Law of Large Numbers, Normal Approximation, and Stirling's Formula

Let $X_1, X_2, \ldots, X_{2n}$ be independent Bernoulli rv's with $P\{X_i = 0\} = P\{X_i = 1\} = 1/2$ for all $i = 1, 2, \ldots, 2n$. Define $S_{2n} = \sum_{i=1}^{2n} X_i$.

Law of Large Numbers. For any $\varepsilon > 0$

$$\lim_{n\to\infty} P\left\{ \left| \frac{S_{2n}}{2n} - \frac{1}{2} \right| \geq \varepsilon \right\} = 0 \text{ or } \lim_{n\to\infty} P\left\{ \left| \frac{S_{2n}}{2n} - \frac{1}{2} \right| < \varepsilon \right\} = 1.$$

Central Limit Theorem (p225). Since $p = 1/2$ and $p(1-p) = 1/4$,

$$\lim_{n\to\infty} P\left\{ a \leq \frac{S_{2n} - n}{\sqrt{n/2}} \leq b \right\} = P\{ a \leq N(0,1) \leq b \}.$$

Normal Approximation. For large n

$$P\left\{ a \leq \frac{S_{2n} - n}{\sqrt{n/2}} \leq b \right\} \approx P\{ a \leq N(0,1) \leq b \}.$$

Question. Does $P\left\{ \frac{S_{2n}}{2n} = \frac{1}{2} \right\} \to 1$ or $\to 0$ as $n \to \infty$?

Answer 1. Use normal approximation: $P\left\{ \frac{S_{2n}}{2n} = \frac{1}{2} \right\} \lesssim \frac{1}{\sqrt{\pi n}}$.

$$P\left\{ \frac{S_{2n}}{2n} = \frac{1}{2} \right\} = P\{ S_{2n} = n \} = P\left\{ n - \tfrac{1}{2} \leq S_{2n} \leq n + \tfrac{1}{2} \right\}$$

$$= P\left\{ \frac{n - \frac{1}{2} - n}{\sqrt{n/2}} \leq \frac{S_{2n} - n}{\sqrt{n/2}} \leq \frac{n + \frac{1}{2} - n}{\sqrt{n/2}} \right\} \approx P\left\{ -\frac{1}{\sqrt{2n}} \leq N(0,1) \leq \frac{1}{\sqrt{2n}} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1/\sqrt{2n}}^{1/\sqrt{2n}} e^{-x^2/2} dx \leq \frac{1}{\sqrt{2\pi}} \int_{-1/\sqrt{2n}}^{1/\sqrt{2n}} 1 \, dx = \frac{1}{\sqrt{2\pi}} \frac{2}{\sqrt{2n}} = \frac{1}{\sqrt{\pi n}}.$$

Answer 2. Use Stirling's formula: $P\left\{ \frac{S_{2n}}{2n} = \frac{1}{2} \right\} \approx \frac{1}{\sqrt{\pi n}}$.

$$P\left\{ \frac{S_{2n}}{2n} = \frac{1}{2} \right\} = P\{ S_{2n} = n \} = \binom{2n}{n} 2^{-2n} = \frac{(2n)!}{n!\, n!} 2^{-2n}$$

$$\approx \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi \cdot 2n}}{n^n e^{-n} \sqrt{2\pi n} \cdot n^n e^{-n} \sqrt{2\pi n}} 2^{-2n} = \frac{1}{\sqrt{\pi n}}.$$

Same as in Answer 1.

Example 4f does $n = 40$: $P\left\{ \frac{S_{40}}{40} = \frac{1}{2} \right\} \approx .1254$ while normal approximation gives $.1272$. See Ross, pp. 225–226.