omega_j).$$
8. Compare with canonical ensemble for Curie-Weiss model:
$$P_{n,\beta}(\omega) = \frac{1}{Z_n(\beta)} \cdot \exp\left[\frac{n\beta}{2} \left(\frac{S_n(\omega)}{n} \right)^2 \right] \prod_{j=1}^n \nu(\omega_j)$$

$\therefore$ Curie-Weiss model $\longrightarrow$ BEG model if $\beta \longrightarrow 2\beta K$ and $\nu \longrightarrow \rho_\beta$.

LDP and Equilibrium Macrostates

$$P_{n,\beta,K}(\omega) = \frac{1}{\tilde{Z}_n(\beta,K)} \cdot \exp\left[n\beta K\left(\frac{S_n(\omega)}{n}\right)^2\right] \prod_{j=1}^{n} \rho_\beta(\omega_j)$$

For $t \in \mathbb{R}$ and $x \in [-1,1]$ define

$$c_\beta(t) = \log \int_{\{-1,0,1\}} e^{t\omega}\, \rho_\beta(d\omega) = \log\left[\frac{1 + e^{-\beta}(e^t + e^{-t})}{1 + 2e^{-\beta}}\right]$$

and the non-explicit

$$I_\beta(x) = \sup_{t \in \mathbb{R}}\{tx - c_\beta(t)\}.$$

Combinatorics or Cramér's Theorem:

$$Q_{n,\beta}\left\{\frac{S_n}{n} \sim x\right\} \asymp e^{-nI_\beta(x)}, \text{ where } Q_{n,\beta}(\omega) = \textstyle\prod_{j=1}^{n} \rho_\beta(\omega_j)$$

Theorem 7. LDP for $P_{n,\beta,K}\left\{\frac{S_n}{n} \sim x\right\}$ with rate function $I_\beta(x) - \beta K x^2$:

$$\begin{aligned} P_{n,\beta,K}\left\{\frac{S_n}{n} \sim x\right\} &= \frac{1}{Z_n(\beta)} \cdot \exp\left(n\beta K x^2\right) Q_{n,\beta}\left\{\frac{S_n}{n} \sim x\right\} \\ &\asymp \frac{1}{Z_n(\beta)} \cdot \exp\left[-n\left(I_\beta(x) - \beta K x^2\right)\right] \end{aligned}$$

For $\beta > 0$ define class of equilibrium macrostates

$$\mathcal{E}_{\beta,K} = \left\{x \in [-1,1] : x \text{ minimizes } I_\beta(x) - \beta K x^2\right\}.$$

Theorem 8. For any $\beta > 0$ and $K > 0$ the following conclusions hold.

(a) Let $K \subset [-1,1]$ be any closed set $\ni K \cap \mathcal{E}_{\beta,K} = \emptyset$. Then

$$P_{n,\beta,K}\left\{\frac{S_n}{n} \in K\right\} \to 0 \text{ exponentially fast.}$$

(b) $P_{n,\beta,K}\left\{\frac{S_n}{n} \in dx\right\}$ concentrates on $\mathcal{E}_{\beta,K}$.

Calculating Equilibrium Macrostates

$$x \in \mathcal{E}_{\beta,K} \iff x \text{ minimizes } I_\beta(x) - \beta K x^2 \Longrightarrow I'_\beta(x) = 2\beta K x$$
$$\iff x = c_\beta{}'(2\beta K x) \Longleftarrow x \text{ minimizes } \beta K x^2 - c_\beta(2\beta K x)$$

Theorem 9. For any $\beta > 0$ and $K > 0$ the following conclusions hold.
(a) x minimizes $I_\beta(x) - \beta K x^2 \iff x$ minimizes $\beta K x^2 - c_\beta(2\beta K x)$.
(b) $\mathcal{E}_{\beta,K} = \Big\{x \in [-1,1] : x \text{ minimizes } \beta K x^2 - c(2\beta K x)\Big\}$.

Minimize $\beta K x^2 - c_\beta(2\beta K x) \iff$ minimize $\frac{w^2}{4\beta K} - c_\beta(w)$ $[w = 2\beta K x]$
$\Longrightarrow \frac{w}{2\beta K} = c_\beta{}'(w)$. Fix β and vary K. Key properties:

(a) for $0 < \beta \leq \beta_c = \log 4$, $c_\beta{}'(w)$ is strictly concave for $w > 0$;
(b) for $\beta > \beta_c$, $c_\beta{}'(w)$ is strictly convex for $0 < w < w_c(\beta)$ [explicitly given] and is strictly concave for $w > w_c(\beta)$.

For $0 < \beta \leq \beta_c$, the situation is the same as in the Curie-Weiss model.

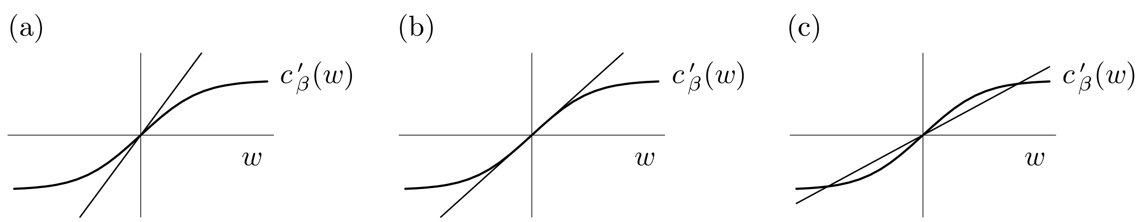

Figure 4: Continuous bifurcation for $\beta = 1 < \beta_c$. (a) $0 < K < K_c^{(2)}(\beta)$, (b) $K = K_c^{(2)}(\beta)$, (c) $K > K_c^{(2)}(\beta)$

Theorem 10. Second-order phase transition for $0 < \beta \leq \beta_c$.
(a) For $0 < K \leq K_c^{(2)}(\beta) = 1/[2\beta c_\beta{}''(0)]$, $\mathcal{E}_\beta = \{0\}$
(b) For $K > K_c^{(2)}(\beta)$, $\exists z(\beta, K) > 0$ such that $\mathcal{E}_\beta = \{\pm z(\beta, K)\}$,
(c) $z(\beta, K) \to 0$ as $K \to (K_c^{(2)}(\beta))^+$.
(d) $\mathcal{E}_{\beta,K}$ has a continuous bifurcation at $K = K_c^{(2)}(\beta)$ [second-order phase transition].

■ 1st-Order Phase Transition for $\beta > \beta_c$

For $\beta > \beta_c$, minimize $F_{\beta,K}(w) = \frac{w^2}{4\beta K} - c_\beta(w)$: solve $\frac{w}{2\beta K} = c_\beta{}'(w)$.

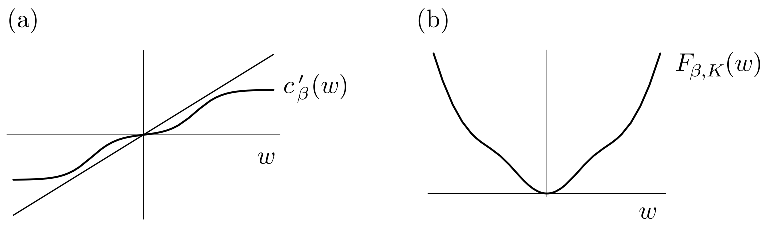

Figure 5: (a) Graph of two components of $F'_{\beta,K}$ and (b) graph of $F_{\beta,K}$ for $0 < K < K_1$

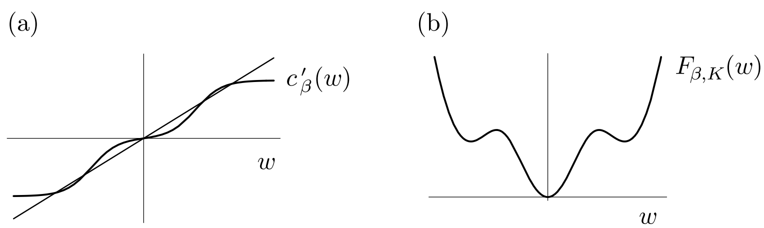

Figure 6: (a) Graph of two components of $F'_{\beta,K}$ and (b) graph of $F_{\beta,K}$ for $K_1 < K < K_c^{(1)}(\beta)$

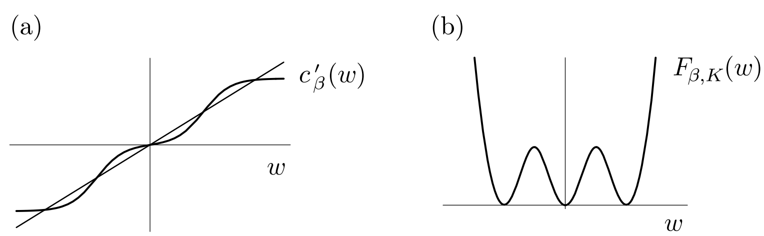

Figure 7: (a) Graph of two components of $F'_{\beta,K}$ and (b) graph of $F_{\beta,K}$ for $K = K_c^{(1)}(\beta)$

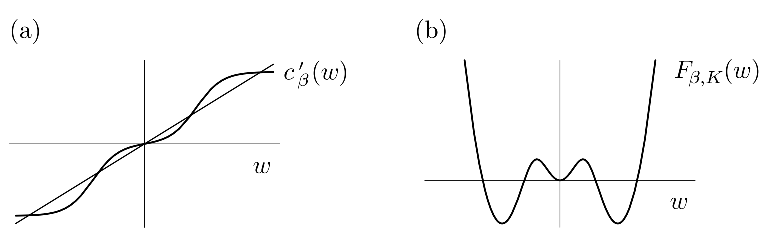

Figure 8: (a) Graph of two components of $F'_{\beta,K}$ and (b) graph of $F_{\beta,K}$ for $K_c^{(1)}(\beta) < K < K_2$

