The birthday problem. Given \( n \) people, let \( p_n \) equal the probability that at least 2 of the \( n \) people have birthday on the same day. Determine \( n \), the smallest value of \( n \) such that \( p_n > \frac{1}{2} \).

Restatement. Define \( g_n = 1 - p_n \). Then \( n \) is the smallest value of \( n \) such that \( g_n < \frac{1}{2} \).

Solution. As we saw in class,

\[
\begin{align*}
\frac{365}{365}, \quad \frac{365 \times 364}{365 \times 365}, \quad \frac{365 \times 364 \times 363}{365 \times 365 \times 365}, \quad \ldots, \quad g_n &= \frac{1}{\prod_{j=0}^{n-1} \frac{365-j}{365}}.
\end{align*}
\]

Calculating \( g_n \) for \( 15 \leq n \leq 30 \) will give \( n \).

Approximate \( g_n \) using Stirling’s formula. We have

\[
\begin{align*}
g_n &= \prod_{j=0}^{n-1} \frac{365-j}{365} = \frac{1}{(365)^n} \prod_{j=0}^{n-1} \frac{365-j}{365} = \frac{1}{(365)^n} \frac{(365)!}{(365-n)!}.
\end{align*}
\]

Stirling’s formula states that \( \lim_{k \to \infty} \frac{k!}{k^k e^{-k} \sqrt{2\pi k}} = 1 \) or that as \( k \to \infty \), \( k! \sim k^k e^{-k} \sqrt{2\pi k} \). This asymptotic relation using Stirling’s formula, we expect that \( g_n \) should be well-approximated by

\[
A_n = \frac{1}{\left(1 - \frac{n}{365}\right)^{365-n+\frac{1}{2}}} e^n.
\]