Notes on Lecture 1 in Math 331

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These notes cover the material presented in class on Wednesday, September 7, 2016. During the semester we will study parts of chapter 1 (first-order ODEs), chapter 2 (second-order, linear ODEs), chapter 6 (Laplace transforms), and chapter 4 (systems of ODEs) in the course textbook *Advanced Engineering Mathematics* by Erwin Kreyszig. I will always let you know the section of the textbook that is related to the material presented in class. However, I will not always follow the book's presentation when I would like to emphasize the material in a different way. If you would like to test your understanding of the material, then I suggest that you work on the practice problems that are available on the course webpage and that you work on the corresponding WeBWorK homework set.

Throughout the course we use standard mathematical notation that includes the following: $\mathbb{R}$ denotes the real line; $\mathbb{N}$ denotes the set of positive integers $\{1, 2, \ldots\}$; if A is a set, then $x \in A$ means that x is a member of A; $\forall$ means "for all"; $\exists$ means "there exists"; $\Rightarrow$ means "implies"; and $\ni$ means "such that".

1 Background in Calculus

We start with the definition of derivative.

Definition 1. *Let $f(x)$ be a function of $x \in \mathbb{R}$. The* **derivative** *of f is denoted by one of the following formulas: f', $f'(x)$, $\frac{df(x)}{dx}$, and $df(x)/dx$. The derivative of f is defined by the formula*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Higher order derivatives are defined similar. Thus we define the second derivative

$$f'' = f''(x) = \frac{d^2 f(x)}{dx^2} = d^2 f(x)/dx^2 \text{ by the formula } (f')'(x) = \frac{d}{dx} f'(x).$$

Review of Derivatives

1. Make sure that you know the derivatives of the following functions: c a real constant, x^n for $n \in \mathbb{N}$, $e^x = \exp(x)$, $\ln(x) = \log_e(x)$ for $x > 0$, $\sin(x)$, and $\cos(x)$. Throughout the course we will write $\ln(x)$ as $\log(x)$.

2. Make sure you know the formula $(fg)' = f'g + fg'$, the formula $(f/g)' = (f'g - fg')/g^2$, and the chain rule, $df(g(x))/dx = f'(g(x)) \cdot g'(x)$.

Review of Integrals

We continue with the definition of the indefinite integral.

Definition 2. *Let $f(x)$ be a continuous function of $x \in \mathbb{R}$. The* **indefinite integral** $\int^x f(t)dt$ *is defined to be any function $F(x)$ satisfying $F'(x) = f(x)$ for all $x \in \mathbb{R}$. The indefinite integral of f is sometimes written as $\int f(x)dx$. Indefinite integrals are defined up to an arbitrary additive constant. Thus, for example, $\int^x e^{3t}dt = (1/3)e^{3x} + c$, where c is an arbitrary real number.*

1. Make sure that you know the indefinite integrals of the following functions: c a real constant, x^n for $n \in \mathbb{N}$, $e^x = \exp(x)$, $1/x$, $\sin(x)$, and $\cos(x)$. Recall that $\int^x (1/t)dt = \log|x| + c$, where c is an arbitrary real number.

2. Make sure that you know how to do integration by parts:

$$\int^x u\,dv = u(x)v(x) - \int^x v\,du.$$

We contrast the indefinite integral $\int^x f(t)dt$ with the **definite integral**. For $x \in \mathbb{R}$ and any fixed $x_0 \in \mathbb{R}$, the definite integral $G(x) = \int_{x_0}^x f(t)dt$ has the following properties:

$$G'(x) = f(x) \text{ for all } x \in \mathbb{R} \text{ and } G(x_0) = 0$$

Thus, for example, $\int_1^x e^{3t}dt = (1/3)e^{3x} - (1/3)e^3$.

2 Introduction to ODEs

Definition 1. (a) *An* **ordinary differential equation (ODE)** *is an equation involving one or several derivatives of an unknown function. In the textbook the unknown function is often written as $y(x)$, $y(t)$, or $x(t)$.*

(b) *The* **order** *of an ODE is the order of the highest derivative of the unknown function. For example, the ODE* $y'(x) + xy(x) = e^x$ *is first order because it involves only the first derivative of the unknown function* $y(x)$*. The ODE* $y''(x) + 6y'(x) - 9y(x) = x^3$ *is second order because it involves both the first and second derivatives of the unknown function* $y(x)$*.*

One can study ODEs by exact solutions, by qualitative methods, and by numerical methods. In this course we will focus on exact solutions.

