Laplace Transform + Green's Function

Solve $y'' + \beta y' + \gamma y = g(t)$, $y(0)=0$, $y'(0)=0$.

Method I. <u>Variation of Parameters</u>

$$y(t) = \int_0^t G(t,u)\,g(u)\,du \quad \text{where} \quad G(t,u) = \frac{y_1(u)\,y_2(t) - y_1(t)\,y_2(u)}{W(u)}$$

Method II. <u>Laplace Transform</u>

$$\mathcal{L}y = \frac{\mathcal{L}g}{s^2+\beta s+\gamma}, \quad y(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+\beta s+\gamma}\,\mathcal{L}g\right)$$

Let $\tilde{G}(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+\beta s+\gamma}\right)$. Then $y(t) = \tilde{G} * g(t)$ or

$$y(t) = \int_0^t \tilde{G}(t-u)\,g(u)\,du.$$

Compare methods: $G(t,u) = \tilde{G}(t-u) = \mathcal{L}^{-1}\left(\frac{1}{s^2+\beta s+\gamma}\right)(t-u)$.

(inverse $\mathcal{L}$ evaluated at $(t-u)$)

$G(t,u)$ = Green's function for initial value problem on $\ell.1$

Case	y_1, y_2	$G(t,u)$	$\tilde{G}(t)$
① $r_1 \neq r_2$ real $(s-r_1)(s-r_2)$ char poly	$e^{r_1 t}, e^{r_2 t}$	$\dfrac{e^{r_2(t-u)} - e^{r_1(t-u)}}{r_2 - r_1}$	$\mathcal{L}^{-1}\left(\frac{1}{(s-r_1)(s-r_2)}\right)$ $= \frac{1}{r_2-r_1}\left(e^{r_2 t} - e^{r_1 t}\right)$
② $r_1 = r_2 = r$ real charpoly $(s-r)^2$	e^{rt}, te^{rt}	$(t-u)\,e^{r(t-u)}$	$\mathcal{L}^{-1}\left(\frac{1}{(s-r)^2}\right) = te^{rt}$
③ $r_1 = \lambda + i\mu$ $r_2 = \lambda - i\mu$ $\mu \neq 0$	$e^{\lambda t}\cos\mu t$, $e^{\lambda t}\sin\mu t$	$\dfrac{e^{\lambda(t-u)}}{\mu}\sin[\mu(t-u)]$	$\mathcal{L}^{-1}\left(\frac{1}{(s-\lambda)^2+\mu^2}\right)$ $= \frac{1}{\mu}e^{\lambda t}\sin\mu t$

<u>Derivation of $\tilde{G}(t)$</u>

① $\mathcal{L}^{-1}\left(\frac{1}{s-r_1}\cdot\frac{1}{s-r_2}\right) = e^{r_1 t} * e^{r_2 t} = \int_0^t e^{r_1(t-u)} e^{r_2 u}\,du = e^{r_1 t}\int_0^t e^{(r_2-r_1)u}\,du$

easier than partial fractions $= e^{r_1 t}\left[\frac{1}{r_2-r_1}e^{(r_2-r_1)t} - \frac{1}{r_2-r_1}\right] = \frac{1}{r_2-r_1}\left(e^{r_2 t} - e^{r_1 t}\right)$

③ Characteristic polynomial

$(s-r_1)(s-r_2) = (s-\lambda-i\mu)(s-\lambda+i\mu) = (s-\lambda)^2 - (i\mu)^2 = (s-\lambda)^2 + \mu^2$