Laplace Transforms + Partial Fractions

Solve $y'' + ay' + by = f(t)$, $y(0)$, $y'(0)$ given : $\mathcal{L}y = (s^2+as+b)^{-1}\left((s+a)\,y(0) + y'(0) + \mathcal{L}f\right)$.

Assume rhs $= C(s) + A(s)/B(s)$, A and B polynomials w/ $\deg A < \deg B$.

How to do $\mathcal{L}^{-1}(A/B)$:

1) Write $B =$ product of poly's deg 1 or 2, where polynomials of degree 2 are irreducible.

2) If $(\alpha s+\beta)$ appears n times, form sum

$$\frac{k_1}{\alpha s+\beta} + \frac{k_2}{(\alpha s+\beta)^2} + \cdots + \frac{k_n}{(\alpha s+\beta)^n}, \quad k_1,\ldots,k_n \text{ to be found}$$

3) If $(\alpha s^2+\beta s+\gamma)$ appears m times, form sum

$$\frac{c_1 s+d_1}{\alpha s^2+\beta s+\gamma} + \frac{c_2 s+d_2}{(\alpha s^2+\beta s+\gamma)^2} + \cdots + \frac{c_m s+d_m}{(\alpha s^2+\beta s+\gamma)^m}, \quad c_i, d_i \text{ to be found}$$

4) Write $A(s)/B(s) = \displaystyle\sum_{\text{linear factors}} \left\{\frac{k_1}{\alpha s+\beta} + \cdots + \frac{k_n}{(\ \ )^n}\right\} + \sum_{\text{quad f.}} \left\{\frac{c_1 s+d_1}{\alpha s^2+\beta s+\gamma} + \cdots + \frac{c_m s+d_m}{(\ \ )^m}\right\}$

5) $\mathcal{L}^{-1}\dfrac{k}{(\alpha s+\beta)^j} = \dfrac{k}{(j-1)!\,\alpha^j}\,\mathcal{L}^{-1}\dfrac{(j-1)!}{(s+\beta/\alpha)^j} = \dfrac{k}{(j-1)!}\,e^{-(\beta/\alpha)t}\,t^{j-1}$.

6) Write $s^2 + \frac{\beta}{\alpha}s + \frac{\gamma}{\alpha} = \left(s+\frac{\beta}{2\alpha}\right)^2 + \delta^2$, $\delta^2 = \frac{\gamma}{\alpha} - \frac{\beta^2}{4\alpha^2}$ (>0 as $\beta^2 - 4\alpha\gamma < 0$ - quad. factor irred)

$$\mathcal{L}^{-1}\left(\frac{cs+d}{\alpha s^2+\beta s+\gamma}\right) = \frac{c}{\alpha}\mathcal{L}^{-1}\frac{s}{\left(s+\frac{\beta}{2\alpha}\right)^2+\delta^2} + \frac{d}{\alpha}\mathcal{L}^{-1}\frac{1}{\left(s+\frac{\beta}{2\alpha}\right)^2+\delta^2}$$

$$= \frac{c}{\alpha}\exp(-\beta t/2\alpha)\cos\delta t + \left(\frac{d}{\alpha\delta} - \frac{c\beta}{2\alpha^2\delta}\right)\exp(-\beta t/2\alpha)\cdot\sin\delta t$$

Thus we can find $\mathcal{L}^{-1}\left(\dfrac{cs+d}{(\alpha s^2+\beta s+\gamma)^j}\right)$ for $j=1$.

Assume we can find $\mathcal{L}^{-1}$ for $j=1,2,\ldots,J-1$, some $J \geq 2$.

Write

$$\frac{cs+d}{(\alpha s^2+\beta s+\gamma)^J} = \underbrace{\frac{cs+d}{(\alpha s^2+\beta s+\gamma)^{J-1}}}_{F}\ \underbrace{\frac{1}{\alpha s^2+\beta s+\gamma}}_{G}$$

Use $\mathcal{L}^{-1}(FG) = f * g$; i.e., $f = \mathcal{L}^{-1}\left(\dfrac{cs+d}{(\alpha s^2+\beta s+\gamma)^{J-1}}\right)$,

$g = \mathcal{L}^{-1}\left(\dfrac{1}{\alpha s^2+\beta s+\gamma}\right)$;