Linear Algebra and $\vec{y}' = A\vec{y}$ in $\mathbb{R}^2$

Math 331 - R. S. Ellis

<u>Defn 1.</u> Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real, 2×2 matrix. We define the <u>determinant</u> of A by $\det A = ad - bc$.
(§7.6 of text)

<u>Thm 1.</u> $A\vec{Y}_0 = 0$ has the unique solution $\vec{Y}_0 = \vec{0}$ if and only if $\det A \neq 0$. (see Theorem 4, p 298 for general case)

<u>Pf.</u> Case 1: $A \neq 0$. Assume $a \neq 0$; similar proof if $b, c,$ or $d \neq 0$.
Write $\vec{Y}_0 = (x_0, y_0)$. Then $A\vec{Y}_0 = 0 \iff \begin{matrix} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0. \end{matrix}$
Then $x_0 = -\frac{b}{a} y_0$ and $c(-\frac{b}{a})y_0 + dy_0 = 0$ or $(ad - bc)y_0 = 0$.
Assume $ad - bc \neq 0$. Then $y_0 = 0$ and $x_0 = 0$, so $\vec{Y}_0 = \vec{0}$.
Assume $ad - bc = 0$. Then any y_0 solves and $x_0 = -\frac{b}{a} y_0$,
so $\vec{Y}_0 = y_0 \begin{pmatrix} -b/a \\ 1 \end{pmatrix} \neq \vec{0}$ if $y_0 \neq 0$.
Case 2. $A = 0$. Then $\det A = 0$ and any $\vec{Y}_0$ solves $A\vec{Y}_0 = \vec{0}$.

<u>Examples.</u> 1) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\det A = -2 \neq 0$. $A\vec{Y}_0 = \vec{0}$ or
$x_0 + 2y_0 = 0$, $3x_0 + 4y_0 = 0$. So $x_0 = -2y_0$ and $3(-2y_0) + 4y_0$
$= -2y_0 = 0 \Rightarrow y_0 = 0$ & $x_0 = 0$

2) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $\det A = 0$. $A\vec{Y}_0 = \vec{0}$ or
$x_0 + 2y_0 = 0$, $+2x_0 + 4y_0 = 0$. $x_0 = -2y_0$, $-2(2y_0) + 4y_0 = 0y_0 = 0$.
So any $y_0 \in \mathbb{R}$ solves and $x_0 = -4y_0$.

<u>Thm 2.</u> If $\det A \neq 0$, then the only equilibrium point for $d\vec{Y}/dt = A\vec{Y}$ is the origin $\vec{Y} = \vec{0}$. <u>Proof.</u> Use Theorem 1.

<u>Thm. 3.</u> If $\det A \neq 0$, then for each $\vec{z} = (z_1, z_2)$ in $\mathbb{R}^2$ $A\vec{y} = \vec{z}$ has a unique solution $\vec{y} = (y_1, y_2)$.
In fact, $\vec{y} = A^{-1}\vec{z}$, where $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
(See Theorem 4, p. 276 for general case.)

<u>Pf.</u> $A\vec{y} = \vec{z}$ or $^{1)}\ ay_1 + by_2 = z_1$, $^{2)}\ cy_1 + dy_2 = z_2$.
Multiply 1) by d and 2) by b, obtaining
$ady_1 + bdy_2 = dz_1$, $bcy_1 + bdy_2 = bz_2$. Subtracting gives $(ad - bc)y_1 = dz_1 - bz_2$ or $y_1 = \frac{1}{\det A}(dz_1 - bz_2)$.
Now multiply 1) by c and 2) by a, obtaining
$acy_1 + bcy_2 = cz_1$, $acy_1 + ady_2 = az_2$. Subtracting gives $(ad - bc)y_2 = -cz_1 + az_2$ or $y_2 = \frac{1}{\det A}(-cz_1 + az_2)$.
It follows that $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = A^{-1}\vec{z}$.

<u>Defn 4.</u> a) Let $\vec{x} = (x_1, y_1)$ and $\vec{y} = (x_2, y_2)$ be two vectors in $\mathbb{R}^2$. We say that $\vec{x}$ and $\vec{y}$ are <u>linearly dependent</u> if there exist real numbers k_1 and k_2 not both 0 such that $k_1\vec{x} + k_2\vec{y} = \vec{0}$.

b) We say that $\vec{x}$ and $\vec{y}$ are <u>linearly independent</u> if they are not linearly dependent, i.e., if the only solution $k_1 \in \mathbb{R}$ and $k_2 \in \mathbb{R}$ of $k_1\vec{x} + k_2\vec{y} = \vec{0}$ is $k_1 = k_2 = 0$ (see pages 128-129 of text).

