Homogeneous Linear ODEs of Second Order
Notes for Math 331

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In this document we consider homogeneous linear ODEs of second order. Sections 1 and 2 cover the material in sections 2.1 and 2.2 in the course textbook *Advanced Engineering Mathematics* by Erwin Kreyszig.

These notes were prepared because of a deficiency in the presentation in the textbook. This deficiency arises from the non-standard and confusing definition of a fundamental set of solutions given on page 50 of the textbook. This point is discussed in detail at the end of section 1 of this document.

1 General Homogeneous Linear ODEs of Second Order

Throughout this section we assume that p and q are continuous functions on $\mathbb{R}$.

Definition 1. *For y a twice continuously differentiable function on $\mathbb{R}$ we define*

$$L[y](x) = y''(x) + p(x)y'(x) + q(x)y(x) \text{ for } x \in \mathbb{R}.$$

Part (a) of the next theorem states that L is linear, a fact that follows from the linearity of the derivative. Part (b) states the superposition principle for the homogeneous ODE $L[y] = 0$. It is an immediate consequence of part (a). Part (b) is proved on page 48 of the textbook.

Theorem 2. Linearity of $L$and Superposition Principle for $L[y] = 0$

(a) *Let f and g be twice continuously differentiable functions on $\mathbb{R}$. Then for $x \in \mathbb{R}$ and any real numbers c_1 and c_2*

$$L[c_1 f + c_2 g](x) = c_1 L[f](x) + c_2 L[g](x).$$

(b) *Let* y_1 *and* y_2 *be solutions of* $L[y] = 0$ *on* $\mathbb{R}$. *Then for any real numbers* c_1 *and* c_2 *the linear combination* $y = c_1y_1 + c_2y_2$ *is also a solution of* $L[y] = 0$ *on* $\mathbb{R}$.

The next definition is basic. We first define the concept of a fundamental set of solution and then the concept of a general solution.

Definition 3. *Let* y_1 *and* y_2 *be solutions of* $L[y] = 0$ *on* $\mathbb{R}$.

(a) *We call* y_1, y_2 *a* **fundamental set of solutions** *of* $L[y] = 0$ *on* $\mathbb{R}$ *if any solution of* $L[y](x) = 0$ *can be written for all* $x \in \mathbb{R}$ *as a linear combination* $y(x) = c_1y_1(x) + c_2y_2(x)$ *for some choice of real numbers* c_1 *and* c_2.

(b) *If* y_1, y_2 *is a fundamental set of solutions of* $L[y] = 0$, *then we call* $c_1y_1 + c_2y_2$ *the* **general solution** *of* $L[y] = 0$ *on* $\mathbb{R}$, *where* c_1 *and* c_2 *are arbitrary real numbers.*

This definition is standard. It differs from the definitions of fundamental set of solutions and general solution on page 50 of the textbook, where the term "fundamental system" is used in place of "fundamental set". Neither of these definitions in the textbook are standard, and they are both confusing. The confusion inherent in the definition on page 50 of the textbook is discussed at the end of this section.

The next step is to give a condition on solutions y_1 and y_2 guaranteeing that two solutions y_1 and y_2 of $L[y] = 0$ are a fundamental set of solutions. First we need several definitions.

Definition 4. *Let* f *and* g *be continuous function on* $\mathbb{R}$ *neither of which is identically* 0.

(a) *We say that* f *is* **proportional** *to* g *on* $\mathbb{R}$ *if there exists a nonzero real number* α *such that* $f(x) = \alpha g(x)$ *for all* $x \in \mathbb{R}$.

(b) *We say that* f *and* g *are* **linearly dependent** *on* $\mathbb{R}$ *if there exists nonzero real numbers* c_1 *and* c_2 *such that* $c_1 f(x) + c_2 g(x) = 0$ *for all* $x \in \mathbb{R}$.

(c) *We say that* f *and* g *are* **linearly independent** *on* $\mathbb{R}$ *if they are not linearly dependent.*

The next lemma presents a number of facts involving the three definitions just given. These facts are easily verified and their proofs are omitted. Part (a) of the next lemma follows from the assumption that the constant α in part (a) of Definition 4 is nonzero.