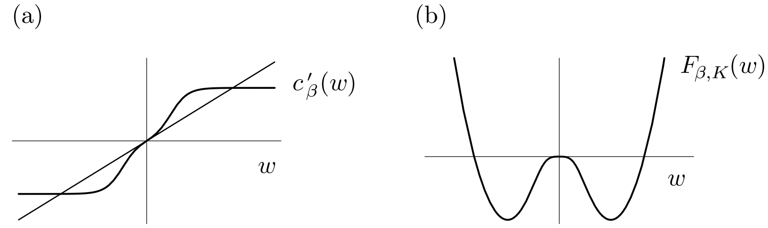

Figure 9: (a) Graph of two components of $F'_{\beta,K}$ and (b) graph of $F_{\beta,K}$ for $K \geq K_2$

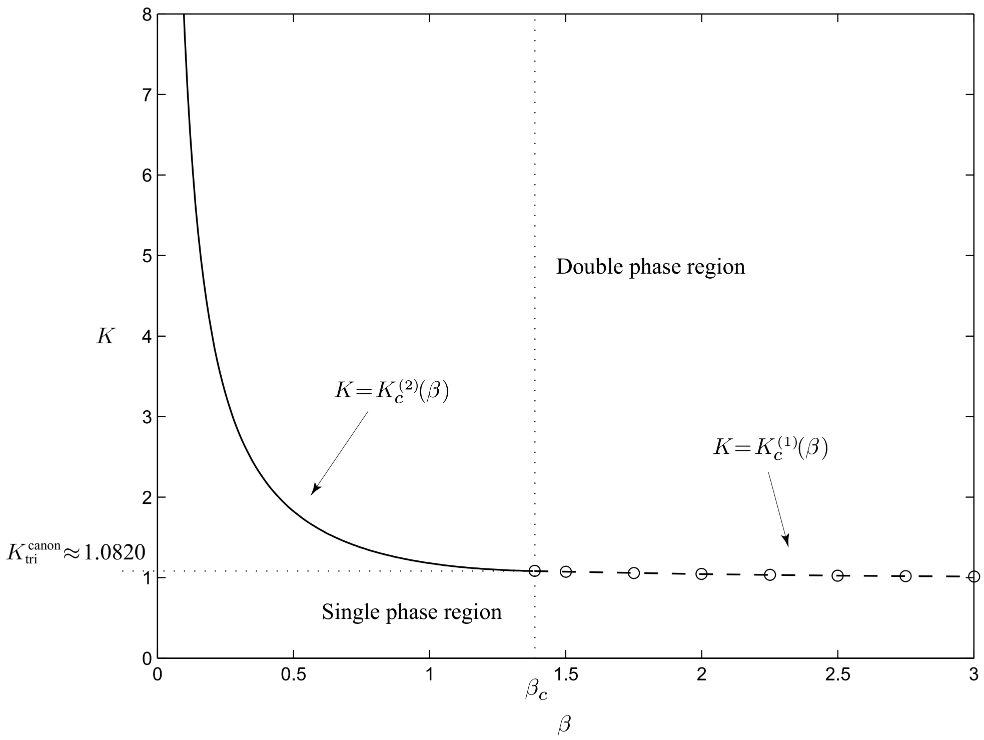

Figure 10: Bifurcation diagram for the BEG model

Theorem 11. First-order phase transition for $\boldsymbol{\beta > \beta_c}$.

(a) For $0 < K \leq K_c^{(1)}(\beta)$ [implicitly defined], $\mathcal{E}_\beta = \{0\}$.

(b) For $K > K_c^{(1)}(\beta)$, $\exists z(\beta, K) > 0$ such that $\mathcal{E}_\beta = \{\pm z(\beta, K)\}$.

(c) For $K = K_c^{(1)}(\beta)$, $\exists z(\beta, K_c^{(1)}(\beta)) > 0$ such that $\mathcal{E}_\beta = \{0, \pm z(\beta, K_c^{(1)}(\beta))\}$.

(d) $z(\beta, K) \to z(\beta, K_c^{(1)}(\beta)) > 0$ as $K \to (K_c^{(1)}(\beta))^+$.

(e) $\mathcal{E}_{\beta,K}$ has a discontinuous bifurcation at $K = K_c^{(1)}(\beta)$ [first-order phase transition].

Scaling Limits for BEG Model

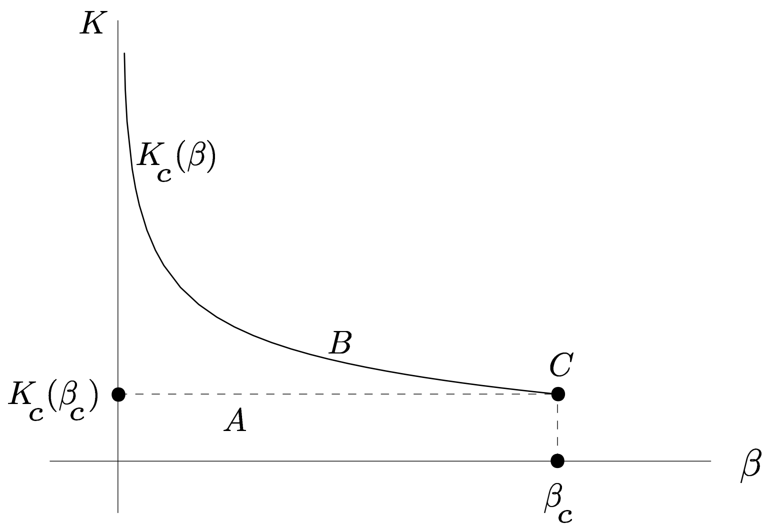

Figure 11: The sets A, B, and C

We write $K_c(\beta)$ instead of $K_c^{(2)}(\beta)$.

- $(\beta_c, K_c(\beta_c))$ is the tricritical point.
- $C = \{(\beta_c, K_c(\beta_c))\}$
- B is the curve of second-order critical points:
$$B = \{(\beta, K) \in \mathbb{R}^2 : 0 < \beta < \beta_c, K = K_c(\beta)\}.$$
- A is the region of non-critical points lying under the curve B:
$$A = \{(\beta, K) \in \mathbb{R}^2 : 0 < \beta \leq \beta_c, K < K_c(\beta)\}.$$
- For all $(\beta, K) \in A \cup B \cup C$, $\mathcal{E}_{\beta,K} = \{0\}$.

Theorem 12. (a) $\exists c_2 = c_2(\beta, K) > 0$, $\exists c_4 = c_4(\beta, K) > 0$, and $c_6 = \frac{9}{40} \ni$
$$P_{n,\beta,K}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \Longrightarrow \begin{cases} \exp(-c_2x^2)\,dx & \text{with } \gamma = \frac{1}{2} \text{ if } (\beta, K) \in A \\ \exp(-c_4x^4)\,dx & \text{with } \gamma = \frac{1}{4} \text{ if } (\beta, K) \in B \\ \exp(-c_6x^6)\,dx & \text{with } \gamma = \frac{1}{6} \text{ if } (\beta, K) \in C. \end{cases}$$
(b) For $(\beta, K) \in A \cup B$, limits as in CW model; for $(\beta, K) \in C$, limit is new.

New topic. Consider positive sequences $(\beta_n, K_n) \to (\beta, K) \in A \cup B \cup C$ to study scaling limits of $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n}.

Theorem 13. Scaling limit for $(\boldsymbol{\beta_n}, \boldsymbol{K_n}) \to (\boldsymbol{\beta}, \boldsymbol{K}) \in \boldsymbol{A}$.
For any $(\beta_n, K_n) \to (\beta, K) \in A$, there exists $c_2 = c_2(\beta, K) > 0$ such that

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1/2}} \in dx\right\} \Longrightarrow \exp(-c_2 x^2)\, dx.$$

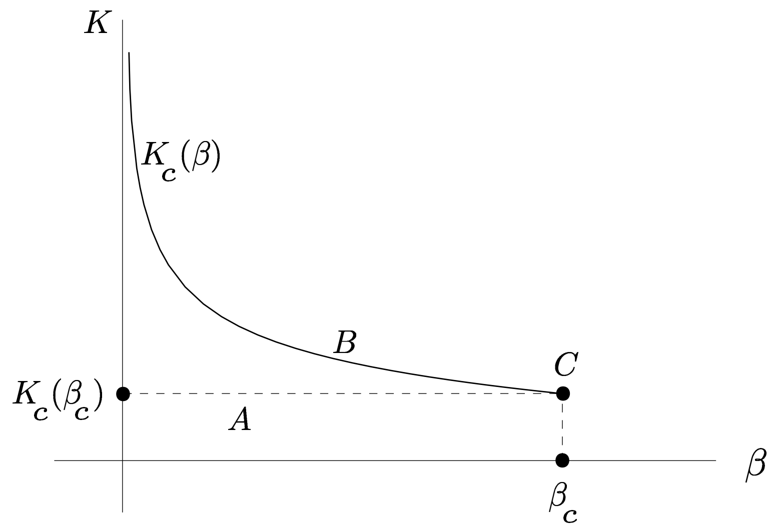

Figure 12: The sets A, B, and C

- For $(\beta, K) \in A$, sequences (β_n, K_n) gives rise to no new phenomena.
- For $(\beta, K) \in B \cup C$, the scaling limits reveal a geometric structure in a model without geometry.
  - For $(\beta, K) \in B$, sequences (β_n, K_n) give rise to 4 scaling limits: arbitrary $\beta_n \to \beta$, special $K_n \to (K_c(\beta))^+$ or $K_n \to (K_c(\beta))^-$.
  - For $(\beta, K) \in C$, sequences (β_n, K_n) give rise to 13 scaling limits: special $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$ from SW, SE, NE, NW.
  - For $(\beta, K) \in B \cup C$, the form of the scaling limit depends on the speed and the direction of approach of (β_n, K_n) to (β, K).