Thm. 5. a) $\vec{x}$ and $\vec{y}$ are linearly independent if they do not lie on the same line through the origin or, equivalently, if neither vector is a multiple of the other, i.e., if there exists no $\lambda \in \mathbb{R}$ such that $\vec{x} = \lambda \vec{y}$ and there exists no $\mu \in \mathbb{R}$ such that $\vec{y} = \mu \vec{x}$.

b) $\vec{x}$ and $\vec{y}$ are linearly dependent if they lie on the same line through the origin or, equivalently, if one vector is a multiple of the other.

Thm. 6. Assume $\vec{x} = (x_1, y_1)$ and $\vec{y} = (x_2, y_2)$ are linearly independent. Then for any $\vec{z} = (z_1, z_2) \in \mathbb{R}^2$ there exist unique values $k_1 \in \mathbb{R}$ and $k_2 \in \mathbb{R}$ such that $k_1 \vec{x} + k_2 \vec{y} = \vec{z}$. We say that $\vec{x}$ and $\vec{y}$ span $\mathbb{R}^2$.

Pf. $k_1 \vec{x} + k_2 \vec{y} = \vec{z}$ is the same as $k_1 x_1 + k_2 x_2 = z_1$ and $k_1 y_1 + k_2 y_2 = z_2$ or $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. We claim that $\det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \neq 0$. If this is true, then we are done by Thm. 3. If $x_1 y_2 - x_2 y_1 = \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = 0$, then $x_1 y_2 = x_2 y_1$. Assume $x_2 \neq 0$ and $y_2 \neq 0$. Then $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \lambda$ or $x_1 = \lambda x_2$, $y_1 = \lambda y_2$ and $\vec{x} = \lambda \vec{y}$. By Thm 5a), this equality would violate the linear independence of $\vec{x}$ and $\vec{y}$. The proof is done if $x_2 \neq 0$ and $y_2 \neq 0$. If $x_2 = 0$ or $y_2 = 0$, the the proof is similar.

Linearity Principle. Suppose $d\vec{Y}/dt = A\vec{Y}$ is a linear system of diff. eqns.
1) If $\vec{Y}(t)$ is a solution and $k \in \mathbb{R}$, then $k\vec{Y}(t)$ is also a solution.
2) If $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are two solutions, then $\vec{Y}_1(t) + \vec{Y}_2(t)$ is also a solution. (p 243 of text).

Note: 1) and 2) are equivalent to the following: if $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are two solutions and $k_1 \in \mathbb{R}$, $k_2 \in \mathbb{R}$, then $k_1\vec{Y}_1(t) + k_2\vec{Y}_2(t)$ is also a solution. We say that linear combinations of solutions are also solution.

Pf of Linearity Principle. On pp 138-139 the text shows that $A(k\vec{Y}) = kA\vec{Y}$ and $A(\vec{Y}_1 + \vec{Y}_2) = A\vec{Y}_1 + A\vec{Y}_2$. The linearity principle now follows from standard rules of differentiation.

Key Theorem

THEOREM 7 Suppose $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions of the linear system $d\mathbf{Y}/dt = \mathbf{AY}$. If $\mathbf{Y}_1(0)$ and $\mathbf{Y}_2(0)$ are linearly independent, then for any initial condition $\mathbf{Y}(0) = (x_0, y_0)$ we can find constants k_1 and k_2 so that $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$ is the solution to the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \mathbf{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \quad \blacksquare$$

In this situation we say that the two-parameter family $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$, where k_1 and k_2 are arbitrary constants, is the **general solution** of the system.

By the Existence and Uniqueness Theorem for systems, we know that each initial-value problem for a linear system has exactly one solution. Given any two solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ of a linear system with linearly independent initial conditions $\mathbf{Y}_1(0)$ and $\mathbf{Y}_2(0)$, we can form the general solution of the system by forming the two-parameter family $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$. By adjusting the constants k_1 and k_2, we can obtain the solution that satisfies any given initial condition.

This is really excellent progress. We now know that to find all the solutions to a linear system, we need to find only two particular solutions with linearly independent initial positions. Two solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ of a linear system for which $\mathbf{Y}_1(0)$ and $\mathbf{Y}_2(0)$ are linearly independent are called **linearly independent solutions** of the linear system. (In the exercises we will see that if $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions of a linear system and the vectors $\mathbf{Y}_1(t_0)$ and $\mathbf{Y}_2(t_0)$ are linearly independent at any particular t_0, then $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent for all values of t—see Exercise 35.) The next step is to find a general way to come up with two linearly independent solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

Illustrate theory via example $\vec{Y}' = A\vec{Y}$, $A = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}$, for which the solution has the form $\vec{Y} = k_1\vec{Y}_1 + k_2\vec{Y}_2$, $\vec{Y}_1 = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$, $\vec{Y}_2 = \begin{pmatrix} -e^{-4t} \\ 2e^{-4t} \end{pmatrix}$

(see Ch. 4)