Lemma 5. *Let* f *and* g *be continuous functions on* $\mathbb{R}$ *neither of which is identically* 0. *The following conclusions hold.*

(a) *The function* f *is proportional to* g *on* $\mathbb{R}$ *if and only if* g *is proportional to* f *on* $\mathbb{R}$, *which is the case if and only if for all* $x \in \mathbb{R}$ *for which* $g(x) \neq 0$ *the ratio* $f(x)/g(x)$ *equals a nonzero real number; in the notation of Definition* 4 *this real number is* α.

(b) *The function f is proportional to g on $\mathbb{R}$ if and only if f and g are linearly dependent on $\mathbb{R}$.*

(c) *Let c_1 and c_2 be real numbers. The functions f and g are linearly independent on $\mathbb{R}$ if and only if $c_1 f(x) + c_2 g(x) = 0$ for all $x \in \mathbb{R}$ implies that $c_1 = c_2 = 0$.*

(d) *The functions f and g are linearly independent on $\mathbb{R}$ if and only if f is not proportional to g on $\mathbb{R}$.*

We now state the main theorem in this document. It gives a condition guaranteeing that two solutions of $L[y] = 0$ are a fundamental set of solutions. As we will see in section 2 of this document, the beauty of the theorem is that this condition can be easily verified. The proof of the theorem is outlined after Theorem 8. The theorem is proved in section 2.6 of the textbook.

Theorem 6. *Let y_1 and y_2 be two solutions of $L[y] = 0$ on $\mathbb{R}$ neither of which is identically 0. Assume that y_1 is not proportional to y_2 on $\mathbb{R}$, or equivalently that y_1 and y_2 are linearly independent on $\mathbb{R}$. Then y_1 and y_2 are a fundamental set of solutions of $L[y] = 0$ on $\mathbb{R}$, and $c_1 y_1 + c_2 y_2$ is the general solution of $L[y] = 0$ on $\mathbb{R}$, where c_1 and c_2 are arbitrary real numbers.*

We now consider two theorems that are the subject of section 2.6 in the textbook. They are Theorem 1 and Theorem 3 in that section, the first of which is not proved and the second of which is proved as a consequence of Theorem 1. These two theorems are stated together in parts (a) and (b) of the next theorem.

Theorem 7. (a) *Let x_0 be a point in $\mathbb{R}$ and K_0 and K_1 real numbers. Then the initial value problem*

$$L[y] = 0 \text{ on } \mathbb{R}, y(x_0) = K_0, y'(x_0) = K_1 \tag{1}$$

has a unique solution on $\mathbb{R}$.

(b) *The homogeneous ODE $L[y](x) = 0$ has a fundamental set of solutions on $\mathbb{R}$.*

Solving the initial value problem (1). Let y_1 and y_2 be a fundamental set of solutions of $L[y] = 0$ on $\mathbb{R}$, the existence of which is guaranteed by part (b) of Theorem 7. We use this fundamental set to solve the initial value problem stated in (1). According to part (a) of Theorem 7, the initial value problem has a unique solution y. Since the solution y of the initial value problem solves $L[y] = 0$ on $\mathbb{R}$, there exists real numbers c_1 and c_2 such that $y(x) = c_1 y_1(x) + c_2 y_2(x)$ for all $x \in \mathbb{R}$. We calculate c_1 and c_2 using the initial conditions $y(x_0) = K_0, y'(x_0) = K_1$. Substituting the initial conditions into $y(x) = c_1 y_1(x) + c_2 y_2(x)$, we obtain the following two linear equations for c_1 and c_2:

$$c_1 y_1(x_0) + c_2 y_2(x_0) = K_0, \tag{2}$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = K_1. \tag{3}$$

Multiply the first equation by $y_2'(x_0)$ and the second equation by $y_2(x_0)$, obtaining

$$\begin{aligned} c_1 y_1(x_0) y_2'(x_0) + c_2 y_2(x_0) y_2'(x_0) &= K_0 y_2'(x_0), \\ c_1 y_1'(x_0) y_2(x_0) + c_2 y_2'(x_0) y_2(x_0) &= K_1 y_2(x_0). \end{aligned}$$

If we subtract the second equation from the first equation, then we find that

$$c_1[y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)] = K_0 y_2'(x_0) - K_1 y_2(x_0).$$

