Weak Law of Large Numbers for Fair Coin Tossing

For $n \in \mathbb{N}$ define $\Sigma_n = \{(\alpha_1, \alpha_2, \dots, \alpha_n) : \text{each } \alpha_j = 0 \text{ or } 1, 1 \le j \le n\}$.
For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Sigma_n$ define $P_n(\alpha) = \frac{1}{2^n} = \frac{1}{|\Sigma_n|}$.
For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Sigma_n$ and $1 \le j \le n$ define $X_j(\alpha) = \alpha_j$, which is the result of the j'th toss. The random variables X_j, $1 \le j \le n$, are independent; that is, for all $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n) \in \Sigma_n$

$$P_n\Big(\bigcap_{1 \le j \le n} \{\alpha \in \Sigma_n : X_j(\alpha) = \bar{\alpha}_j\}\Big) = P_n\Big(\bigcap_{1 \le j \le n} \{X_j = \bar{\alpha}_j\}\Big) = \prod_{1 \le j \le n} P_n(X_j = \bar{\alpha}_j).$$

<u>Proof</u>. $P_n\Big(\bigcap_{1 \le j \le n} \{X_j = \bar{\alpha}_j\}\Big) = P_n\big((\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n)\big) = \frac{1}{2^n}$

$$= \left(\frac{1}{2}\right)^n = \prod_{1 \le j \le n} P_n(X_j = \bar{\alpha}_j)$$

because $P_n(X_j = \bar{\alpha}_j) = \frac{2^{n-1}}{2^n} = \frac{1}{2}$. QED

How should we formulate a law of large numbers for $S_n = \sum_{j=1}^n X_j$? Is it true that $\lim_{n\to\infty} P_{2n}\left(\frac{S_{2n}}{2n} = \frac{1}{2}\right) = 1$? The answer is <u>no</u> because $P_{2n}\left(\frac{S_{2n}}{2n} = \frac{1}{2}\right) = P_{2n}(\alpha \in \Sigma_{2n} : n\ \alpha_j = 0,\ n\ \alpha_j = 1)$
$= \frac{(2n)!}{n!\,n!} \frac{1}{2^{2n}}$. Apply Stirling's formula: $k! \sim k^k e^{-k} \sqrt{2\pi k}$. We have

$$P_{2n}\left(\frac{S_{2n}}{2n} = \frac{1}{2}\right) = \frac{(2n)!}{n!\,n!} \frac{1}{2^{2n}} \sim \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi \cdot 2n}}{(n^n e^{-n} \sqrt{2\pi n})^2\, 2^{2n}} = \frac{1}{\sqrt{\pi n}} \to 0.$$

<u>Theorem (Weak Law of Large Numbers).</u> Define $S_n = \sum_{j=1}^n X_j$. For any $\varepsilon > 0$

$$\lim_{n\to\infty} P_n\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| < \varepsilon\right) = 1.$$

<u>Proof</u>. Using Chebyshev's Inequality, we prove that

$$0 \le P_n\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| \ge \varepsilon\right) \le \frac{1}{\varepsilon^2} \frac{\sigma^2}{n} \to 0, \text{ where } \sigma^2 = Var(X_1) = \frac{1}{4}.$$

Define $A_n = \{j \in \mathbb{N} : \frac{1}{2} - \varepsilon < \frac{j}{n} < \frac{1}{2} + \varepsilon\} = \{j \in \mathbb{N} : n(\frac{1}{2} - \varepsilon) < j < n(\frac{1}{2} + \varepsilon)\}$.
We have the exact formula $P_n\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| < \varepsilon\right) = \sum_{j \in A_n} P_n\left(\frac{S_n}{n} = j\right)$
$= \sum_{j \in A_n} \frac{n!}{j!\,(n-j)!} \frac{1}{2^n}$, but it is too complicated to prove that $P_n\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| < \varepsilon\right)$ converges to 1.