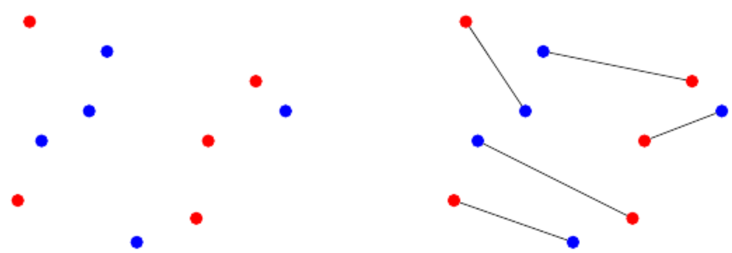
Only the Most Galvanizing Problems 3

On Wednesday May 17, there will be a quiz in class containing one of these questions.

1. Suppose that n red dots and n blue dots are drawn in the plane (for some integer n) with no three dots in a line. Prove that it is possible to draw n non-intersecting line segments to connect each red dot to a different blue dot. For example, if the dots were arranged as in the picture on the left, you might pair them up as in the picture on the right.



2. The Fibonacci numbers are defined recursively as follows: set $F_0 = F_1 = 1$ and for $n \geq 2$, set $F_n = F_{n-1} + F_{n-2}$. The sequence begins with 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... Consider the chain of inequalities $a_1 \leq a_2 \geq a_3 \leq a_4 \geq a_5 \ldots$ with n variables. You would like to assign values 0 or 1 to each of a_1 through a_n satisfying these inequalities. For example, if $n = 5$, one solution is $1 \leq 1 \geq 0 \leq 1 \geq 0$. Prove that the number of ways to do it is F_{n+1}.

3. Show that any positive integer can be written as a sum of distinct Fibonacci numbers.

4. Suppose that the numbers from 1 to n are written on a blackboard. At any step, you can erase two numbers a and b, and write the number $ab + a + b$ on the board. Eventually, only one number will remain on the board. What is that number and why?