Under the assumption that the factor $y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)$ multiplying c_1 is nonzero, we obtain the formula

$$c_1 = \frac{K_0 y_2'(x_0) - K_1 y_2(x_0)}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)}. \tag{4}$$

Similarly, under the assumption that $y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0$, we obtain the formula

$$c_2 = \frac{K_1 y_1(x_0) - K_0 y_1'(x_0)}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)}. \tag{5}$$

With these formulas for c_1 and c_2, the function $y = c_1 y_1 + c_2 y_2$ solves the initial value problem

$$L[y] = 0 \text{ on } \mathbb{R}, y(x_0) = K_0, y'(x_0) = K_1.$$

Under the assumption that $y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0$, the quantities c_1 and c_2 given in (4) and (5) are the unique solutions of (2) and (3). This completes the discussion of the initial value problem (1).

Properties of the Wronskian. The quantity $y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)$ appearing in the numerators of the formulas for c_1 and c_2 in (4) and (5) plays a central role. It is called the **Wronskian** of y_1 and y_2 at x_0 and is denoted by $W(y_1, y_2)(x_0)$. In the next theorem we state the main facts about the Wronskian. These facts are stated and proved in equivalent form in Theorem 2 in section 2.6 of the textbook.

Theorem 8. *Let y_1 and y_2 be solutions of $L[y] = 0$ on $\mathbb{R}$. The following conclusions hold.*

(a) *The solutions y_1 and y_2 are linearly independent on $\mathbb{R}$ if and only if the Wronskian $W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is not 0 for all $x \in \mathbb{R}$.*

(b) *If for some point $x_0 \in \mathbb{R}$ the Wronskian $W(y_1, y_2)(x_0) \neq 0$, then for all $x \in \mathbb{R}$ the Wronskian $W(y_1, y_2)(x) \neq 0$.*

The formulas for c_1 and c_2 in (4) and (5) are derived under the assumption that y_1 and y_2 are a fundamental set of solutions of $L[y] = 0$ on $\mathbb{R}$. One can easily prove that this property of y_1

and y_2 implies that these two functions are linearly independent on $\mathbb{R}$. It then follows from part (a) of Theorem 8 that the Wronskian appearing in the denominators of c_1 and c_2 in (4) and (5) is not 0. It follows that the formulas for c_1 and c_2 are well defined.

We return to this idea in the following outline of the proof of Theorem 6. This theorem states that two nonzero solutions y_1 and y_2 of $L[y] = 0$ are a fundamental set of solutions if y_1 is not proportional to y_2 on $\mathbb{R}$.

Outline of Proof of Theorem 6. Let $y = Y(x)$ be any solution of $L[y] = 0$ on $\mathbb{R}$. In order to prove that y_1, y_2 is a fundamental set of $L[y] = 0$ on $\mathbb{R}$, we must find suitable values of c_1 and c_2 such that $Y(x) = c_1 y_1(x) + c_2 y_2(x)$ for all $x \in \mathbb{R}$. Let $y(x) = c_1 y_1(x) + c_2 y_2(x)$. We start by choosing any point $x_0 \in \mathbb{R}$ and seek real numbers c_1 and c_2 such that $y(x_0) = Y(x_0)$ and $y'(x_0) = Y'(x_0)$. Written out in terms of the definition of y, these equations take the form

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= Y(x_0), \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= Y'(x_0). \end{aligned}$$

These equations are the same as (2) and (3) with K_0 and K_1 replaced respectively by $Y(x_0)$ and $Y'(x_0)$. As we saw when solving (2) and (3), the two equations in the last display have a unique solution provided the Wronskian $W(y_1, y_2)(x_0) = y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)$ is nonzero.

In order to prove that $W(y_1, y_2)(x_0) \neq 0$, we use the hypothesis of Theorem 6, which is that the solution y_1 is not proportional to the solution y_2 on $\mathbb{R}$. By part (d) of Lemma 5, it follows that y_1 and y_2 are linearly independent on $\mathbb{R}$. Part (a) of Theorem 8 then guarantees that $W(y_1, y_2)(x) \neq 0$ for all $x \in \mathbb{R}$ and thus for x_0. We conclude that the two equations in the last display have a unique solution c_1 and c_2 given by (4) and (5) with K_0 and K_1 replaced respectively by $Y(x_0)$ and $Y'(x_0)$.