Theorem 14. Scaling limits for $(\beta_n, K_n) \longrightarrow (\beta, K_c(\beta)) \in B$.
Given $\beta \in (0, \beta_c)$, any $\beta_n \to \beta$, $\theta > 0$, $\gamma > 0$, $k \neq 0$, define

$$K_n = K(\beta_n) - \frac{k}{n^\theta} \quad \left[K(\beta) = \frac{e^\beta+2}{4\beta}\right],$$

$$G(x) = \delta(v, 2\gamma + \theta - 1)k\beta x^2 + \delta(v, 4\gamma - 1)c_4 x^4 \quad [c_4 = c_4(\beta, K)].$$

(a) Assume that $v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}$ equals 0. Then

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \Longrightarrow \exp[-G(x)]\, dx.$$

(b) $v = 0$ if and only case 1, 2, 3, or 4 holds: 4 forms of G.

case	**values of θ**	**values of γ**	**scaling limit of $S_n/n^{1-\gamma}$**
influence	**speed**		
1	$\theta > \frac{1}{2}$	$\gamma = \frac{1}{4}$	$\exp(-c_4 x^4)\, dx$
B	fast		$c_4 > 0$, $k \in \mathbb{R}$
2	$\theta \in (0, \frac{1}{2})$	$\gamma = \frac{1-\theta}{2} \in (\frac{1}{4}, \frac{1}{2})$	$\exp(-k\beta x^2)\, dx$
A	slow		$k > 0$
3–4	$\theta = \frac{1}{2}$	$\gamma = \frac{1}{4}$	$\exp(-k\beta x^2 - c_4 x^4)\, dx$
$A + B$	critical		$k > 0$ or $k < 0$

Table 14: Values of θ and γ and scaling limits in part (b) of Theorem 14

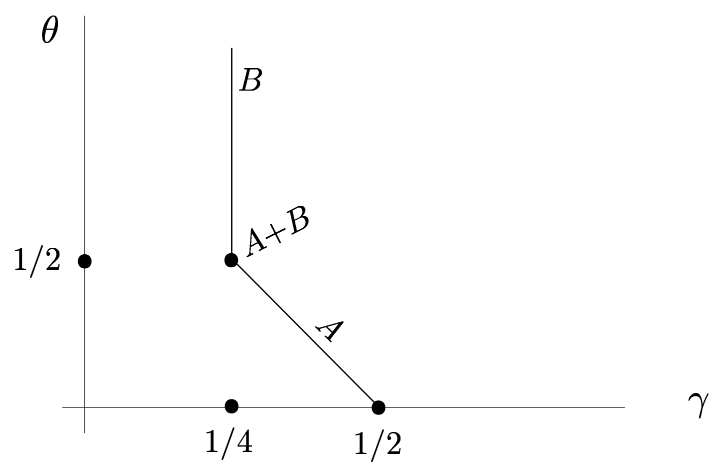

Figure 13: Influence of A and B on scaling limits when $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$

Theorem 15. Scaling limits for $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c)) \in C$.
Given $\alpha > 0$, $\theta > 0$, $\gamma > 0$, $b \neq 0$, and $k \neq 0$, define

$$\beta_n = \log\left(4 - \frac{b}{n^\alpha}\right) = \log\left(e^{\beta_c} - \frac{b}{n^\alpha}\right), K_n = K(\beta_n) - \frac{k}{n^\theta},$$

$$G(x) = \delta(w, 2\gamma + \theta - 1)k\beta_c x^2 + \delta(w, 4\gamma + \alpha - 1)\tfrac{3b}{16}x^4 + \delta(w, 6\gamma - 1)\tfrac{9}{40}x^6.$$

(a) Assume that $w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}$ equals 0. Then

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \Longrightarrow \exp[-G(x)]\, dx.$$

(b) $w = 0$ if and only one of the cases 1–13 holds: 13 forms of G.

case / **influence**	**values of α** / **values of θ**	**values of γ** / **speed**	**scaling limit of $S_n/n^{1-\gamma}$**
1	$\alpha > \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-c_6 x^6)\, dx$
C	$\theta > \frac{2}{3}$	fast	$c_6 > 0$, $b \in \mathbb{R}$, $k \in \mathbb{R}$
2	$\alpha \in (0, \frac{1}{3})$	$\gamma = \frac{1-\alpha}{4} \in (\frac{1}{6}, \frac{1}{4})$	$\exp(-b\bar{c}_4 x^4)\, dx$
B	$\theta > \frac{\alpha+1}{2}$	intermediate	$\bar{c}_4 > 0$, $b > 0$, $k \in \mathbb{R}$
3a	$\alpha > 0$	$\gamma = \frac{1-\theta}{2} \in (\frac{1}{4}, \frac{1}{2})$	$\exp(-k\beta_c x^2)\, dx$
A	$\theta \in (0, \frac{1}{2})$	slow	$k > 0$, $b \in \mathbb{R}$
3b	$\theta \in [\frac{1}{2}, \frac{2}{3})$	$\gamma = \frac{1-\theta}{2} \in (\frac{1}{6}, \frac{1}{4}]$	$\exp(-k\beta_c x^2)\, dx$
A	$\alpha > 2\theta - 1$	slow	$k > 0$, $b \in \mathbb{R}$
4–5	$\alpha = \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-b\bar{c}_4 x^4 - c_6 x^6)\, dx$
$B + C$	$\theta > \frac{2}{3}$	critical	$b > 0$ or $b < 0$. $k \in \mathbb{R}$
6–7	$\alpha > \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-k\beta_c x^2 - c_6 x^6)\, dx$
$A + C$	$\theta = \frac{2}{3}$	critical	$k > 0$ or $k < 0$, $b \in \mathbb{R}$
8–9	$\alpha \in (0, \frac{1}{3})$	$\gamma = \frac{1-\alpha}{4} \in (\frac{1}{6}, \frac{1}{4})$	$\exp(-k\beta_c x^2 - b\bar{c}_4 x^4)\, dx$
$A + B$	$\theta = \frac{\alpha+1}{2} \in (\frac{1}{2}, \frac{2}{3})$	critical	$k > 0$ or $k < 0$, $b > 0$
10–13	$\alpha = \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-k\beta_c x^2 - b\bar{c}_4 x^4 - c_6 x^6)\, dx$
$A + B + C$	$\theta = \frac{2}{3}$	critical	$k > 0$ or $k < 0$, $b > 0$ or $b < 0$

Table 15: Values of α, θ, and γ and scaling limits in part (b) of Theorem 15

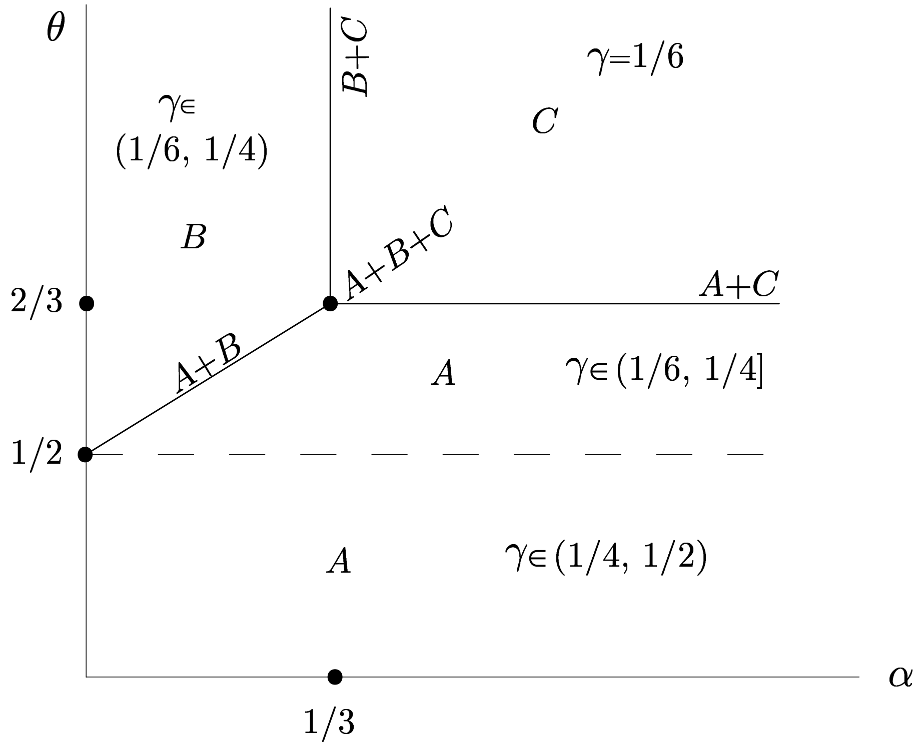

Figure 14: Influence of C, B, and A when $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$

- Scaling limits reveal a geometric structure in a model without geometry.
- Each point in α-θ plane corresponds to a curve in β-K plane.
- Point $A + B + C$ is a multiple critical point, analogous to the tricritical point in the bifurcation diagram of the BEG model.

