Law of Large Numbers, Normal Approximation, and Stirling’s Formula

Let \( X_1, X_2, \ldots, X_{2n} \) be independent Bernoulli r.v.'s with 
\[ P(X_i = 0) = P(X_i = 1) = \frac{1}{2} \text{ for all } i = 1, 2, \ldots, 2n. \]
Define \( S_{2n} = \sum_{i=1}^{2n} X_i. \)

**Law of Large Numbers.** For any \( \varepsilon > 0 \)
\[ \lim_{n \to \infty} P\left\{ \left| \frac{S_{2n}}{2n} - \frac{1}{2} \right| \geq \varepsilon \right\} = 0 \text{ or } \lim_{n \to \infty} P\left\{ \left| \frac{S_{2n}}{2n} - \frac{1}{2} \right| < \varepsilon \right\} = 1. \]

**Central Limit Theorem (p.225).** Since \( p = \frac{1}{2} \) and \( p(1-p) = \frac{1}{4} \),
\[ \lim_{n \to \infty} P\left\{ \frac{n \cdot S_{2n} - n}{\sqrt{2n}} \leq b \right\} = P\left\{ a \leq N(0,1) \leq b \right\}. \]

**Normal Approximation.** For large \( n \)
\[ P\left\{ a \leq \frac{S_{2n} - n}{\sqrt{n/2}} \leq b \right\} \approx P\left\{ a \leq N(0,1) \leq b \right\}. \]

**Question.** Does \( P\left\{ \frac{S_{2n}}{2n} = \frac{1}{2} \right\} \to 1 \text{ or } \to 0 \text{ as } n \to \infty? \)

**Answer 1.** Use normal approximation: \( P\left\{ \frac{S_{2n}}{2n} = \frac{1}{2} \right\} \approx \frac{1}{\sqrt{\pi n}}. \)
\[ P\left\{ \frac{S_{2n}}{2n} = \frac{1}{2} \right\} = P\left\{ S_{2n} = n \right\} = P\left\{ n - \frac{1}{2} \leq S_{2n} \leq n + \frac{1}{2} \right\} \]
\[ = P\left\{ \frac{n - \frac{1}{2} - n}{\sqrt{2n}} \leq \frac{S_{2n} - n}{\sqrt{2n}} \leq \frac{n + \frac{1}{2} - n}{\sqrt{2n}} \right\} \approx P\left\{ \frac{-1}{\sqrt{2n}} \leq N(0,1) \leq \frac{1}{\sqrt{2n}} \right\} \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-1/\sqrt{2n}}^{1/\sqrt{2n}} e^{-x^2/2} \, dx \leq \frac{1}{\sqrt{2\pi}} \int_{-1/\sqrt{2n}}^{1/\sqrt{2n}} 1 \, dx \leq \frac{1}{\sqrt{2\pi}} \frac{2}{\sqrt{2n}} = \frac{1}{\sqrt{\pi n}}. \]

**Answer 2.** Use Stirling's formula: \( P\left\{ \frac{S_{2n}}{2n} = \frac{1}{2} \right\} \approx \frac{1}{\sqrt{\pi n}}. \)
\[ P\left\{ \frac{S_{2n}}{2n} = \frac{1}{2} \right\} = P\left\{ S_{2n} = n \right\} = \binom{2n}{n} \frac{2^{-2n}}{n^n n!} \]
\[ \approx \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi n}}{n^n e^{-n} \sqrt{2\pi n} n^n e^{-n} \sqrt{2\pi n}} \cdot \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{\pi n}}. \] Same as in Answer 1.

**Example 48.** Does \( n = 40 \).
\[ P\left( S_{40} = 20 \right) = \frac{1}{2} \approx 0.125 \% \text{ while normal approximation gives } 0.1272. \text{ See Ross, pp. 274-276.} \]
Although one cannot apply the normal approximation to prove the law of large numbers, they are consistent in the following sense:

\[
P\left( \left| \frac{S_n - n}{\sqrt{n}} \right| > \frac{3}{2} \right) = P\left( \left| \frac{S_n - n}{\sqrt{n}} \right| > \frac{2 \sqrt{n} \varepsilon}{\varepsilon} \right)
\]

\[
= P\left( \left| \frac{S_n - n}{\sqrt{n}} \right| > \frac{2 \sqrt{n} \varepsilon}{\sqrt{n}} \right)
\]

\[
\approx P\left( |N(0,1)| > 2\sqrt{n} \varepsilon \right)
\]

\[
= \frac{2}{\sqrt{2\pi}} \int_{2\sqrt{n} \varepsilon}^{\infty} e^{-x^2/2} dx \to 0 \text{ as } n \to \infty.
\]

One cannot apply the normal approximation in this way to study the \(n \to \infty\) behavior of \(P\left( \left| \frac{S_n - n}{\sqrt{n}} \right| > \frac{2 \sqrt{n} \varepsilon}{\varepsilon} \right)\).

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**How accurate is the normal approximation?**

Let \(X_1, X_2, \ldots, X_n\) be independent, i.i.d. r.v.'s having the same distribution function. Assume that \(E[X_i] = \mu\) and \(\text{Var}(X_i) = \sigma^2\) for all \(i = 1, 2, \ldots, n\). Then for any \(b\)

\[
\left| P\left( \frac{S_n - n\mu}{\sigma \sqrt{n}} \leq b \right) - P\left( |N(0,1)| \leq b \right) \right| \leq \frac{C \sigma}{\sigma^3 \sqrt{n}}
\]

where \(C\) is a fixed positive constant and for all \(i\)

\[
y = E(|X_i|^3).
\]

This is called the Berry–Esseen Theorem. One can prove that \(0 < C \leq 4\).

**Example.** To make the error less than \(\varepsilon\), choose \(\frac{C \sigma}{\sigma^3 \sqrt{n}} \leq \varepsilon\) or \(n \geq \left( \frac{C \sigma}{\sigma^3 \varepsilon} \right)^2\).