We have succeeded in proving that with this choice of c_1 and c_2, the function $y(x) = c_1 y_1(x) + c_2 y_2(x)$ solves the initial value problem

$$L[y] = 0, y(_0) = Y(x_0), y'(x_0) = Y'(x_0).$$

The function Y introduced at the beginning of the proof satisfies the same initial value problem. Since by Theorem 7 this initial value problem has a unique solution, it follows that $Y(x) = y(x) = c_1 y_1(x) + c_2 y_2(x)$ for all $x \in \mathbb{R}$. This completes the proof that y_1, y_2 are a fundamental set of solutions of $L[y] = 0$ on $\mathbb{R}$. ■

Set of solutions of $L[y] = 0$ as a vector space. We next point out an interesting analogy between the theory of homogeneous linear ODEs of second order and the two-dimensional Euclidean space $\mathbb{R}^2$ consisting of all vectors $X = (x_1, x_2)$, where x_1 and x_2 are real numbers. We add vectors $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ by the formula $X + Y = (x_1 + y_1, x_2 + y_2)$, and we multiply vectors $X = (x_1, x_2)$ by real numbers α by the formula $\alpha X = (\alpha x_1, \alpha x_2)$. As explained in section 7.9 of the textbook, with these operations $\mathbb{R}^2$ is a real vector space. Nonzero vectors X and Y in $\mathbb{R}^2$ are said to be **linearly dependent** if there exist nonzero real numbers c_1 and c_2 such that $c_1X + c_2Y = 0$. Nonzero vectors X and Y are said to be **linearly independent** if they are not linearly dependent. A well known fact about $\mathbb{R}^2$ is that if X and Y are linearly independent vectors in $\mathbb{R}^2$, then any vector $Z \in \mathbb{R}^2$ can be written as a linear combination $Z = c_1X + c_2Y$ for some choice of real numbers c_1 and c_2. Because of this property, any two linear independent vectors X and Y in $\mathbb{R}^2$ are said to be a **basis** of $\mathbb{R}^2$, and the vector space $\mathbb{R}^2$ is said to be **two dimensional**.

These facts about $\mathbb{R}^2$ parallel the analogous facts about the set of solutions of the ODE $L[y] = 0$ on $\mathbb{R}$. By part (b) of Theorem 2 the set of solutions of $L[y] = 0$ on $\mathbb{R}$ is a vector space, where addition of solutions is defined by $(y_1 + y_2)(x) = y_1(x) + y_2(x)$ and multiplication of a solution y by real numbers α is defined by $(\alpha y)(x) = \alpha y(x)$. The role of the linearly independent vectors X and Y in $\mathbb{R}^2$ is played by a fundamental set of solutions y_1, y_2 of $L[y] = 0$ on $\mathbb{R}$. In fact, just as any vector in $\mathbb{R}^2$ can be written as a linear combination of X and Y, so, as specified in Theorem 6, any solution of $L[y] = 0$ can be written as a linear combination of the solutions y_1, y_2. Because of this property of the set of solutions of $L[y] = 0$ on $\mathbb{R}$, we say that this set is two dimensional.

Why definition of fundamental set of solutions on page 50 of textbook is confusing. On page 50 of the textbook the concept of a fundamental set or fundamental system of solutions is defined as follows.

Two solutions y_1, y_2 of $L[y] = 0$ on $\mathbb{R}$ are called a fundamental set if y_1 is not proportional to y_2.

This definition is confusing and non-standard. It should be contrasted with Definition 3 in this document, according to which two solutions y_1, y_2 of $L[y] = 0$ on $\mathbb{R}$ are called a fundamental set of solutions if any solution of $L[y] = 0$ can be written as a linear combination of y_1, y_2. The fact that a fundamental set of solutions y_1, y_2 generates all solutions as linear combinations of y_1, y_2 is the main property of a fundamental set. As we have just seen, it is used to solve the initial value problem (1) in part (a) of Theorem 7. A condition guaranteeing that solutions y_1, y_2 are a fundamental set of solutions is given in Theorem 6. The confusion inherent in the definition on page 50 of the textbook is that it does not mention the main property of a fundamental set at all. This main property is postponed until page 78, where it is stated in Theorem 4. A similar comment applies to the definition of general solution on page 50 of the textbook.

