The Birthday Problem, Exponential Approximation, and Poisson Approx.

Let g_n denote the probability that in a group of n people no 2 people have a birthday on the same day. Then $g_n = \prod_{j=1}^{n-1} \frac{365-j}{365} = \prod_{j=1}^{n-1}\left(1-\frac{j}{365}\right)$. For small x, $1-x \approx e^{-x}$.

Thus for n small compared to 365, we expect that

$$g_n \approx \prod_{j=1}^{n-1} \exp\left(-\frac{j}{365}\right) = \exp\left(-\frac{1}{365}\sum_{j=1}^{n-1} j\right) = \exp\left(\frac{-n(n-1)}{730}\right).$$

On p. 164 of the text, Ross uses the Poisson approximation to get the same answer.

For a second illustration of the strength of the Poisson approximation when the trials are weakly dependent, let us reconsider the birthday problem presented in Example 5i of Chapter 2. In this example we suppose that each of n people is equally likely to have any of the 365 days of the year as their birthday, and the problem is to determine the probability that a set of n independent people all have different birthdays. A combinatorial argument was used to determine this probability and it was then computed that when $n = 23$ this probability was less than $\frac{1}{2}$.

We can approximate the above probability by using the Poisson approximation as follows. Imagine that we have a trial for each of the $\binom{n}{2}$ pairs of individuals i and $j, i \neq j$, and say that trial i, j is a success if persons i and j have the same birthday. If we let E_{ij} denote the event that trial i, j is a success, then whereas the $\binom{n}{2}$ events E_{ij}, $1 \le i < j \le n$ are not independent (see Theoretical Exercise 21) their dependence appears to be rather weak. (Indeed, these events are even *pairwise independent* in that any 2 of the events E_{ij} and E_{kl} are independent—again see Theoretical Exercise 21). As $P(E_{ij}) = 1/365$ it is thus reasonable to suppose that the number of successes should approximately have a Poisson distribution with mean $\binom{n}{2}\Big/365 = n(n-1)/730$. Therefore,

$$P\{\text{no 2 people have the same birthday}\} = P\{0 \text{ successes}\} \approx \exp\left\{\frac{-n(n-1)}{730}\right\}$$

To determine the smallest integer n for which this probability is less than $\frac{1}{2}$ note that

$$\exp\left\{\frac{-n(n-1)}{730}\right\} \le \frac{1}{2}$$

is equivalent to

$$\exp\left\{\frac{n(n-1)}{730}\right\} \ge 2$$

or, taking logarithms of both sides, we obtain

$$n(n-1) \ge 730 \log 2 \approx 505.997$$

which yields the solution $n = 23$, in agreement with the result of Example 5i of Chapter 2.