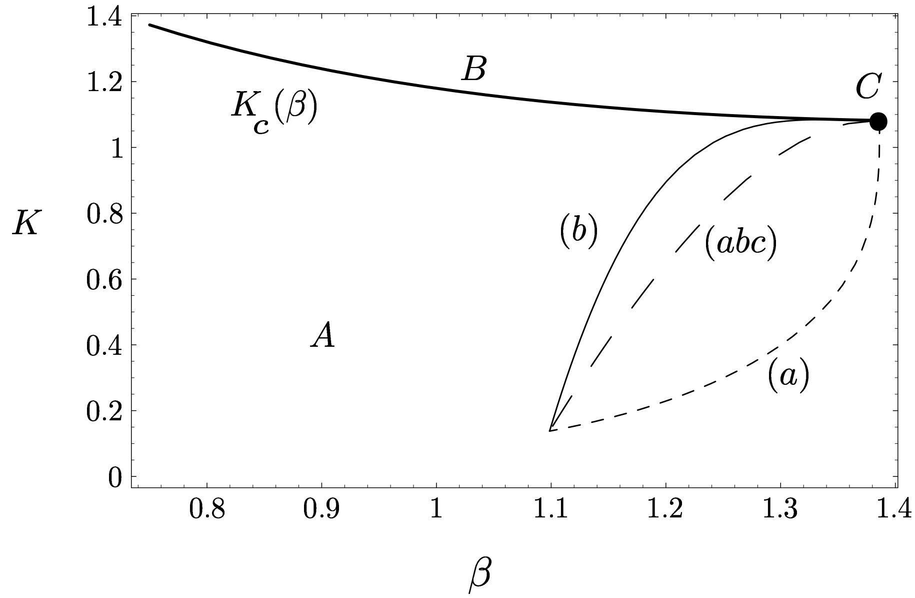

Figure 15: Three choices of (β_n, K_n) that show the influence of A, of B, and of A, B, and C

- $\beta_n = \log\left(e^{\beta_c} - \dfrac{b}{n^\alpha}\right)$ and $K_n = K(\beta_n) - \dfrac{k}{n^\theta}$
- Curve labeled (a): $\alpha = 1$, $b > 0$, $\theta = \frac{1}{3}$, $k > 0$ (case 3); influence of A
- Curve labeled (b): $\alpha = \frac{1}{4}$, $b > 0$, $\theta = 1$, $k > 0$ (case 2); influence of B
- Curve labeled (abc): $\alpha = \frac{1}{3}$, $b > 0$, $\theta = \frac{2}{3}$, $k > 0$ (case 10); influence of A, B, and C

MDPs for BEG Model

Theorem 16. $(\boldsymbol{\beta_n}, \boldsymbol{K_n}) \longrightarrow (\boldsymbol{\beta}, \boldsymbol{K}) \in \boldsymbol{A}$.
(a) **Scaling limit.** For any $(\beta_n, K_n) \to (\beta, K) \in A$, $\exists c_2 = c_2(\beta, K) > 0 \ni$

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1/2}} \in dx\right\} \Longrightarrow \exp(-c_2 x^2)\, dx.$$

(b) **MDP.** For any $(\beta_n, K_n) \to (\beta, K) \in A$ and any $\gamma \in (0, \frac{1}{2})$

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \asymp \exp(-n^{1-2\gamma} c_2 x^2)\, dx.$$

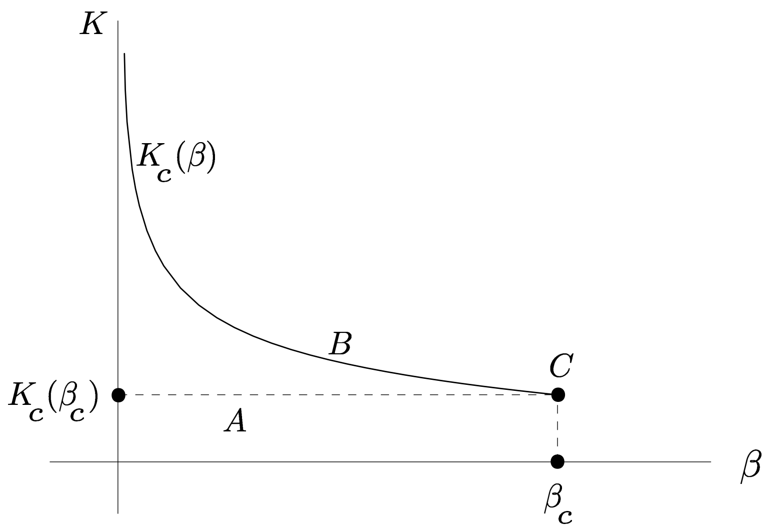

Figure 16: The sets A, B, and C

- For $(\beta, K) \in B \cup C$, the MDPs reveal a geometric structure in a model without geometry.
  - For $(\beta, K) \in B$, sequences (β_n, K_n) give rise to 4 MDPs: arbitrary $\beta_n \to \beta$, special $K_n \to (K_c(\beta))^+$ or $K_n \to (K_c(\beta))^-$.
  - For $(\beta, K) \in C$, sequences (β_n, K_n) give rise to 13 MDPs: special $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$ from SW, SE, NE, NW.
  - For $(\beta, K) \in B \cup C$, the form of the MDP depends on the speed and the direction of approach of (β_n, K_n) to (β, K).

Given $\beta \in (0, \beta_c)$, any $\beta_n \to \beta$, $\theta > 0$, $\gamma > 0$, and $k \neq 0$, define

$$K_n = K(\beta_n) - \frac{k}{n^\theta},$$

$$G(x) = \delta(v, 2\gamma + \theta - 1)k\beta x^2 + \delta(v, 4\gamma - 1)c_4 x^4 \quad [c_4 = c_4(\beta, K)].$$

Theorem 17. $(\boldsymbol{\beta_n, K_n}) \to (\boldsymbol{\beta, K_c(\beta)}) \in \boldsymbol{B}$.
(a) **Scaling limits.** Assume that $v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}$ equals 0. Then

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \Longrightarrow \exp[-G(x)]\,dx.$$

(b) $v = 0$ if and only case 1, 2, 3, or 4 holds.

case / influence	values of θ	values of γ	scaling limit of $S_n/n^{1-\gamma}$
1 / B	$\theta > \frac{1}{2}$	$\gamma = \frac{1}{4}$	$\exp(-c_4 x^4)\,dx$ $c_4 > 0$, $k \in \mathbb{R}$
2 / A	$\theta \in (0, \frac{1}{2})$	$\gamma = \frac{1-\theta}{2} \in (\frac{1}{4}, \frac{1}{2})$	$\exp(-k\beta x^2)\,dx$ $k > 0$
3–4 / $A + B$	$\theta = \frac{1}{2}$	$\gamma = \frac{1}{4}$	$\exp(-k\beta x^2 - c_4 x^4)\,dx$ $k > 0$ or $k < 0$

Table 14: Values of θ and γ and scaling limits in part (b) of Theorem 14

(c) **MDPs.** Assume that $v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}$ satisfies $v < 0$. Then

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \asymp \exp[-n^{-v}\Gamma(x)]\,dx \quad [\Gamma(x) = G(x) - \inf_{y \in \mathbb{R}} G(y)].$$

(d) $v < 0$ if and only case 1, 2, 3, or 4 holds.

case / influence	values of γ	values of θ	exp'l speed	function G in rate function Γ
1 / B	$\gamma \in (0, \frac{1}{4})$	$\theta > 2\gamma$	$n^{1-4\gamma}$	$c_4 x^4$ $c_4 > 0$, $k \in \mathbb{R}$
2a / A	$\gamma \in (0, \frac{1}{4}]$	$\theta \in (0, 2\gamma)$	$n^{1-2\gamma-\theta}$	$k\beta x^2$ $k > 0$
2b / A	$\gamma \in (\frac{1}{4}, \frac{1}{2})$	$\theta \in (0, 1 - 2\gamma)$	$n^{1-2\gamma-\theta}$	$k\beta x^2$ $k > 0$
3–4 / $A + B$	$\gamma \in (0, \frac{1}{4})$	$\theta = 2\gamma$	$n^{1-4\gamma}$	$k\beta x^2 + c_4 x^4$ $k > 0$ or $k < 0$

Table 17: Values of γ and θ, exponential speeds, and rate functions in part (c) of Theorem 17

Theorem 18. MDPs for $(\boldsymbol{\beta_n}, \boldsymbol{K_n}) \longrightarrow (\boldsymbol{\beta}, \boldsymbol{K_c}(\boldsymbol{\beta})) \in \boldsymbol{B}$. Given any $\beta_n \to \beta$, $\theta > 0$, $\gamma > 0$, and $k \neq 0$, define

$$K_n = K(\beta_n) - \frac{k}{n^\theta}.$$

(a) Assume that $v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}$ satisfies $v < 0$. Then

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \asymp \exp[-n^{-v}\Gamma(x)]\,dx \quad [\Gamma(x) = G(x) - \inf_{y\in\mathbb{R}} G(y)].$$

(b) $v < 0$ if and only case 1, 2, 3, or 4 holds.

