Facts about $E[X]$ and $Var(X)$ - #2

Let $X, Y, X_1, X_2, \ldots, X_n$ be discrete or continuous r.v.'s.

1) For α and β real $E[\alpha X+\beta Y]=\alpha\cdot E[X]+\beta\cdot E[Y]$. (p330)

2) For $\alpha_1,\ldots,\alpha_n$ real $E\left[\sum_{i=1}^{n}\alpha_i X_i\right]=\sum_{i=1}^{n}\alpha_i E[X_i]$. (p 330)

3) For α real $Var(\alpha X)=\alpha^2 Var(X)$.

4) X and Y are said to be independent if for all $A\subset\mathbb{R}, B\subset\mathbb{R}$
$P\{X\in A, Y\in B\}=P\{X\in A\}\cdot P\{Y\in B\}$ (p 267). In this case
$Var(X+Y)=Var(X)+Var(Y)$ (p 357).

Proof. Let $E[X]=\mu$, $E[Y]=\nu$. Then $Var(X+Y)=E[(X-\mu+Y-\nu)^2]$
$=E[(X-\mu)^2+2(X-\mu)(Y-\nu)+(Y-\nu)^2]=E[(X-\mu)^2]+E[(Y-\nu)^2]$
$+2E[(X-\mu)(Y-\nu)]=Var(X)+Var(Y)+2\underbrace{E[X-\mu]\cdot E[Y-\nu]}_{0}$.

5) Let $X_1,\ldots,X_n$ be pairwise independent. Then
$Var\left(\sum_{i=1}^{n}X_i\right)=\sum_{i=1}^{n}Var(X_i)$. (p 357).

6) $X_1,\ldots,X_n$ are said to be independent if for all $A_1\subset\mathbb{R}$, $\ldots, A_n\subset\mathbb{R}$ $\quad P\left\{\bigcap_{i=1}^{n}\{X_i\in A_i\}\right\}=\prod_{i=1}^{n}P\{X_i\in A_i\}$.

7) Let $X_1,\ldots,X_n$ be pairwise independent and satisfy
$E[X_i]=\mu$ and $Var(X_i)=\sigma^2$ for all i. Define

$$W=\frac{1}{\sqrt{n}\,\sigma}\left(\sum_{i=1}^{n}X_i-n\mu\right)=\frac{1}{\sqrt{n}\,\sigma}\sum_{i=1}^{n}(X_i-\mu).$$

Then $E[W]=0$ and $Var(W)=1$. Thus W has the same expected value and variance as $N(0,1)$.

Proof. $E[W]=\frac{1}{\sqrt{n}\,\sigma}\sum_{i=1}^{n}E[X_i-\mu]=\frac{1}{\sqrt{n}\,\sigma}\sum_{i=1}^{n}(E[X_i]-\mu)=0.$

$Var(W)=\frac{1}{n\sigma^2}\sum_{i=1}^{n}Var(X_i-\mu)=\frac{1}{n\sigma^2}n\sigma^2=1$ (use 3) + 5)).

Property 7) is related to the central limit theorem (p 225, p 434)