# Method of Variation of Parameters

Let $p(x)$, $s(x)$, $g(x)$ be continuous function on $I = (a, b)$.
Define $L[y] = y''(x) + p(x)y'(x) + s(x)y(x)$. Let $\{y_1, y_2\}$ be a fundamental set of solutions for $L[y] = 0$. It follows that $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ is nonzero for $x \in I$.
See my handout "Facts about the Wronskian."

**Problem.** Find a particular solution $Y$ of $L[y] = g$.

**Method.** Seek $Y$ in the form $Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$.

**Calculation.** We want $L[Y] = Y'' + pY' + sY = g$.

$$Y = u_1 y_1 + u_2 y_2, \quad Y' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'$$

$$Y'' = (u_1' y_1 + u_2' y_2)' + u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Set $\boxed{u_1' y_1 + u_2' y_2 = 0.}$ Then $Y = u_1 y_1 + u_2 y_2$,

$$Y' = u_1 y_1' + u_2 y_2', \quad Y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.$$

$$L[Y] = u_1(y_1'' + py_1' + sy_1) + u_2(y_2'' + py_2' + sy_2) + u_1' y_1' + u_2' y_2'$$
$$= u_1 L[y_1] + u_2 L[y_2] + u_1' y_1' + u_2' y_2' = u_1' y_1' + u_2' y_2'$$

since $L[y_1] = 0$, $L[y_2] = 0$. Thus if $u_1' y_1 + u_2' y_2 = 0$ then $L[Y] = g$ if and only if $u_1' y_1' + u_2' y_2' = g$.

**Summary.** Let $u_1$ and $u_2$ solve
$$\left.\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g \end{aligned}\right\} \circledast$$
Then $Y = u_1 y_1 + u_2 y_2$ solves $L[Y] = g$.

**Find $u_1$ and $u_2$.** Solve $\circledast$ by elimination: since $W(y_1, y_2) \neq 0$,

$$u_1' = \frac{-y_2 g}{W(y_1, y_2)}, \quad u_2' = \frac{y_1 g}{W(y_1, y_2)}.$$ Integrate to get $u_1$, $u_2$.