case / influence	values of γ	values of θ	exp'l speed	function G in rate function Γ
1 / B	$\gamma \in (0, \frac{1}{4})$	$\theta > 2\gamma$	$n^{1-4\gamma}$	$c_4 x^4$, $c_4 > 0$, $k \in \mathbb{R}$
2a / A	$\gamma \in (0, \frac{1}{4}]$	$\theta \in (0, 2\gamma)$	$n^{1-2\gamma-\theta}$	$k\beta x^2$, $k > 0$
2b / A	$\gamma \in (\frac{1}{4}, \frac{1}{2})$	$\theta \in (0, 1-2\gamma)$	$n^{1-2\gamma-\theta}$	$k\beta x^2$, $k > 0$
3–4 / $A + B$	$\gamma \in (0, \frac{1}{4})$	$\theta = 2\gamma$	$n^{1-4\gamma}$	$k\beta x^2 + c_4 x^4$, $k > 0$ or $k < 0$

Table 18: Values of γ and θ, exponential speeds, and rate functions in part (b) of Theorem 18

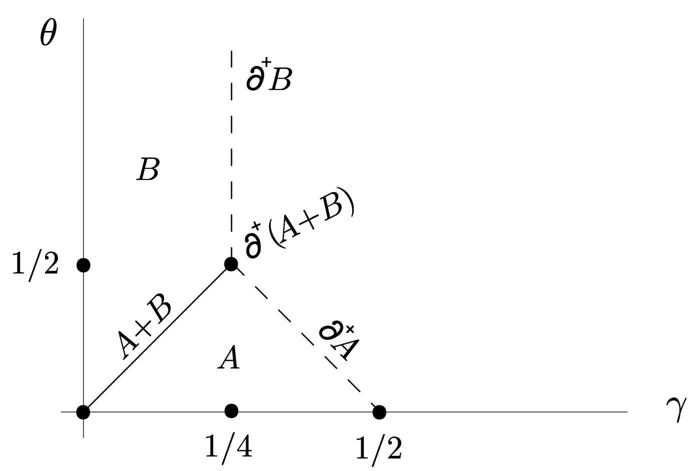

Figure 17: Influence of B and A on MDPs when $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$

- The sets labeled $\partial^+ B$, $\partial^+ A$, and $\partial^+(A+B)$ are the subsets leading to the influence of B, A, and $A + B$, respectively, in the scaling limits for $(\beta_n, K_n) \to (\beta, K) \in B$.

Phase Transitions in BEG Model

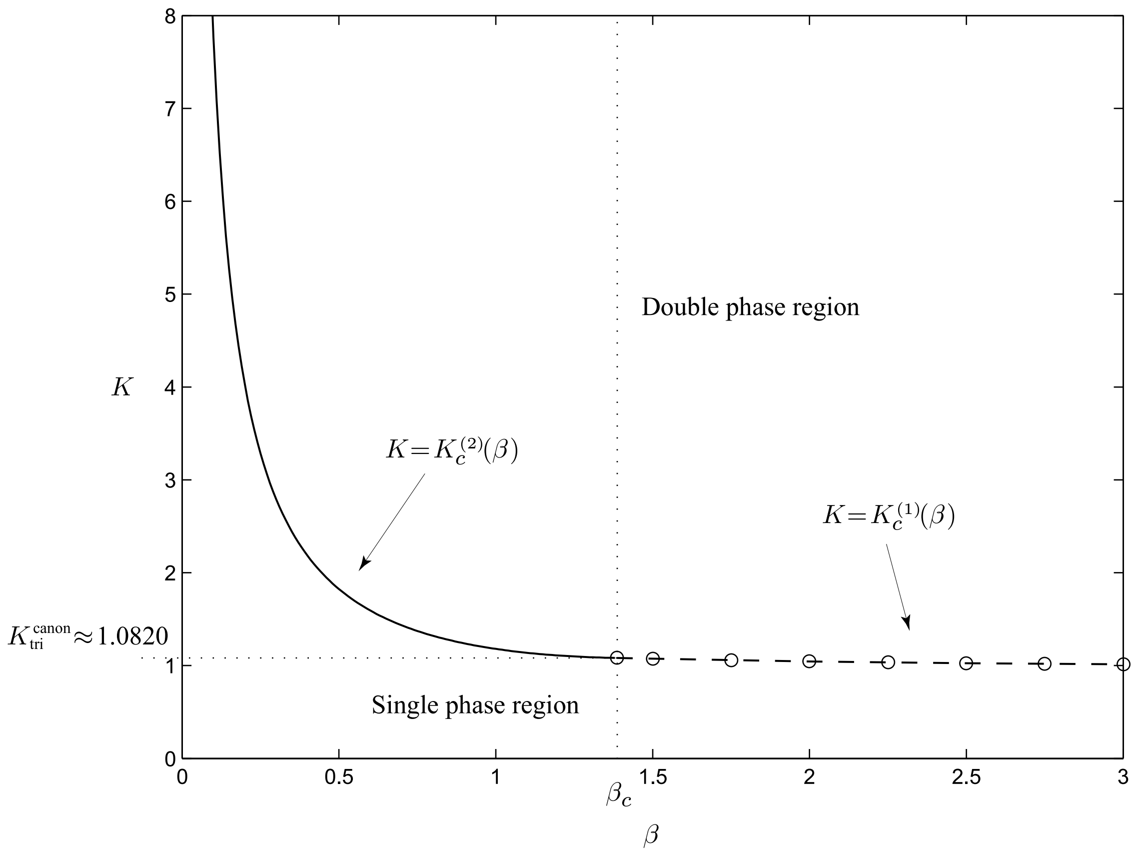

Figure 18: Bifurcation diagram for the BEG model

- Continuous bifurcation (second-order phase transition) for $0 < \beta < \beta_c = \log 4$ as $K \nearrow K_c^{(2)}(\beta)$
- Discontinuous bifurcation (first-order phase transition) for $\beta > \beta_c$ as $K \nearrow K_c^{(1)}(\beta)$
- Tricritical point at $(\beta_c, K_c^{(2)}(\beta_c))$ separates the two phase transitions
- Unique phase for (β, K) under the two phase-transition curves
- Double phase for (β, K) above the two phase-transition curves

Corollary 19. Unexpected: trace of double phase in MDP for $(\boldsymbol{\beta_n}, \boldsymbol{K_n}) \longrightarrow (\boldsymbol{\beta}, \boldsymbol{K_c(\beta)}) \in \boldsymbol{B}$. Given any $\beta_n \to \beta$, $\theta > 0$, $\gamma > 0$, and $k \neq 0$, define

$$K_n = K(\beta_n) - k/n^\theta.$$

Assume that $v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}$ satisfies $v < 0$. Then

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \asymp \exp[-n^{-v}\Gamma(x)]\, dx \quad [\Gamma(x) = G(x) - \inf_{y \in \mathbb{R}} G(y)].$$

case influence	values of γ	values of θ	exp'l speed	function G in rate function Γ
1 B	$\gamma \in (0, \frac{1}{4})$	$\theta > 2\gamma$	$n^{1-4\gamma}$	$c_4 x^4$ $c_4 > 0$, $k \in \mathbb{R}$
2a A	$\gamma \in (0, \frac{1}{4}]$	$\theta \in (0, 2\gamma)$	$n^{1-2\gamma-\theta}$	$k\beta x^2$ $k > 0$
2b A	$\gamma \in (\frac{1}{4}, \frac{1}{2})$	$\theta \in (0, 1-2\gamma)$	$n^{1-2\gamma-\theta}$	$k\beta x^2$ $k > 0$
3–4 $A + B$	$\gamma \in (0, \frac{1}{4})$	$\theta = 2\gamma$	$n^{1-4\gamma}$	$k\beta x^2 + c_4 x^4$ $k > 0$ or $k < 0$

Table 18: Values of γ and θ, exponential speeds, and rate functions in part (b) of Theorem 18

(a) For any $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$,

$$P_{n,\beta_n,K_n}\{S_n/n \in dx\} \Longrightarrow \delta_0.$$

(b) In case 4, for any $\beta_n \to \beta$, $\gamma \in (0, \frac{1}{4})$, and $k < 0$, define

$$K_n = K(\beta_n) + |k|/n^{2\gamma}.$$

Then $(\beta_n, K_n) \to (\beta, K)$ from the double-phase region above the curve B, $\Gamma(x)$ has global minimum points at $\pm x(\beta, k) = \pm\sqrt{|k|\beta/[2c_4]}$, and

$$P_{n,\beta_n,K_n}\left\{S_n/n^{1-\gamma} \in dx\right\} \Longrightarrow \tfrac{1}{2}\left(\delta_{x(\beta,k)} + \delta_{-x(\beta,k)}\right).$$

- Even more fascinating behavior in MDPs for $(\beta_c, K_c(\beta_c)) \in C$ when Γ has multiple global min. points.