We now focus on homogeneous linear ODEs with constant coefficients, for which a fundamental set of solutions can always be explicitly found.

2 Homogeneous Linear ODEs with Constant Coefficients

Let a and b be two real numbers. We consider the special case of the general homogeneous linear ODE $L[y] = 0$ obtained by choosing $p(x)$ and $q(x)$ to be the respective constants a and b. Thus the homogeneous ODE $L[y] = 0$ takes the form

$$L[y](x) = y''(x) + ay'(x) + by(x).$$

The purpose of this section is to give the explicit form of the fundamental set of solutions of all homogeneous ODEs with constant coefficients. The starting point is to evaluate $L[e^{\lambda x}]$, where λ is a real number. We have

$$L[e^{\lambda x}] = e^{\lambda x}(\lambda^2 + a\lambda + b),$$

from which it follows that $e^{\lambda x}$ solves $L[y] = 0$ if and only if $\lambda^2 + a\lambda + b = 0$. The solutions of this equation, which is called the **characteristic equation**, are given by the quadratic formula. In terms of the notation

$$r = -\frac{a}{2} \text{ and } D = a^2 - 4b,$$

the solutions of the characteristic equation are given by

$$\lambda_1 = r + \frac{1}{2}\sqrt{D},\ \lambda_2 = r - \frac{1}{2}\sqrt{D}.$$

The solutions of the characteristic equation are the roots of the quadratic $\lambda^2 + a\lambda + b$, which is called the **characteristic polynomial**. The roots of the characteristic polynomial have three different forms depending on whether $D > 0$, $D = 0$, or $D < 0$.

Case I. If $D > 0$, there are two real unequal roots $\lambda_1 = r + \frac{1}{2}\sqrt{D}$ and $\lambda_2 = r - \frac{1}{2}\sqrt{D}$ if $D > 0$.

Case II. If $D = 0$, there is one real double root $\lambda_1 = r$ if $D = 0$.

Case III. If $D < 0$, there are two complex conjugate, unequal roots $\lambda_1 = r + i\omega$ and $\lambda_2 = r - i\omega$, where $\omega = \frac{1}{2}\sqrt{|D|}$ and $i = \sqrt{-1}$.

In each of the three cases the quadratic $\lambda^2 + a\lambda + b$ has a different factorization.

Case I. If $D > 0$, then $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$.

Case II. If $D = 0$, then $\lambda^2 + a\lambda + b = (\lambda - r)^2$.

Case III. If $D < 0$, then $\lambda^2 + a\lambda + b = (\lambda - (r + i\omega))(\lambda - (r - i\omega))$.

The fundamental set for Case I is given in Theorem 9, for Case II in Theorem 10, and for Case III in Theorem 11. We next consider each of the three cases.

Case I. If $D > 0$, the characteristic polynomial $\lambda^2 + a\lambda + b$ has two real unequal roots λ_1 and λ_2, which generate the two solutions of $L[y] = 0$ given by

$$y_1 = e^{\lambda_1 x} \text{ and } y_2 = e^{\lambda_2 x}.$$

As shown in the next theorem, these two solutions are a fundamental set of solutions of $L[y] = 0$ in Case I.

Theorem 9. *In Case* I *the two solutions* $y_1 = e^{\lambda_1 x}$ *and* $y_2 = e^{\lambda_2 x}$ *are a fundamental set of solutions of* $L[y] = 0$ *on* $\mathbb{R}$.

