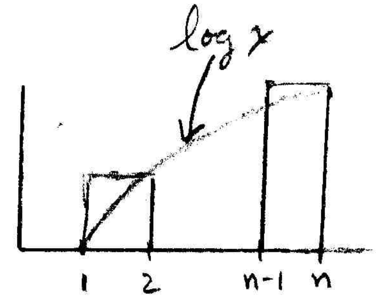
Proof of Stirling's Formula

Stirling's Formula. $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$



Motivation. Note that

$$\log(n!) = \log \prod_{k=1}^{n} k = \sum_{k=1}^{n} \log k .$$

By comparing areas, we see that

$$\log(n!) = \sum_{k=1}^{n} \log k > \int_1^n (\log x)\, dx = (x\log x - x)\Big|_1^n$$
$$= n\log n - n + 1.$$

Hence $n! > \exp[n\log n - n + 1] = n^n e^{-n} e$, which is consistent with Stirling's formula. In order to prove this formula, we must work harder.

Proof. Integrating by parts n times gives

$$n! = \int_0^\infty e^{-x} x^n \, dx. \qquad (1)$$

We prove that

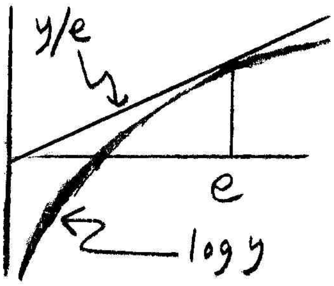
$$\lim_{n\to\infty} \frac{1}{n^n e^{-n}\sqrt{2\pi n}} \int_0^\infty e^{-x} x^n \, dx = 1. \qquad (2)$$

In the integral

$$\frac{1}{n^n e^{-n}\sqrt{2\pi n}} \int_0^\infty e^{-x} x^n \, dx = \frac{1}{\sqrt{2\pi n}} \int_0^\infty e^{-x} \left(\frac{ex}{n}\right)^n dx, \qquad (3)$$

make the change of variables $y = ex/n$. Since $x = ny/e$ and $dx = \frac{n}{e}\,dy$, we obtain from (3)

$$\frac{1}{n^n e^{-n}\sqrt{2\pi n}}\int_0^\infty e^{-x}x^n\,dx = \frac{n/e}{\sqrt{2\pi n}}\int_0^\infty e^{-ny/e}y^n\,dy$$



$$= \sqrt{\frac{n}{2\pi}}\,\frac{1}{e}\int_0^\infty \exp\left[-n\left(\frac{y}{e}-\log y\right)\right]dy \qquad (4)$$

$$= \sqrt{\frac{n}{2\pi}}\,\frac{1}{e}\int_0^\infty e^{-nf(y)}\,dy,$$

where $f(y) = \frac{y}{e} - \log y$. This function has the properties that for $y > 0$ $f(y) \geq 0$, $f(y) = 0$ iff $y = e$, $f'(y) = \frac{1}{e} - \frac{1}{y} = 0$ iff $y = e$, and $f''(e) = 1/e^2$. By Taylor's Theorem, for any $0 < \delta < e$ and all $|y-e| < \delta$ $f(y)$ can be approximated by the quadratic

$$f(e) + f'(e)\cdot(y-e) + \frac{1}{2}f''(e)\cdot(y-e)^2 = \frac{1}{2}\frac{1}{e^2}(y-e)^2. \qquad (5)$$

I claim that (see p. 3 for proof) there exist $\varepsilon_n \to 0$ such that

$$\frac{1}{n^n e^{-n}\sqrt{2\pi n}}\int_0^\infty e^{-x}x^n\,dx = \sqrt{\frac{n}{2\pi}}\,\frac{1}{e}\int_0^\infty e^{-nf(y)}\,dy \qquad (6)$$

$$= \sqrt{\frac{n}{2\pi}}\,\frac{1}{e}\int_{-\infty}^{\infty}\exp\left[-\frac{n}{2e^2}(y-e)^2\right]dy + \varepsilon_n.$$

Since the last integral equals

$$\sqrt{\frac{n}{2\pi}}\,\frac{1}{e}\int_{-\infty}^{\infty}\exp\left[-\frac{n}{2e^2}y^2\right]dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}z^2\right]dz = 1, \qquad (7)$$

(6) implies that $\frac{1}{n^n e^{-n}\sqrt{2\pi n}}\int_0^\infty e^{-x}x^n\,dx = 1 + \varepsilon_n$, where $\varepsilon_n \to 0$.
This proves (2) and hence Stirling's formula.