Preliminaries Concerning Normal Random Variables

(§5.4 of Ross)

By definition, X is a normal random variable with parameters μ and σ^2 if the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-(x-\mu)^2/2\sigma^2\right] \quad \text{for } -\infty < x < \infty.$$

Ross proves that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ by use of polar coordinates. He also proves that $E[X] = \mu$ and $Var(X) = \sigma^2$ by calculating the relevant integrals. In calculating $Var(X)$, he integrates by parts, which is based on the formula

$$\int_a^b uv'\,dx = (uv)\Big|_a^b - \int_a^b u'v\,dx, \text{ where } u = u(x),\ v = v(x),$$

and $(uv)\Big|_a^b = u(b)v(b) - u(a)v(a)$. Here is a proof.

$$\int_a^b (uv)'\,dx = (uv)\Big|_a^b = \int_a^b (u'v + uv')\,dx = \int_a^b u'v\,dx + \int_a^b uv'\,dx.$$

Solving for $\int_a^b uv'\,dx$ gives $\int_a^b uv'\,dx = (uv)\Big|_a^b - \int_a^b u'v\,dx$.

A normal random variable with parameters μ and σ^2 is also called an $N(\mu, \sigma^2)$ random variable.

Example 4a. Find (a) $E[X]$ and (b) $\mathrm{Var}(X)$ when X is a normal random variable with parameters μ and σ^2.

Solution (a) $E[X] = \dfrac{1}{\sqrt{2\pi}\,\sigma}\displaystyle\int_{-\infty}^{\infty} x e^{-(x-\mu)^2/2\sigma^2}\,dx$

Writing x as $(x - \mu) + \mu$ yields

$$E[X] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}(x-\mu)e^{-(x-\mu)^2/2\sigma^2}\,dx + \mu\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Letting $y = x - \mu$ in the first integral yields

$$E[X] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} y e^{-y^2/2\sigma^2}\,dy + \mu\int_{-\infty}^{\infty} f(x)\,dx$$

where $f(x)$ is the normal density. By symmetry, the first integral must be 0, so

$$E[X] = \mu\int_{-\infty}^{\infty} f(x)\,dx = \mu$$

(b) Since $E[X] = \mu$, we have that

$$\mathrm{Var}(X) = E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}\,dx \tag{4.1}$$

Substituting $y = (x - \mu)/\sigma$ in Equation (4.1) yields

$$\begin{aligned}
\mathrm{Var}(X) &= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 e^{-y^2/2}\,dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\left[-ye^{-y^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right] \text{ by integration by parts} \\
&= \sigma^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}\,dy \\
&= \sigma^2
\end{aligned}$$

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An important fact about normal random variables is that if X is normally distributed with parameters μ and σ^2, then $Y = \alpha X + \beta$ is normally distributed with parameters $\alpha\mu + \beta$ and $\alpha^2\sigma^2$. To show this, suppose that $\alpha > 0$. (The verification when $\alpha < 0$ is similar.) Now, F_Y,† the cumulative distribution function of the random variable Y, is given by