Proof. By Theorem 6 it suffices to prove that y_1 is not proportional to y_2 on $\mathbb{R}$. We prove this by contradiction by assuming that y_1 is proportional to y_2 on $\mathbb{R}$ and showing that this assumption leads to a contradiction. If y_1 is proportional to y_2 on $\mathbb{R}$, then there exists a nonzero real number α such that $e^{\lambda_1 x} = \alpha e^{\lambda_2 x}$ for all $x \in \mathbb{R}$. This equation implies that for all $x \in \mathbb{R}$

$$\alpha = e^{\lambda_1 x} \cdot e^{-\lambda_2 x} = e^{(\lambda_1 - \lambda_2)x}.$$

However, because $\lambda_1 \neq \lambda_2$, the last equation is impossible because α is a fixed real number while $e^{(\lambda_1 - \lambda_2)x}$ is a non-constant function of x. This completes the proof. ■

Case II. If $D = 0$, the characteristic polynomial $\lambda^2 + a\lambda + b$ has one real double root $\lambda_1 = r = -a/2$, which generates the solution $y_1 = e^{rx}$ of $L[y] = 0$. We seek a second solution of $L[y] = 0$ in the form $y_2 = ue^{rx}$, where $u = u(x)$ is a function that must be calculated. Substituting y_2 into $L[y] = 0$ yields

$$\begin{aligned} L[y_2] &= L[ue^{rx}] = (ue^{rx})'' + a(ue^{rx})' + bue^{rx} \\ &= (u''e^{rx} + 2ru'e^{rx} + r^2e^{rx}) + a(u'e^{rx} + ure^{rx}) + bue^{rx} \\ &= e^{rx}\left[u'' + u'(2r + a) + u(r^2 + ar + b)\right]. \end{aligned}$$

Since $r^2 + ar + b = 0$ and $2r + a = 0$, we see that ue^{rx} solves $L[y] = 0$ if and only if $u'' = 0$. Integrating twice gives $u = \alpha x + \beta$, where α and β are real numbers, and $y_2 = ue^{rx} = (\alpha x + \beta)e^{rx}$. We make the choice $\alpha = 1$ and $\beta = 0$, obtaining $u = x$ and $y_2 = xe^{rx}$.

We now have two solutions of $L[y] = 0$ given by

$$y_1 = e^{rx} \text{ and } y_2 = xe^{rx}.$$

As shown in the next theorem, these two solutions are a fundamental set of solutions of $L[y] = 0$ in Case II.

Theorem 10. *In Case* II *the two solutions* $y_1 = e^{rx}$ *and* $y_2 = xe^{rx}$ *are a fundamental set of solutions of* $L[y] = 0$ *on* $\mathbb{R}$.

Proof. By Theorem 6 is suffices to prove that y_1 is not proportional to y_2 on $\mathbb{R}$. We prove this by contradiction by assuming that y_1 is proportional to y_2 on $\mathbb{R}$ and showing that this assumption leads to a contradiction. If y_1 is proportional to y_2 on $\mathbb{R}$, then there exists a nonzero real number α such that $e^{rx} = \alpha x e^{rx}$ for all $x \in \mathbb{R}$. This equation implies that for all

$$\alpha x = 1 \text{ for all } x \in \mathbb{R} \text{ or } \alpha = \frac{1}{x} \text{ for all } x \neq 0.$$

However, the last equation is impossible because α is a fixed real number while $1/x$ is a non-constant function of $x \neq 0$. This completes the proof. ■

Case III. If $D < 0$, the characteristic polynomial $\lambda^2 + a\lambda + b$ has two complex conjugate unequal roots $\lambda_1 = r + i\omega$ and $\lambda_2 = r - i\omega$, where $r = -a/2$ and $\omega = \frac{1}{2}\sqrt{|D|} = \frac{1}{2}\sqrt{b^2 - 4a}$. In order to find a fundamental set of real solutions of $L[y] = 0$ we consider a sequence of six steps. In these steps we prove that

$$y_1 = e^{rx}\cos(\omega x) \text{ and } y_2 = e^{rx}\sin(\omega x)$$

are solutions of $L[y] = 0$ on $\mathbb{R}$. As shown in the next theorem, these two solutions are a fundamental set of solutions of $L[y] = 0$ on $\mathbb{R}$.

Theorem 11. *In Case* III *the two solutions*

$$y_1 = e^{rx}\cos(\omega x) \textit{ and } y_2 = e^{rx}\sin(\omega x)$$

determined in Steps 1–6 *are a fundamental set of solutions of* $L[y] = 0$ *on* $\mathbb{R}$.

