Constructing Binomial Random Variables

For $n \in \mathbb{N}$, define $S = \{(e_1, e_2, \ldots, e_n) : \text{each } e_i = 0 \text{ or } 1\}$.

For $0 < p < 1$, define

$$P(\{e_1, \ldots, e_n\}) = p^\alpha (1-p)^{n-\alpha}, \text{ where } \alpha = \#\{i : e_i = 1\}.$$

For $A \subset S$ define $P(A) = \sum_{y \in A} P(\{y\})$. Clearly $0 \le P(A)$ for any $A \subset S$ and P is additive on unions of disjoint sets. To prove that P is a probability, we must show that $P(S) = \sum_{y \in S} P(\{y\}) = 1$. It then follows that $P(A) \le P(S) = 1$ for any $A \subset S$.

In order to do this, define $S_n(y) = \sum_{i=1}^{n} e_i$ for $y \in S$.

Then as we discussed in class, for $k = 0, 1, \ldots, n$

$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Thus S_n is binomial with parameters (n, p). Now prove $P(S) = 1$:

$$P(S) = \sum_{y \in S} P(\{y\}) = \sum_{k=0}^{n} \sum_{\{y \in S : S_n(y) = k\}} P(\{y\}) = \sum_{k=0}^{n} P(S_n = k)$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1.$$

The next-to-last inequality uses the binomial theorem.

For $y \in S$ and $i = 1, 2, \ldots, n$, define $X_i(y) = e_i$.

The claim is that $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables with $P(X_i = 1) = p$, $P(X_i = 0) = 1 - p$ for each i. It then follows that $S_n = \sum_{i=1}^{n} X_i$.

We prove that $X_1, X_2, \ldots, X_n$ are independent by showing that for any choice of $e_i = 0$ or 1,

$$(*) \quad P(X_1 = e_1, X_2 = e_2, \ldots, X_n = e_n) = \prod_{i=1}^{n} P(X_i = e_i).$$

<u>Claim</u>. Each X_i is Bernoulli with parameter p.

If the claim is true, then $(*)$ is true because both sides of $(*)$ equal $p^{\alpha}(1-p)^{n-\alpha}$.

<u>Proof of Claim</u>. We prove that $P(X_1 = 0) = 1-p$. A similar ~~proof~~ shows that $P(X_i = 0) = 1-p$ for $i = 1, 2, \ldots, n$ and that $P(X_i = 1) = p$ for $i = 1, 2, \ldots, n$.

$$P(X_1 = 0) = P(\{(e_1, e_2, \ldots, e_n) \in S : e_1 = 0\})$$

$$= P(\{(0, e_2, \ldots, e_n)\} : e_i = 0 \text{ or } 1,\ i = 2, \ldots, n)$$

$$= (1-p) \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$= (1-p)\,(p + (1-p))^{n-1}$$

$$= 1-p.$$

This completes the proof that $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables with parameter p.