1. A diagonal of a convex polygon is a line segment connecting two non-adjacent vertices of the polygon. Let \( p_n \) denote the number of ways to decompose a convex polygon having \( n \) vertices by drawing \( n - 3 \) diagonals that do not cross inside the polygon. Assume that the vertices of the polygon are labeled, so that triangulations with different orientations are counted separately. Determine \( p_3, p_4, p_5 \) and \( p_6 \) by showing all possible triangulations, and then find and prove a recursive formula for \( p_n \).

2. A Dyck path of length \( 2n \) is a staircase walk from \((0, 0)\) to \((n, n)\) that never rises above \( y = x \). Let \( p_n \) denote the number of Dyck paths of length \( 2n \). Determine \( p_3, p_4, p_5 \) and \( p_6 \) by showing all possible Dyck paths, and then find and prove a recursive formula for \( p_n \).

3. Let \( p_n \) be the number of rooted (strictly) binary trees with \( n \) internal vertices (i.e., non-leaves). Determine \( p_3, p_4, p_5 \) and \( p_6 \) by showing all possible trees, and then find and prove a recursive formula for \( p_n \).

4. The Fibonacci numbers are defined recursively as follows: set \( F_0 = F_1 = 1 \) and for \( n \geq 2 \), set \( F_n = F_{n-1} + F_{n-2} \). The sequence begins with \( 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots \). Consider the chain of inequalities \( a_1 \leq a_2 \geq a_3 \leq a_4 \geq a_5 \ldots \) with \( n \) variables. You would like to assign values 0 or 1 to each of \( a_1 \) through \( a_n \) satisfying these inequalities. For example, if \( n = 5 \), one solution is \( 1 \leq 1 \geq 0 \leq 1 \geq 0 \). Prove that the number of ways to do it is \( F_{n+1} \).

5. Show that any positive integer can be written as a sum of distinct Fibonacci numbers.

6. Let \( F_k \) be the \( k \)th Fibonacci number. Find and prove a general formula for

   (a) \( \sum_{k=0}^{n} F_k \),
   
   (b) \( \sum_{k=0}^{n} F_{2k} \),
   
   (c) \( \sum_{k=1}^{n} F_{2k-1} \) if \( n \geq 1 \),
   
   (d) \( F_{n+1}F_{n-1} - F_n^2 \) if \( n \geq 1 \).
**Hints:**

1. Let $v$ be a fixed vertex of your polygon. Consider two cases: either $v$ is the end point of one of the $n - 3$ diagonals, or it is not.

2. Think about the first place where a path meets the line $y = x$.

3. How many internal vertices are to the right of the root and how many are to the left?

4. How can you make a valid solution longer by one? By two?

5. Suppose not and consider the smallest positive integer that cannot be written as a sum of distinct Fibonacci numbers.

6. Use induction.