Proof. By Theorem 6 it suffices to prove that y_1 is not proportional to y_2 on $\mathbb{R}$. As in Cases I and II, we prove this by contradiction by assuming that y_1 is proportional to y_2 on $\mathbb{R}$ and showing that this assumption leads to a contradiction. If y_1 is proportional to y_2 on $\mathbb{R}$, then there exists a nonzero real number α such that $e^{rx}\cos(\omega x) = \alpha e^{rx}\sin(\omega x)$ for all $x \in \mathbb{R}$. This equation implies that for all $x \in \mathbb{R}$ for which $\sin(\omega x) \neq 0$

$$\alpha = \frac{\cos(\omega x)}{\sin(\omega x)} = \cot(\omega x).$$

However, the last equation is impossible because α is a fixed real number while $\cot(\omega x)$ is a non-constant function of x. This completes the proof. ■

We now proceed with the sequence of six steps showing that the functions y_1 and y_2 appearing in Theorem 11 are solutions of $L[y] = 0$ on $\mathbb{R}$.

Step 1. Define $e^{\alpha+i\beta}$, where α and β are real numbers and $\beta \neq 0$.

To do this, we recall the Maclaurin series

$$e^{\alpha} = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!},$$

which converges for all $\alpha \in \mathbb{R}$. We use the same Maclaurin series to define $e^{\alpha+i\beta}$:

$$e^{\alpha+i\beta} = \sum_{j=0}^{\infty} \frac{(\alpha+i\beta)^j}{j!},$$

which also converges for all α and β in $\mathbb{R}$. This completes Step 1.

Step 2. Let $\gamma = \alpha + i\beta$, where α and β are real numbers and $\beta \neq 0$. Verify that for all α, β, and x in $\mathbb{R}$

$$\frac{d}{dx} e^{\gamma x} = \gamma e^{\gamma x}.$$

To do this, we again use the Maclaurin series

$$e^{\gamma x} = \sum_{j=0}^{\infty} \frac{(\gamma x)^j x^j}{j!} = \sum_{j=0}^{\infty} \frac{\gamma^j x^j}{j!} = 1 + \sum_{j=1}^{\infty} x^j \frac{\gamma^j}{j!},$$

and we differentiate term-by-term. Since $d1/dx = 0$, we obtain

$$\frac{d}{dx} e^{\gamma x} = \frac{d}{dx} \sum_{j=1}^{\infty} x^j \frac{\gamma^j}{j!} = \sum_{j=1}^{\infty} \frac{d(x^j)}{dx} \frac{\gamma^j}{j!} = \sum_{j=1}^{\infty} j x^{j-1} \frac{\gamma^j}{j!}.$$

We now write $\gamma^j = \gamma \cdot \gamma^{j-1}$, move the common factor of γ outside the summation sign, use the fact that $j/j! = 1/(j-1)!$, and let $k = j - 1$. As j goes from 1 to ∞, k goes from 0 to ∞. Using these steps, we find that

$$\frac{d}{dx} e^{\gamma x} = \gamma \sum_{j=1}^{\infty} x^{j-1} \frac{\gamma^{j-1}}{(j-1)!} = \gamma \sum_{k=0}^{\infty} x^k \frac{\gamma^k}{k!} = \gamma e^{\gamma x}.$$

This completes Step 2. Of course, if $\beta = 0$ and $\gamma = \alpha$, the formula $de^{\alpha x}/dx = \alpha e^{\alpha x}$ is well known from calculus.

Step 3. Conclude that

$$Y_1 = e^{\lambda_1 x} = e^{(r+i\omega)x} \text{ and } Y_1 = e^{\lambda_2 x} = e^{(r-i\omega)x}$$

are complex-valued solutions of $L[y] = 0$ and are a fundamental set of solutions.

To show that Y_1 and Y_2 solve $L[y] = 0$, we repeat the calculation given at the beginning of this section. By step 2, since λ_1 and λ_2 are the roots of the characteristic polynomial, it follows that $L[e^{\lambda_1 x}] = 0$ and $L[e^{\lambda_2 x}] = 0$. To show that Y_1 and Y_2 are a fundamental set of solutions, we must verify that Y_1 is not proportional to Y_2. This is proved as in Case I using the fact that

$$e^{\lambda_1 x} e^{-\lambda_2 x} = e^{(\lambda_1 - \lambda_2)x},$$

a formula that can be proved using the Maclaurin series for e^{γ}, where γ is a complex number. The details are omitted because we will not need that fact that the complex-valued solutions Y_1 and Y_2 are a fundamental set of solutions of $L[y] = 0$. This completes Step 3.

