Poisson Limit Theorems and Poisson Approximation

In Theorems 1 and 2 we fix $\lambda > 0$ and obtain the Poisson distribution for λ as a limit of the binomial distribution of S_{n,p_n} where $p_n = \lambda/n$ in Theorem 1 and $np_n \to \lambda$ in Theorem 2. In Theorems 3 and 4 we fix n and p and obtain an approximation to the Poisson distribution for $\lambda = np$ by the binomial distribution of $S_{n,p}$.

<u>Theorem 1.</u> Given $\lambda > 0$ let $S_{n,\lambda/n}$ be a sequence of binomial rv's with parameters n and λ/n. Let N_λ be a Poisson rv with parameter λ. Then for any $i \in \mathbb{N} \cup \{0\}$

$$\lim_{n\to\infty} P(S_{n,\lambda/n} = i) = e^{-\lambda}\frac{\lambda^i}{i!} = P(N_\lambda = i).$$

[Proved in class]

<u>Theorem 2.</u> Same notation as in Theorem 1 with λ/n replaced by p_n where $np_n \to \lambda$. Then for any $i \in \mathbb{N} \cup \{0\}$ [Proved like Thm 1]

$$\lim_{n\to\infty} P(S_{n,p_n} = i) = e^{-\lambda}\frac{\lambda^i}{i!} = P(N_\lambda = i).$$

[If $p_n = \lambda/n$, then get Thm 1]

<u>Theorem 3.</u> Given $n \in \mathbb{N}$ and $0 < p < 1$, let $S_{n,p}$ be a binomial rv with parameters n and p. Let N_{np} be a Poisson rv with parameter $\lambda = np$. Then for any $i \in \mathbb{N} \cup \{0\}$

$$|P(S_{n,p} = i) - P(N_{np} = i)| \le np^2.$$

[Proved in §8.6 of text]

If $p = \lambda/n$, then Thm 3 yields Thm 1 because $np^2 = \frac{\lambda^2}{n} \to 0$. In general, this approximation is useful if np^2 is small; similarly in Theorem 4.

<u>Theorem 4.</u> Same notation as in Theorem 3. Then for any subset $A \subset \mathbb{N} \cup \{0\}$

$$|P(S_{n,p} \in A) - P(N_{np} \in A)| = \left|\sum_{i\in A} P(S_{n,p} = i) - \sum_{i \in A} P(N_{np} = i)\right| \le np^2.$$

[Proved in §8.6 of text]