$P(X \in \text{interval})$ for Continuous RVs X

Let X be a continuous rv with pdf $f(x)$ and cumulative distribution function F_X. Thus $f(x) \geq 0$ for all $x \in \mathbb{R}$, $\int_{-\infty}^{\infty} f(x)\,dx = 1$, for any subset B of $\mathbb{R}$ $P(X \in B) = \int_B f(x)\,dx$, and for $y \in \mathbb{R}$ $F_X(y) = P(X \leq y) = P(X \in (-\infty, y]) = \int_{-\infty}^{y} f(x)\,dx$.

For the purpose of this discussion we add the assumption that f is a continuous function on $\mathbb{R}$.

A basic fact concerning such random variables is that for any $y \in \mathbb{R}$

(1) $P(X = y) = P(X \in \{y\}) = \int_y^y f(x)\,dx = 0.$

We want to calculate $P(X \in I)$ for any interval $I \subset \mathbb{R}$.
There are 3 classes of intervals.

Class 1. $I = \mathbb{R}$. Then $P(X \in \mathbb{R}) = \int_{-\infty}^{\infty} f(x)\,dx = 1.$

Class 2. I a semi-infinite integral. There are 4 cases:
for $b \in \mathbb{R}$, $I = (-\infty, b], (-\infty, b), [b, \infty), (b, \infty)$. Recall that $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$, $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$, $[b, \infty) = (-\infty, b)^c$, $(b, \infty) = (-\infty, b]^c$.

Theorem 1. $P(X \in (-\infty, b]) = P(X \in (-\infty, b)) = P(X \leq b) = F_X(b)$.

Proof. By definition of F_X $P(X \in (-\infty, b]) = P(X \leq b) = F_X(b)$.
Also $P(X \in (-\infty, b]) = P(X \in (-\infty, b)) + P(X = b) = P(X \in (-\infty, b)) = F_X(b)$. QED

Check that $P(X \in [b, \infty)) = P(X \in (b, \infty)) = 1 - F_X(b)$.

On page 2, we take $-\infty < a < b < \infty$ and consider the 4 intervals with endpoints a and b:
$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$,
$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

Class 3. I a bounded interval. There are 4 cases: for $-\infty < a < b < \infty$, $I = (a,b]$, $[a,b]$, $[a,b)$, (a,b).

Theorem 2. $P(X \in (a,b]) = P(X \in [a,b]) = P(X \in [a,b)) = P(X \in (a,b))$
$$= P(X \le b) - P(X \le a) = F_X(b) - F_X(a).$$

Proof. Recall that $P(X \in (a,b]) = P(a < X \le b)$, $P(X \in [a,b]) = P(a \le X \le b)$, $P(X \in (a,b]) = P(a < X \le b)$, $P(X \in (a,b)) = P(a < X < b)$. We first consider $P(X \in (a,b])$. If $A \subset B \subset \mathbb{R}$, then by finite additivity $P(X \in B) = P(X \in A) + P(X \in B \setminus A)$. Thus $P(X \in B \setminus A) = P(X \in B) - P(X \in A)$. Now let $A = (-\infty, a]$ and $B = (-\infty, b]$, where $-\infty < a < b < \infty$. We have
$$P(X \in (a,b]) = P(B \setminus A) = P(B) - P(A) = P(X \le b) - P(X \le a)$$
$$= F_X(b) - F_X(a),$$
as claimed. Since $P(X = a) = P(X = b) = 0$, it is not hard to prove that
$$P(X \in [a,b]) = P(X \in [a,b)) = P(X \in (a,b)) = P(X \in (a,b]) = F_X(b) - F_X(a).$$
See if you can fill in the details. QED

Examples. See "Normal Random Variables" handout. A standard $N(0,1)$ rv has the continuous pdf $f_{0,1}(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ and cumulative distribution function Φ. Use $\frac{N(1,4)-1}{2} = N(0,1)$.

a) $P(0 < N(1,4) < 3) = P\left(-\frac{1}{2} < \frac{N(1,4)-1}{2} < \frac{3-1}{2}\right) = P\left(-\frac{1}{2} < N(0,1) < 1\right)$
$$= \Phi(1) - \Phi(-1/2) = \Phi(1) - (1 - \Phi(1/2)). \text{ Use table}$$

b) $P(|N(1,4) - 1| \le 2) = P(-1 < N(1,4) \le 3) = P(-1 < N(0,1) < 1)$
$$= \Phi(1) - \Phi(-1) = \Phi(1) - (1 - \Phi(1)). \text{ Use table.}$$

c) Since $N(0,2)/\sqrt{2} = N(0,1)$, $P(N(0,2) < 2) = P(N(0,1) < \sqrt{2})$
$$= \Phi(\sqrt{2}). \text{ Not in Table. Use interpolation.}$$