Theorem 20. MDPs for $(\boldsymbol{\beta_n}, \boldsymbol{K_n}) \longrightarrow (\boldsymbol{\beta_c}, \boldsymbol{K_c}(\boldsymbol{\beta_c})) \in \boldsymbol{C}$.
Given $\alpha > 0$, $\theta > 0$, $\gamma > 0$, $b \neq 0$, and $k \neq 0$, define

$$\beta_n = \log\left(4 - \frac{b}{n^\alpha}\right) = \log\left(e^{\beta_c} - \frac{b}{n^\alpha}\right), K_n = K(\beta_n) - \frac{k}{n^\theta},$$

$$G(x) = \delta(w, 2\gamma + \theta - 1)k\beta_c x^2 + \delta(w, 4\gamma + \alpha - 1)\tfrac{3b}{16}x^4 + \delta(w, 6\gamma - 1)\tfrac{9}{40}x^6.$$

(a) Assume that $w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}$ satisfies $w < 0$. Then

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \asymp \exp[-n^{-w}\Gamma(x)]\,dx \quad [\Gamma(x) = G(x) - \inf_{y\in\mathbb{R}} G(y)].$$

(b) $w < 0$ if and only one of the cases 1–13 holds.

case **influence**	**values of γ**	**values of α** **values of θ**	**exp'l** **speed**	**function G in** **rate function Γ**
1 C	$\gamma \in (0, \frac{1}{6})$	$\alpha > 2\gamma$ $\theta > 4\gamma$	$n^{1-6\gamma}$	$c_6 x^6$ $c_6 > 0$, $b \in \mathbb{R}$, $k \in \mathbb{R}$
2 B	$\gamma \in (0, \frac{1}{4})$	$\alpha \in (0, \min\{2\gamma, 1 - 4\gamma\})$ $\theta > 2\gamma + \alpha$	$n^{1-4\gamma-\alpha}$	$b\bar{c}_4 x^4$ $b > 0$, $\bar{c}_4 > 0$, $k \in \mathbb{R}$
3 A	$\gamma \in (0, \frac{1}{2})$	$\theta \in (0, \min\{4\gamma, 1 - 2\gamma\})$ $\alpha > \max(\theta - 2\gamma, 0)$	$n^{1-2\gamma-\theta}$	$k\beta_c x^2$ $k > 0$, $b \in \mathbb{R}$
4–5 $B + C$	$\gamma \in (0, \frac{1}{6})$	$\alpha = 2\gamma$ $\theta > 4\gamma$	$n^{1-6\gamma}$	$b\bar{c}_4 x^4 + c_6 x^6$ $b > 0$ or $b < 0$, $k \in \mathbb{R}$
6–7 $A + C$	$\gamma \in (0, \frac{1}{6})$	$\alpha > 2\gamma$ $\theta = 4\gamma$	$n^{1-6\gamma}$	$k\beta_c x^2 + c_6 x^6$ $k > 0$ or $k < 0$, $b \in \mathbb{R}$
8–9 $A + B$	$\gamma = (0, \frac{1}{4})$	$\alpha \in (0, \min\{2\gamma, 1 - 4\gamma\})$ $\theta = 2\gamma + \alpha$	$n^{1-4\gamma-\alpha}$	$k\beta_c x^2 + b\bar{c}_4 x^4$ $k > 0$ or $k < 0$, $b > 0$
10–13 $A + B + C$	$\gamma \in (0, \frac{1}{6})$	$\alpha = 2\gamma$ $\theta = 4\gamma$	$n^{1-6\gamma}$	$k\beta_c x^2 + b\bar{c}_4 x^4 + c_6 x^6$ $k > 0$ or $k < 0$ $b > 0$ or $b < 0$

Table 20: Values of γ, α, θ, exponential speeds, and rate functions in part (b) of Theorem 20

- In each of the cases 1–13 the function G appearing in the last column equals the function G appearing in the corresponding case of the limiting density $\exp[-G(x)]dx$ in the scaling limit in Theorem 15.
- In general, the values of α, θ, and γ leading to the 13 cases of the MDPs differ from the values of α, θ, and γ leading to the corresponding case of the scaling limit in Theorem 15.

Phase Transitions in BEG Model

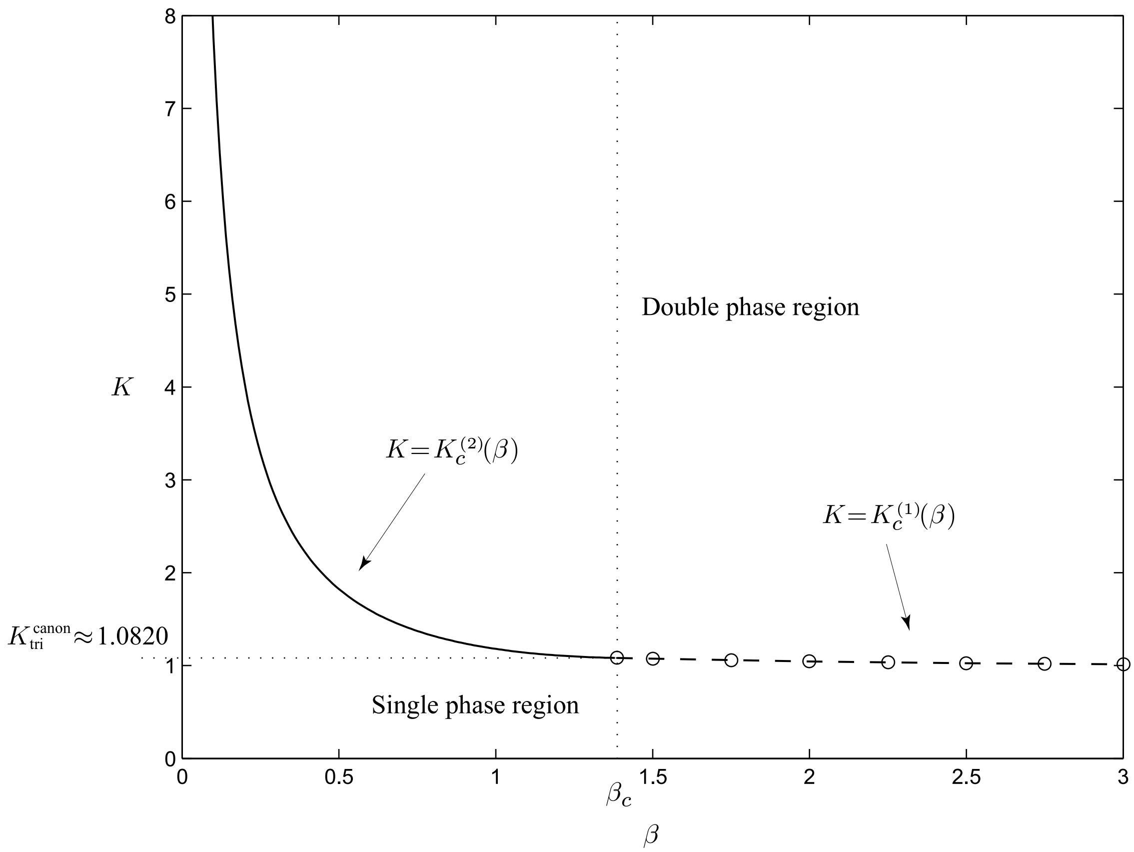

Figure 19: Bifurcation diagram for the BEG model

- Continuous bifurcation (second-order phase transition) for $0 < \beta < \beta_c = \log 4$ as $K \nearrow K_c^{(2)}(\beta)$
- Discontinuous bifurcation (first-order phase transition) for $\beta > \beta_c$ as $K \nearrow K_c^{(1)}(\beta)$
- Tricritical point at $(\beta_c, K_c^{(2)}(\beta_c))$ separates the two phase transitions
- Unique phase for (β, K) under the two phase-transition curves
- Double phase for (β, K) above the two phase-transition curves

Corollary 21. Unexpected: traces of double phase and triple phase in MDPs for $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$.

$$\beta_n = \log(e^{\beta_c} - b/n^{\alpha}), \quad K_n = K(\beta_n) - k/n^{\theta},$$

$$G(x) = \delta(w, 2\gamma + \theta - 1) k\beta_c x^2 + \delta(w, 4\gamma + \alpha - 1)\tfrac{3b}{16}x^4 + \delta(w, 6\gamma - 1)\tfrac{9}{40}x^6.$$

$w < 0$ implies $P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in dx\} \asymp \exp[-n^{-w}\Gamma(x)]\, dx \quad [\Gamma(x) = G(x) - \inf_{y\in\mathbb{R}} G(y)]$.

case / **influence**	**values of γ**	**values of α** / **values of θ**	**exp'l speed**	**function G in rate function Γ**
1	$\gamma \in (0, \frac{1}{6})$	$\alpha > 2\gamma$	$n^{1-6\gamma}$	$c_6 x^6$
C		$\theta > 4\gamma$		$c_6 > 0$, $b \in \mathbb{R}$, $k \in \mathbb{R}$
2	$\gamma \in (0, \frac{1}{4})$	$\alpha \in (0, \min\{2\gamma, 1 - 4\gamma\})$	$n^{1-4\gamma-\alpha}$	$b\bar{c}_4 x^4$
B		$\theta > 2\gamma + \alpha$		$b > 0$, $\bar{c}_4 > 0$, $k \in \mathbb{R}$
3	$\gamma \in (0, \frac{1}{2})$	$\theta \in (0, \min\{4\gamma, 1 - 2\gamma\})$	$n^{1-2\gamma-\theta}$	$k\beta_c x^2$
A		$\alpha > \max(\theta - 2\gamma, 0)$		$k > 0$, $b \in \mathbb{R}$
4–5	$\gamma \in (0, \frac{1}{6})$	$\alpha = 2\gamma$	$n^{1-6\gamma}$	$b\bar{c}_4 x^4 + c_6 x^6$
$B + C$		$\theta > 4\gamma$		$b > 0$ or $b < 0$, $k \in \mathbb{R}$
6–7	$\gamma \in (0, \frac{1}{6})$	$\alpha > 2\gamma$	$n^{1-6\gamma}$	$k\beta_c x^2 + c_6 x^6$
$A + C$		$\theta = 4\gamma$		$k > 0$ or $k < 0$, $b \in \mathbb{R}$
8–9	$\gamma = (0, \frac{1}{4})$	$\alpha \in (0, \min\{2\gamma, 1 - 4\gamma\})$	$n^{1-4\gamma-\alpha}$	$k\beta_c x^2 + b\bar{c}_4 x^4$
$A + B$		$\theta = 2\gamma + \alpha$		$k > 0$ or $k < 0$, $b > 0$
10–13	$\gamma \in (0, \frac{1}{6})$	$\alpha = 2\gamma$	$n^{1-6\gamma}$	$k\beta_c x^2 + b\bar{c}_4 x^4 + c_6 x^6$
$A + B + C$		$\theta = 4\gamma$		$k > 0$ or $k < 0$
				$b > 0$ or $b < 0$

