Variation of Parameters and Green's Functions

Let $p(x)$, $s(x)$, $h(s)$ be continuous functions on $I=(a,b)$. Define $L[y] = y''(x) + p(x)\,y'(x) + s(x)\,y(x)$. Let $\{y_1, y_2\}$ be a fundamental set of solutions of $L[y] = 0$.

On the handout entitled "Variation of Parameters and Initial Value Problems," we saw that the initial value problem $L[y]=h$, $y(x_0)=0$, $y'(x_0)=0$ is solved by $\bar{y}(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$, where

$$u_1(x) = -\int_{x_0}^{x} \frac{y_2(s)\,h(s)}{W(s)}\,ds\,, \quad u_2(x) = \int_{x_0}^{x} \frac{y_1(s)\,h(s)}{W(s)}\,ds,$$

and $W(s) = y_1(s)\,y_2'(s) - y_1'(s)\,y_2(s)$.

We can rewrite $\bar{y}(x)$ as a single definite integral; namely,

$$\bar{y}(x) = \int_{x_0}^{x} \frac{y_1(s)\,y_2(x) - y_1(x)\,y_2(s)}{W(s)}\,h(s)\,ds.$$

The function $G(x,s) = \dfrac{y_1(s)\,y_2(x) - y_1(x)\,y_2(s)}{W(s)}$ is called the <u>Green's function</u> for the ivp $L[y]=h$, $y(x_0)=0$, $y'(x_0)=0$.

<u>Example</u>. $y''=0$ has $\{1, x\}$ as a fundamental set. Since $W(1,x)=1$, $G(x,s) = x-s$.

<u>Summary</u>. The defining property of the Green's function $G(x,s)$ is that $\bar{y}(x) = \int_{x_0}^{x} G(x,s)\,h(s)\,ds$ solves the ivp $L[y]=h$, $y(x_0)=0$, $y'(x_0)=0$.