1. Find the minimum size of a maximal matching in $C_n$.

I cannot have a path of length three on $C_n$ where none of these edges are in the matching. In that case, the matching would not be maximal since I could add the middle edge of that path and still have a matching. Therefore, since at least one of every three edges must be present, the minimum size of a maximal matching in $C_n$ is $\left\lceil \frac{n}{3} \right\rceil$.

2. Let $G$ be a bipartite graph. Show that $G$ has a matching of size at least $\frac{|E(G)|}{\Delta(G)}$.

Any vertex is adjacent to at most $\Delta(G)$ edges. So any edge cover must have size at least $\frac{|E(G)|}{\Delta(G)}$. By König’s theorem, we know that the size of the minimum edge cover is equal to the size of the maximum matching. Thus the maximum matching in the graph must have size at least $\frac{|E(G)|}{\Delta(G)}$.

3. Let $k$ be some fixed integer between 1 and $n$. Let $G$ be some subgraph of $K_{n,n}$ with more than $(k - 1)n$ edges. Prove that $G$ has a matching of size at least $k$.

Note that any vertex is adjacent to at most $n$ vertices since $G$ is a subgraph of $K_{n,n}$. By the previous question, we thus get that there is a matching of size at least $\frac{(k - 1)n + 1}{n}$ since $|E(G)| \geq (k - 1)n + 1$. Thus, since this fraction is strictly greater than $k - 1$, the max matching must have size at least $k$.

4. Draw a connected, 3-regular graph that has both a cut vertex and a perfect matching.

![Diagram of a 3-regular graph with a perfect matching]

This graph is 3-regular. The red edges form a perfect matching. The middle vertices are cut vertices—removing either disconnects the graphs.

5. Determine how many different perfect matchings there are in $K_{n,n}$.

There are $n!$ since the first vertex in the part $X$ can be matched to any of the $n$ vertices in part $Y$, then the second vertex of part $X$ can be matched to any of the remaining unmatched $n - 1$ vertices in part $Y$, and so on.