Table 20: Values of γ, α, θ, exponential speeds, and rate functions in part (b) of Theorem 20

(a) In cases 5, 7, 9, 11, and 13, $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$ from the double-phase region, Γ has global minimum points at $\pm x(b, k)$, $x(b, k) > 0$,

$$P_{n,\beta_n,K_n}\left\{S_n/n^{1-\gamma} \in dx\right\} \Longrightarrow \tfrac{1}{2}\left(\delta_{x(\beta,k)} + \delta_{-x(\beta,k)}\right).$$

(b) In case 12, for fixed $b < 0$ and $n \in \mathbb{N}$, as $k > 0$ decreases, (β_n, K_n) crosses the first-order critical curve from below, Γ has global minimum points at (i) 0, (ii) 0 and $\pm x(b, k)$, and (iii) $\pm x(b, k)$ [see Figures 5–9], and

$$\text{(i) } P_{n,\beta_n,K_n}\left\{S_n/n^{1-\gamma} \in dx\right\} \Longrightarrow \delta_0,$$

$$\text{(ii) } P_{n,\beta_n,K_n}\left\{S_n/n^{1-\gamma} \in dx\right\} \Longrightarrow \tfrac{1}{2}\left(\delta_{x(\beta,k)} + \delta_{-x(\beta,k)}\right),$$

$$\text{(iii) } P_{n,\beta_n,K_n}\left\{S_n/n^{1-\gamma} \in dx\right\} \Longrightarrow \lambda_0\delta_0 + \lambda_1\left(\delta_{x(\beta,k)} + \delta_{-x(\beta,k)}\right).$$

Ideas Behind the Proofs

All the magic is in $G_{\beta,K}(x) = \beta K x^2 - c_\beta(2\beta K x)$.

1. **Equilibrium macrostates.** $\mathcal{E}_{\beta,K} = \{x \in [-1,1] : x \text{ minimizes } G_{\beta,K}(x)\}$.
2. **Distribution of $S_n/n^{1-\gamma}$.** Let $W_n = N(0,(2\beta_n K_n)^{-1})$ on $(\Omega,\mathcal{F},Q)$. Then
$$P_{n,\beta_n,K_n} \times Q\left\{\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}} \in dx\right\} \propto \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\,dx.$$
3. **Taylor expansions and scaling limits.** Let $(\beta_n,K_n) \to (\beta,K) \in A \cup B \cup C$. To determine scaling limits of $S_n/n^{1-\gamma}$, use a Taylor expansion of $G_{\beta_n,K_n}(x/n^\gamma)$.
   - If $\mathcal{E}_{\beta_n,K_n} = \{0\}$, then expand $G_{\beta_n,K_n}(x/n^\gamma)$ around its unique global minimum point 0.
   - **Key technical issue.** What if $\mathcal{E}_{\beta_n,K_n} = \{\pm z(\beta_n,K_n)\}$?
     - Should $G_{\beta_n,K_n}(x/n^\gamma)$ be expanded around 0 or around $\pm z(\beta_n,K_n)$, which converge to 0 as $n \to \infty$?
     - If expand around $\pm z(\beta_n,K_n)$, then must determine rate at which $z(\beta_n,K_n) \to 0$. Complicated.
     - Because of error estimate in item 6, always expand $G_{\beta_n,K_n}(x/n^\gamma)$ around 0.
   - **Type of $0 \in \mathcal{E}_{\beta,K}$**
     - For $(\beta,K) \in A$, $G^{(2)}_{\beta,K}(0) > 0$. Write
     $$G_{\beta_n,K_n}(x/n^\gamma) = \text{2nd-order Taylor expansion around 0.}$$
     - For $(\beta,K) = (\beta,K_c(\beta)) \in B$, $G^{(2)}_{\beta,K}(0) = 0$, $G^{(4)}_{\beta,K}(0) > 0$. Write
     $$G_{\beta_n,K_n}(x/n^\gamma) = \text{4th-order Taylor expansion around 0.}$$
     - For $(\beta,K) = (\beta_c,K_c(\beta_c)) \in C$, $G^{(2)}_{\beta,K}(0) = 0$, $G^{(4)}_{\beta,K}(0) = 0$, $G^{(6)}_{\beta,K}(0) > 0$. Write
     $$G_{\beta_n,K_n}(x/n^\gamma) = \text{6th-order Taylor expansion around 0.}$$

4. **Determining special sequence K_n.** Take $(\beta, K) = (\beta, K_c(\beta)) \in B$.

- Define $K(\beta_n) = (e^\beta + 2)/(4\beta)$. Expand

$$nG_{\beta_n,K_n}(x/n^\gamma) = \frac{1}{n^{2\gamma-1}} \frac{G^{(2)}_{\beta_n,K_n}(0)}{2!} x^2 + \frac{1}{n^{4\gamma-1}} \frac{G^{(4)}_{\beta_n,K_n}(0)}{4!} x^4 + \text{error},$$

$$G^{(2)}_{\beta_n,K_n}(0) = \frac{2\beta_n K_n[K(\beta_n) - K_n]}{K(\beta_n)} \to G^{(2)}_{\beta,K_c(\beta)}(0) = 0,$$

$$\frac{G^{(4)}_{\beta_n,K_n}(0)}{4!} \to c_4 > 0.$$

- **Key insight.** Choose K_n so that $G^{(2)}_{\beta_n,K_n}(0) \to 0$ at a rate $1/n^\theta$. Let $\beta_n \to \beta$ be arbitrary. For $k \neq 0$ define

$$K_n = K(\beta_n) - \frac{k}{n^\theta}.$$

  Then $(\beta_n, K_n) \to (\beta, K(\beta)) = (\beta, K_c(\beta))$.

- Substitute into Taylor expansion:

$$nG_{\beta_n,K_n}(x/n^\gamma) \approx \frac{1}{n^{2\gamma+\theta-1}} k\beta x^2 + \frac{1}{n^{4\gamma-1}} c_4 x^4.$$

- Define $v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}$. Then

$$\lim_{n\to\infty} nG_{\beta_n,K_n}(x/n^\gamma) = G(x) = \delta(v, 2\gamma + \theta - 1)k\beta x^2 + \delta(v, 4\gamma - 1)c_4 x^4.$$

- Assume that $v = 0$. If $\gamma \in (0, \frac{1}{2})$, then $W_n/n^{1/2-\gamma} \to 0$, and

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \propto \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx \Longrightarrow \exp[-G(x)]\, dx.$$

- $v = 0$ if and only if case 1, 2, 3, or 4 holds.

case / **influence**	**values of θ** / **speed**	**values of γ**	**scaling limit of $S_n/n^{1-\gamma}$**
1 / B	$\theta > \frac{1}{2}$ / fast	$\gamma = \frac{1}{4}$	$\exp(-c_4 x^4)\, dx$ $c_4 > 0$, $k \in \mathbb{R}$
2 / A	$\theta \in (0, \frac{1}{2})$ / slow	$\gamma = \frac{1-\theta}{2} \in (\frac{1}{4}, \frac{1}{2})$	$\exp(-k\beta x^2)\, dx$ $k > 0$
3–4 / $A + B$	$\theta = \frac{1}{2}$ / critical	$\gamma = \frac{1}{4}$	$\exp(-k\beta x^2 - c_4 x^4)\, dx$ $k > 0$ or $k < 0$

Values of θ and γ and scaling limits for $(\beta, K) = (\beta, K_c(\beta)) \in B$

5. Take $(\beta, K) = (\beta_c, K_c(\beta_c)) \in C$.

- For $b \neq 0$ define

$$\beta_n = \log\left(4 - \frac{b}{n^\alpha}\right) = \log\left(e^{\beta_c} - \frac{b}{n^\alpha}\right).$$

Then $G^{(4)}_{\beta_n,K_n}(0) \to G^{(4)}_{\beta_c,K_c}(0) = 0$ at a rate $1/n^\alpha$.