Step 4. Let $\gamma = \alpha + i\beta$, where α and β are real numbers and $\beta \neq 0$. Verify Euler's formula

$$e^{\gamma} = e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta).$$

To do this, we recall the two Maclaurin series

$$\cos\beta = \sum_{k=0}^{\infty} (-1)^k \frac{\beta^{2k}}{(2k)!} \quad \text{and} \quad \sin\beta = \sum_{k=0}^{\infty} (-1)^k \frac{\beta^{2k+1}}{(2k+1)!}.$$

Using the Maclaurin series for e^{γ}, e^{α}, and $e^{i\beta}$, one can prove that $e^{\gamma} = e^{\alpha+i\beta} = e^{\alpha}e^{i\beta}$. It follows that

$$e^{\gamma} = e^{\alpha}e^{i\beta} = e^{\alpha} \sum_{j=0}^{\infty} \frac{(i\beta)^j}{j!} = e^{\alpha} \sum_{j=0}^{\infty} i^j \frac{\beta^j}{j!}.$$

We now write the sum over all nonnegative integers j as a sum of two terms: a sum over all nonnegative even integers k and a sum over all positive odd integers k. We obtain

$$e^{\gamma} = e^{\alpha} \left(\sum_{k=0}^{\infty} i^{2k} \frac{\beta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i^{2k+1} \frac{\beta^{2k+1}}{(2k+1)!} \right) = e^{\alpha} \sum_{k=0}^{\infty} i^{2k} \frac{\beta^{2k}}{(2k)!} + e^{\alpha} \sum_{k=0}^{\infty} i^{2k+1} \frac{\beta^{2k+1}}{(2k+1)!}.$$

In the second sum we write $i^{2k+1} = i \cdot i^{2k}$. Since $i^2 = -1$, we have $i^{2k} = (-1)^k$. It follows that

$$e^{\gamma} = e^{\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{\beta^{2k}}{(2k)!} + ie^{\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{\beta^{2k+1}}{(2k+1)!} = e^{\alpha}\cos\beta + ie^{\alpha}\sin\beta = e^{\alpha}(\cos\beta + i\sin\beta).$$

This completes Step 4.

Step 5. Let Y_1 and Y_2 be the complex-valued solutions of $L[y] = 0$ considered in Step 3. Then

$$Y_1 = e^{(r+i\omega)x} = e^{rx}(\cos(\omega x) + i\sin(\omega x) \text{ and } Y_2 = e^{(r-i\omega)x} = e^{rx}(\cos(\omega x) - i\sin(\omega x)$$

These formulae follow immediately from Step 4.

Step 6. The functions

$$y_1 = e^{rx}\cos(\omega x) \text{ and } y_2 = e^{rx}\sin(\omega x)$$

are solutions of $L[y] = 0$ on $\mathbb{R}$.

To do this, we start with part (b) of Theorem 2, which uses the linearity of L in part (a) to prove that if Y_1 and Y_2 are solutions of $L[y] = 0$, then for any real numbers c_1 and c_2 the linear combination $c_1Y_1 + c_2Y_2$ is also a solution of $L[y] = 0$. Now let Y_1 and Y_2 be the solutions of $L[y] = 0$ given in Step 5. The choices $c_1 = c_2 = 1/2$ show that

$$\frac{1}{2}Y_1 + \frac{1}{2}Y_2 = e^{rx}\cos(\omega x) = y_1$$

is a solution of $L[y] = 0$. This function y_1 equals the real part of the solution Y_1. As one easily verifies, part (a) and (b) of Theorem 2 are valid if the real numbers c_1 and c_2 are replaced by complex numbers. The choices $c_1 = 1/(2i)$ and $c_2 = -1/(2i)$ show that

$$\frac{1}{2i}Y_1 - \frac{1}{2i}Y_2 = e^{rx}\sin(\omega x) = y_2$$

is a solution of $L[y] = 0$. This function y_2 equals the imaginary part of the solution Y_1.This completes Step 6 and the discussion of the fundamental set of $L[y] = 0$ in Case III.