De Morgan's Laws

Theorem. Let $E_1, E_2, \ldots, E_n$ be subsets of a sample space S. Then

(1) $\left(\bigcup_{j=1}^{n} E_j\right)^c = \bigcap_{j=1}^{n} (E_j^{\,c})$ and (2) $\left(\bigcap_{j=1}^{n} E_j\right)^c = \bigcup_{j=1}^{n} (E_j^{\,c})$.

Proof. (1) $x \in \left(\bigcup_{j=1}^{n} E_j\right)^c \iff x \notin \bigcup_{j=1}^{n} E_j \iff x \in E_j^{\,c} \ \forall\, 1 \le j \le n$

$\iff x \in \bigcap_{j=1}^{n} (E_j^{\,c})$.

(2) This can be proved like (1) (try it). However, it is instructive to derive it from (1). First rewrite (1) for arbitrary subsets $F_1, F_2, \ldots, F_n$ in S:

$\left(\bigcup_{j=1}^{n} F_j\right)^c = \bigcap_{j=1}^{n} (F_j)^c$. Then substitute $F_j = E_j^{\,c}$,

obtaining $\left(\bigcup_{j=1}^{n} (E_j)^c\right)^c = \bigcap_{j=1}^{n} E_j$. Replacing each side of this equation by its complement gives

$$\left(\bigcap_{j=1}^{n} E_j\right)^c = \left(\left(\bigcup_{j=1}^{n} (E_j)^c\right)^c\right)^c.$$

This yields (2) since for any set A we have $(A^c)^c = A$.