- Choose K_n as in item 4 and proceed as in item 4.
- Define $w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}$. Then

$$\lim_{n\to\infty} nG_{\beta_n,K_n}(x/n^\gamma) = G(x)$$
$$= \delta(w, 2\gamma + \theta - 1)k\beta_c x^2 + \delta(w, 4\gamma + \alpha - 1)\tfrac{3b}{16}x^4 + \delta(w, 6\gamma - 1)\tfrac{9}{40}x^6.$$

- Assume that $w = 0$. If $\gamma \in (0, \frac{1}{2})$, then $W_n/n^{1/2-\gamma} \to 0$, and

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \propto \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\,dx \Longrightarrow \exp[-G(x)]\,dx.$$

- $w = 0$ if and only if one of the cases 1–13 holds.

case	**values of α**	**values of γ**	**scaling limit of $S_n/n^{1-\gamma}$**
influence	**values of θ**	**speed**	
1	$\alpha > \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-c_6 x^6)\,dx$
C	$\theta > \frac{2}{3}$	fast	$c_6 > 0$, $b \in \mathbb{R}$, $k \in \mathbb{R}$
2	$\alpha \in (0, \frac{1}{3})$	$\gamma = \frac{1-\alpha}{4} \in (\frac{1}{6}, \frac{1}{4})$	$\exp(-b\bar{c}_4 x^4)\,dx$
B	$\theta > \frac{\alpha+1}{2}$	intermediate	$\bar{c}_4 > 0$, $b > 0$, $k \in \mathbb{R}$
3a	$\alpha > 0$	$\gamma = \frac{1-\theta}{2} \in (\frac{1}{4}, \frac{1}{2})$	$\exp(-k\beta_c x^2)\,dx$
A	$\theta \in (0, \frac{1}{2})$	slow	$k > 0$, $b \in \mathbb{R}$
3b	$\theta \in [\frac{1}{2}, \frac{2}{3})$	$\gamma = \frac{1-\theta}{2} \in (\frac{1}{6}, \frac{1}{4}]$	$\exp(-k\beta_c x^2)\,dx$
A	$\alpha > 2\theta - 1$	slow	$k > 0$, $b \in \mathbb{R}$
4–5	$\alpha = \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-b\bar{c}_4 x^4 - c_6 x^6)\,dx$
$B + C$	$\theta > \frac{2}{3}$	critical	$b > 0$ or $b < 0$. $k \in \mathbb{R}$
6–7	$\alpha > \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-k\beta_c x^2 - c_6 x^6)\,dx$
$A + C$	$\theta = \frac{2}{3}$	critical	$k > 0$ or $k < 0$, $b \in \mathbb{R}$
8–9	$\alpha \in (0, \frac{1}{3})$	$\gamma = \frac{1-\alpha}{4} \in (\frac{1}{6}, \frac{1}{4})$	$\exp(-k\beta_c x^2 - b\bar{c}_4 x^4)\,dx$
$A + B$	$\theta = \frac{\alpha+1}{2} \in (\frac{1}{2}, \frac{2}{3})$	critical	$k > 0$ or $k < 0$, $b > 0$
10–13	$\alpha = \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-k\beta_c x^2 - b\bar{c}_4 x^4 - c_6 x^6)\,dx$
$A + B + C$	$\theta = \frac{2}{3}$	critical	$k > 0$ or $k < 0$, $b > 0$ or $b < 0$

Values of α, θ, and γ and scaling limits for $(\beta, K) = (\beta_c, K_c(\beta_c)) \in C$

6. Main technical lemma to prove scaling limits

- Using Taylor expansions and the dominated convergence theorem, we prove that for $(\beta, K) \in A \cup B \cup C$, any $f \in C_b(\mathbb{R})$, and a suitable $R > 0$

$$\lim_{n\to\infty} \int_{\{|x|<Rn^\gamma\}} f(x)\, \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx = \int_{\mathbb{R}} f(x)\, \exp[-G(x)]\, dx.$$

- In order to complete the proof, we must show that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \int_{\{|x|\geq Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx = 0.$$

  - Originally we could prove this only when $\mathcal{E}_{\beta_n,K_n} = \{0\}$, an assumption that would restrict the choice of $(\beta_n, K_n) \to (\beta, K)$. Let $\varphi(\beta_n, K_n) = \inf_{x\in\mathbb{R}} G_{\beta_n,K_n}(x)$. When $\mathcal{E}_{\beta_n,K_n} = \{\pm z(\beta_n, K_n)\}$, we could prove that the left-hand side of the last display multiplied by $\exp[n\varphi(\beta_n, K_n)]$ converges to 0. But this would have forced us to consider the Taylor expansions of $G_{\beta_n,K_n}(x/n^\gamma)$ around $\pm z(\beta_n, K_n)$, which in turn would have introduced further complications (e.g., determining the rate at which $z(\beta_n, K_n) \to 0$).

- Prove this using the LDP

$$P_{n,\beta_n,K_n} \times Q\left\{\frac{S_n}{n} + \frac{W_n}{n^{1/2}} \in dx\right\} \propto \exp[-nG_{\beta_n,K_n}(x)]\, dx \asymp \exp[-nG_{\beta,K}(x)]\, dx.$$

- Let $y_n = \int_{\{|x|<Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx$. Since $\mathcal{E}_{\beta,K} = \{0\}$, there exists $a_9 > 0$ such that for all $n \in \mathbb{N}$

$$\begin{aligned}
\exp(-na_9) &\geq P_{n,\beta_n,K_n} \times Q\left\{\frac{S_n}{n} + \frac{W_n}{n^{1/2}} \not\in (-R, R)\right\} \\
&= P_{n,\beta_n,K_n} \times Q\left\{\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}} \not\in (-Rn^\gamma, Rn^\gamma)\right\} \\
&= \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx} \cdot \int_{\{|x|\geq Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx \\
&= \frac{z_n}{y_n + z_n}.
\end{aligned}$$

- Since $\lim_{n\to\infty} y_n$ exists, $z_n = \mathrm{O}(\exp(-na_9)) \to 0$. QED

7. Proof of MDPs is done by similar method, but much more delicate.

- For $(\beta, K) \in B$ or $(\beta, K) \in C$, let $u = v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}$ or $u = w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}$. Assume that $u < 0$. Then

$$P_{n,\beta_n,K_n}\left\{\frac{S_n}{n^{1-\gamma}} \in dx\right\} \asymp \exp[-n^{-u}\Gamma(x)]\, dx \quad [\Gamma(x) = G(x) - \inf_{y\in\mathbb{R}} G(y)].$$

- To prove this, use the fact that when $u < 0$,

$$\lim_{n\to\infty} n^{1+u} G_{\beta_n,K_n}(x/n^{\gamma}) = G(x)$$

and $W_n/n^{1/2-\gamma} \to 0$ at a rate faster than $\exp(-n^{-u})$.

- Formally

$$\begin{aligned} &P_{n,\beta_n,K_n}\left\{\tfrac{S_n}{n^{1-\gamma}} \in dx\right\} \\ &\quad\approx P_{n,\beta_n,K_n} \times Q\left\{\tfrac{S_n}{n^{1-\gamma}} + \tfrac{W_n}{n^{1/2-\gamma}} \in dx\right\} \\ &\quad\propto \exp[-nG_{\beta_n,K_n}(x/n^{\gamma})]\, dx \\ &\quad= \exp\left[-n^{-u} \cdot n^{1+u} G_{\beta_n,K_n}(x/n^{\gamma})\right] dx \\ &\quad\approx \exp[-n^{-u}\, G(x)]\, dx \end{aligned}$$

- **Prove MDPs via equivalent Laplace principle.** For any $\psi \in C_b(\mathbb{R})$

$$\lim_{n\to\infty} \frac{1}{n^{-u}} \log \int_{\Lambda^n} \exp\left[n^{-u}\, \psi\left(\frac{S_n}{n^{1-\gamma}}\right)\right] dP_{n,\beta_n,K_n} = \sup_{x\in\mathbb{R}}\{\psi(x) - \Gamma(x)\}.$$

- Formal calculation transformable into a proof:

$$\begin{aligned} &\int_{\Lambda^n} \exp\left[n^{-u}\, \psi\left(\tfrac{S_n}{n^{1-\gamma}}\right)\right] dP_{n,\beta_n,K_n} \\ &\quad\approx \int_{\Lambda^n\times\Omega} \exp\left[n^{-u}\, \psi\left(\tfrac{S_n}{n^{1-\gamma}} + \tfrac{W_n}{n^{1/2-\gamma}}\right)\right] d(P_{n,\beta_n,K_n} \times Q) \\ &\quad\propto \int_{\mathbb{R}} \exp[n^{-u}\psi(x) - nG_{\beta_n,K_n}(x/n^{\gamma})]\, dx \\ &\quad= \int_{\mathbb{R}} \exp\left[n^{-u}\left\{\psi(x) - n^{1+u} G_{\beta_n,K_n}(x/n^{\gamma})\right\}\right] dx \\ &\quad\approx \int_{\mathbb{R}} \exp\left[n^{-u}\{\psi(x) - G(x)\}\right] dx \\ &\quad\approx \exp[n^{-u} \sup_{x\in\mathbb{R}}\{\psi(x) - G(x)\}]\,. \end{aligned}$$

Summary

- We prove scaling limits and MDPs for the BEG model, a mean-field approximation to important spin models in statistical mechanics.
- These limit results are a tool for studying the fine structure of the phase transitions in the model: second-order for $0 < \beta < \beta_c$ and first-order for $\beta > \beta_c$.
- Our goal is to highlight the complex behavior in the neighborhood of the tricritical point, which separates the second-order phase transition and the first-order phase transition.
  - For $(\beta, K) \in A$, 1 scaling limit and 1 MDP.
  - For $(\beta, K) = (\beta, K_c(\beta)) \in B$, 4 scaling limits and 4 MDPs.
  - For $(\beta, K) = (\beta_c, K_c(\beta_c)) \in C$, 13 scaling limits and 13 MDPs.
- The 36 limit theorems reveal a geometric structure in this mean-field model that has no geometry.
- All the magic is in $G_{\beta,K} = \beta K x^2 - c_\beta(2\beta K x)$.
- Similar methods should be applicable to other models, including the mean-field XY Heisenberg model and the Hopfield model of spin glasses and neural networks.
- Statistical mechanics is a source of beautiful problems in probability theory.
- New physical phenomena to be studied with Jonathan Machta
  - Connections with critical exponents and the renormalization group
  - Trace of double phase in case 4 in Corollary 19 [MDP for $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$].
  - Traces of double phase in cases 5, 7, 9, 11, 13 in Corollary 21 [MDP for $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c)) \in C$] and trace of triple phase in case 12 in Corollary 21.