Worksheet 2.6 (Part 2) - Math 455

1. A *diagonal* of a convex polygon is a line segment connecting two non-adjacent vertices of the polygon. Let p_n denote the number of ways to decompose a convex polygon having n vertices by drawing $n-3$ diagonals that do not cross inside the polygon. Assume that the vertices of the polygon are labeled, so that triangulations with different orientations are counted separately. Determine p_3, p_4, p_5 and p_6 by showing all possible triangulations, and then find and prove a recursive formula for p_n.

2. A Dyck path of length $2n$ is a staircase walk from $(0,0)$ to (n,n) that never rises above $y=x$. Let p_n denote the number of Dyck paths of length $2n$. Determine p_3, p_4, p_5 and p_6 by showing all possible Dyck paths, and then find and prove a recursive formula for p_n.

3. Let p_n be the number of rooted (strictly) binary trees with n internal vertices (i.e., non-leaves). Determine p_3, p_4, p_5 and p_6 by showing all possible trees, and then find and prove a recursive formula for p_n.

4. The Fibonacci numbers are defined recursively as follows: set $F_0 = F_1 = 1$ and for $n \geq 2$, set $F_n = F_{n-1} + F_{n-2}$. The sequence begins with 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... Consider the chain of inequalities $a_1 \leq a_2 \geq a_3 \leq a_4 \geq a_5 \ldots$ with n variables. You would like to assign values 0 or 1 to each of a_1 through a_n satisfying these inequalities. For example, if $n=5$, one solution is $1 \leq 1 \geq 0 \leq 1 \geq 0$. Prove that the number of ways to do it is F_{n+1}.

5. Show that any positive integer can be written as a sum of distinct Fibonacci numbers.

6. Let F_k be the kth Fibonacci number. Find and prove a general formula for
   (a) $\sum_{k=0}^{n} F_k$,
   (b) $\sum_{k=0}^{n} F_{2k}$,
   (c) $\sum_{k=1}^{n} F_{2k-1}$ if $n \geq 1$,
   (d) $F_{n+1}F_{n-1} - F_n^2$ if $n \geq 1$.

Hints:

1. Let v be a fixed vertex of your polygon. Consider two cases: either v is the end point of one of the $n-3$ diagonals, or it is not.
2. Think about the first place where a path meets the line $y = x$.
3. How many internal vertices are to the right of the root and how many are to the left?
4. How can you make a valid solution longer by one? By two?
5. Suppose not and consider the smallest positive integer that cannot be written as a sum of distinct Fibonacci numbers.
6. Use induction.