The Sum of Independent Normal Random Variables Is Normal

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Here are two proofs of Proposition 3.2 in chapter 6 of the textbook. The proposition states that if $X_i = N(\mu_i, \sigma_i^2)$ for $i=1,2,\ldots,n$ and are independent, then $\sum_{i=1}^n X_i = N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$. As the book points out, by induction it suffices to prove the case $n=2$. It also suffices to prove the case where $\mu_1 = \mu_2 = 0$.

Proof 1. The central limit theorem (CLT) states that if $\{Y_i, i=1,2,\ldots,n\}$ are iid with mean 0 and variance σ^2 then $\forall -\infty < a < b < \infty$

$$\lim_{n\to\infty} P(a \le S_n/\sqrt{n}\,\sigma \le b) = P(a \le N(0,1) \le b), \text{ where } S_n = \sum_{i=1}^n Y_i;$$

equivalently, $\lim_{n\to\infty} P(a \le S_n/\sqrt{n} \le b) = P(a \le N(0,\sigma^2) \le b)$.

We summarize this by the notation $S_n/\sqrt{n} \Rightarrow N(0,\sigma^2)$.

Now let $\{Y_i, i=1,2,\ldots,n\}$ and $\{Z_i, i=1,2,\ldots,n\}$ be $2n$ iid rv's, where Y_i has mean 0 and variance σ_1^2 and Z_i has mean 0 and variance σ_2^2. Define $S_n = \sum_{i=1}^n Y_i$ and $T_n = \sum_{i=1}^n Z_i$. By the CLT $S_n/\sqrt{n} \Rightarrow N(0,\sigma_1^2)$ and $T_n/\sqrt{n} \Rightarrow N(0,\sigma_2^2)$. Since $S_n/\sqrt{n}$ and $T_n/\sqrt{n}$ are independent, it is plausible that $N(0,\sigma_1^2)$ and $N(0,\sigma_2^2)$ are independent. By advanced techniques it follows that $S_n/\sqrt{n} + T_n/\sqrt{n} \Rightarrow N(0,\sigma_1^2) + N(0,\sigma_2^2)$.

But $S_n/\sqrt{n} + T_n/\sqrt{n} = \sum_{i=1}^n (Y_i + Z_i)/\sqrt{n} \Rightarrow N(0, \sigma_1^2 + \sigma_2^2)$ by the CLT since $\{Y_i + Z_i, i=1,2,\ldots,n\}$ are iid with mean 0 and variance $\sigma_1^2 + \sigma_2^2$. Since the limits are unique it follows that $N(0,\sigma_1^2) + N(0,\sigma_2^2) = N(0,\sigma_1^2+\sigma_2^2)$, as claimed. This proof is compelling, but several details must be filled in.

Proof 2. Let $X_1 = N(0, \sigma_1^2)$ and $X_2 = N(0, \sigma_2^2)$ be independent. By equation (3.2) $X+Y$ has the pdf for $a \in \mathbb{R}$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} \frac{1}{\sqrt{2\pi}\,\sigma_2} \int_{-\infty}^{\infty} \exp\{-c_1 (a-y)^2] \exp\{-c_2 y^2] dy,$$

where $c_1 = 1/2\sigma_1^2$ and $c_2 = 1/2\sigma_2^2$. It suffices to prove that there exist constants $\alpha > 0$ and $\beta > 0$ such that $f_{X+Y}(a) = \alpha \exp[-\beta a^2]$. This implies that $X+Y = N(0, \sigma^2)$ for $\sigma^2 = \frac{1}{2\beta}$ and $\alpha = \frac{1}{\sqrt{2\pi}\,\sigma}$. But since $E[X+Y] = E[X] + E[Y] = 0$ and $Var(X+Y) = Var(X) + Var(Y)$ $= \sigma_1^2 + \sigma_2^2$, we must have $\sigma^2 = \sigma_1^2 + \sigma_2^2$. Thus $X+Y = N(0, \sigma_1^2 + \sigma_2^2)$, as claimed. From line 4 on this page we have

$$f_{X+Y}(a) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \frac{1}{\sqrt{2\pi}\,\sigma_2} \int_{-\infty}^{\infty} \exp\{-c_1(a^2 - 2ay + y^2) - c_2 y^2] dy$$

$$= \text{const} \cdot \exp[-c_1 a^2] \int_{-\infty}^{\infty} \exp[-(c_1+c_2) y^2 + 2c_1 a y]\, dy$$

$$= \text{Const} \cdot \exp[-c_1 a^2] \int_{-\infty}^{\infty} \exp\left[-(c_1+c_2)\left(y^2 - \frac{2c_1}{c_1+c_2}\, a y\right)\right] dy$$

$$= \text{const} \cdot \exp[-c_1 a^2] \int_{-\infty}^{\infty} \exp\left[-(c_1+c_2)\left(y - \frac{c_1}{c_1+c_2}\, a\right)^2\right] dy \times \exp\left[(c_1+c_2)\left(\frac{c_1}{c_1+c_2}\right)^2 a^2\right]$$

$$= \text{const} \cdot \exp\left\{-c_1 a^2 + \frac{c_1^2}{c_1+c_2} a^2\right] \int_{-\infty}^{\infty} \exp[-(c_1+c_2) y^2]\, dy$$

$$= \text{const} \cdot \exp\left\{-\frac{c_1 c_2}{c_1+c_2} a^2\right)$$

Thus $f_{X+Y}(a) = \alpha \exp\{-\beta a^2)$ for some $\alpha > 0$ and $\beta > 0$. QED.

Compare this proof with the proof of Proposition 3.2 in the textbook. Note that $\frac{c_1 c_2}{c_1+c_2} = \frac{1}{2(\sigma_1^2 + \sigma_2^2)}$, which shows that $X+Y = N(0, \sigma_1^2 + \sigma_2^2)$.