3 Application to Newton's Second Law of Motion

The textbook presents a number of interesting applications of ODEs. Some of these applications are discussed in section 1.1. Here is an important application arising from Newton's second law of motion, which states the force F on a body of mass m having acceleration a is given by $F = ma$. In order to convert this equation into an ODE, we introduce some notation. Let $x(t)$ be the location of a body of mass m at time $t \geq 0$. Then the first derivative $x'(t)$ denotes the velocity of the body at time t, and the second derivative $x''(t)$ denotes the acceleration of the body at time t. We also make the reasonable assumption that the force F at time t is a function of the position $x(t)$ and the velocity $x'(t)$; we express this assumption using the notation $F = F(x'(t), x''(t))$. With this notation, Newton's second law of motion takes the form of the second-order ODE

$$F(x(t), x'(t)) = mx''(t).$$

Here is a simple yet important example, which will also give us insight into ODEs. We let the body of mass m fall, assuming that it is subject only to gravity and that there is no air friction. In this case the force $F(x(t), x'(t))$ equals the constant mg, where g is the gravitational constant 9.80 m/sec^2. Let $x(t)$ denote the distance that the object has fallen as a function of $t \geq 0$. Since at time 0 the body has not yet fallen, we have $x(0) = 0$. Newton's second law of motion takes the form $mg = mx''(t)$ or $x''(t) = g$.

To solve this ODE, we integrate once to obtain

$$x'(t) = gt + c_1,$$

where c_1 is an arbitrary constant of integration. We then integrate a second time to obtain

$$x(t) = \frac{1}{2}gt^2 + c_1 t + c_2,$$

where c_2 is another arbitrary constant of integration. We call

$$x(t) = \frac{1}{2}gt^2 + c_1 t + c_2$$

the **general solution** of the ODE $x''(t) = g$ because for all $t > 0$ this function satisfies the ODE $x''(t) = g$ and because any solution of $x''(t) = g$ has this form. The function $x(t) = \frac{1}{2}gt^2 + c_1 t + c_2$ satisfies the ODE $x''(t) = g$ because

$$\frac{d^2}{dt^2}\left(\frac{1}{2}gt^2 + c_1 t + c_2\right) = \frac{d^2}{dt^2}\left(\frac{1}{2}gt^2\right) + \frac{d^2}{dt^2}c_1 t + \frac{d^2}{dt^2}c_2 = g.$$

It is not a surprise that the general solution involves two arbitrary constants of integration. This follows from the fact that ODE $x''(t) = g$ is solved by integrating twice.

We now consider the **initial value problem** consisting of the ODE $x''(t) = g$ and the two initial conditions $x(0) = 0$ and $x'(0) = v_0$, v_0 is a given real number. The first initial condition expresses the fact that at time 0 the body has fallen 0 meters; the second initial condition allows for the body at time 0 to have an initial velocity v_0.

Substituting $x(0) = 0$ into the general solution $x(t) = \frac{1}{2}gt^2 + c_1 t + c_2$, we obtain

$$0 = x(0) = \frac{1}{2} \cdot g \cdot 0^2 + c_1 \cdot 0 + c_2 = c_2,$$

which implies that $c_2 = 0$. Substituting $x'(0) = v_0$ into the derivative of the general solution $x'(t) = gt + c_1$, we obtain

$$v_0 = x'(0) = g \cdot 0 + c_1 = c_1,$$

which implies that $c_1 = v_0$. We summarize this as follows.

Solution of the initial value problem. The initial value problem $x''(t) = g$, $x(0) = 0$, and $x'(0) = v_0$ has the unique solution $x(t) = \frac{1}{2}gt^2 + v_0 t$.

In this course we will study many different initial value problems. They will share features with the simple initial value problem that we have just considered.

Here is an example of such a feature, which we will study in chapter 2 when we consider second order, linear ODEs. The form of the general solution of the ODE $x''(t) = g$ is

$$x(t) = \frac{1}{2}gt^2 + c_1 \cdot t + c_2 \cdot 1,$$

where we write $c_2 = c_2 \cdot 1$. The first function $\frac{1}{2}gt^2$ satisfies the ODE $x''(t) = g$. We call $\frac{1}{2}gt^2$ a **particular solution** of $x''(t) = g$. The remaining part of the general solution is $c_1 \cdot t + c_2 \cdot 1$, which is a linear combination of the function t and the constant function 1. Note that each of these functions t and 1 solve the ODE $x''(t) = 0$, where we replace the constant g in the original ODE $x''(t) = g$ by 0. The ODE $x''(t) = 0$ is called the **homogeneous equation** corresponding to **nonhomogeneous** equation $x''(t) = g$.

The next topic is first-order, linear ODEs, which are discussed in section 1.5 in the textbook.