Normal Random Variables (§5.4 of Ross)

Let X be a continuous r.v. with probability density function (p.d.f.) f. Thus for any $x \in \mathbb{R}$

$$F_X(x) = P\{X \le x\} = \int_{-\infty}^{x} f(y)\,dy, \quad F_X'(x) = f(x).$$

<u>Defn.</u> X is an $N(\mu, \sigma^2)$ r.v. if X has the p.d.f.

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-(x-\mu)^2/2\sigma^2\right], \quad \mu \in \mathbb{R}, \ \sigma^2 > 0.$$

See Ross, p. 218.

<u>Facts</u> (Ross, p. 221). $E[X] = \mu$, $Var(X) = \sigma^2$ if $X = N(\mu, \sigma^2)$.

<u>Another Fact</u> (special case of Ross, pp. 219–220). Let $X = N(\mu, \sigma^2)$.

Then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$.

<u>Proof.</u>

$$F_Y(x) = P\{Y \le x\} = P\left\{\frac{X-\mu}{\sigma} \le x\right\} = P\{X \le \sigma x + \mu\} = \int_{-\infty}^{\sigma x+\mu} f_{\mu,\sigma^2}(y)\,dy.$$

Change variables: let $z = \dfrac{y-\mu}{\sigma}$. Then when $y = \sigma x + \mu$, $z = x$.
Also $y = \sigma z + \mu$ and so $dy = \sigma\,dz$. Thus

$$F_Y(x) = \int_{-\infty}^{\sigma x+\mu} f_{\mu,\sigma^2}(y)\,dy = \sigma \int_{-\infty}^{x} f_{\mu,\sigma^2}(\sigma z + \mu)\,dz = \int_{-\infty}^{x} f_{0,1}(z)\,dz$$

Thus Y has pdf $F_Y'(x) = f_{0,1}(x)$, and so $Y = N(0,1)$.