Homework 6 - Math 409

In preparation of Quiz 6 on May 16

1. Show whether or not \( M = (E(M), F(M)) \) is a matroid when

   - \( E(M) = E \) for some graph \( G = (V, E) \) and independent sets are matchings in \( G \)
   - \( E(M) = V \) for some graph \( G = (V, E) \) and independent sets are vertex covers in \( G \)
   - \( E(M) = V \) for some graph \( G = (V, E) \) and independent sets are stable sets of \( G \)
   - \( E(M) = E_1 \cup \ldots \cup E_l \) where \( E_1, \ldots, E_l \) are disjoint, and \( F(M) = \{ S \subseteq E : |S \cap E_i| \leq k_i \forall i = 1, \ldots, l \} \)
   - \( E(M) = E_1 \cup \ldots \cup E_l \) where \( E_1, \ldots, E_l \) are not necessarily disjoint, and \( F(M) = \{ S \subseteq E : |S \cap E_i| \leq k_i \forall i = 1, \ldots, l \} \)

2. One class of matroids we discussed in class is the class of graphic matroids, i.e. matroids where the ground set is composed of the edges of a graph \( G = (V, E) \) and the independent sets are the edge sets of \( G \)-forests. We also discussed linear matroids, i.e. matroids where the ground set is composed of the indices of the columns of a matrix \( A \) and where we say a set of these indices is independent if the corresponding columns are linearly independent.

   (a) Show that any graphic matroid is also a linear matroid by constructing a matrix \( A \) where the rows are indexed by the vertices of \( V \) and the columns are indexed by the edges of \( E \), and where a column vector indexed by \((i, j)\) has 0's in every row, except for a 1 in the \( i \)th or \( j \)th row and a \(-1\) in the other.

   (b) Show that any such matrix \( A \) is totally unimodular.

3. Let \( M = (E, F) \) be a matroid. Let \( k \in \mathbb{N} \) and define

   \[ F_k = \{ X \in F : |X| \leq k \} . \]

   (a) Show that \( M_k = (E, F_k) \) is also a matroid.

   (b) What is the rank function of \( M_k \) if \( M \) has rank function \( r \)?

4. We are given \( n \) jobs that each take one unit of processing time. All jobs are available at time 0, and job \( j \) has a profit of \( c_j \) and a deadline \( d_j \). The profit for job \( j \) will only be earned if the job completes by time \( d_j \). The problem is to find an ordering of the jobs that maximizes the total profit. First, prove that if a subset of the jobs can be completed on time, then they can also be completed on time if they are scheduled in the order of their deadlines. Now, let \( E(M) = \{ 1, 2, \ldots, n \} \) and let \( F(M) = \{ S \subseteq E(M) : S \text{ can be completed on time} \} \). Prove that \( M \) is a matroid and describe how to find an optimal ordering for the jobs.