1. Show that the dual of the dual is the primal.

2. Show that, if the primal is feasible but the dual is infeasible, then the primal will be unbounded.

3. (a) List all the faces of the tetrahedron $P = \{ x \in \mathbb{R}^3 | x_i \geq 0 \ \forall i \in [3], x_1 + x_2 + x_3 \leq 1 \}$.

   (b) Let $Q = \{ x \in \mathbb{R}^n | x_i \geq 0 \ \forall i \in [n], \sum_{i=1}^n x_i \leq 1 \}$. How many faces does $Q$ have? How many faces of dimension $k$ does $Q$ have?

4. Show that $P = \{ x \in \mathbb{R}^n | Ax \leq b \}$ has no vertices if rank($A$) $< n$.

5. Let $G$ be any graph. Let $P$ be the convex hull of all perfect matchings in $G$. Show that the vertices corresponding to matchings $M_1$ and $M_2$ are adjacent on $P$ if and only if $M_1 \Delta M_2$ has exactly one connected component. To do so, proceed in two steps.

   First, show that if $M_1 \Delta M_2$ has more than one connected components, then you can build two new matchings $M_3$ and $M_4$ that contain together the same edges (with the same multiplicity) as $M_1$ and $M_2$ together. What does this imply about faces containing $M_1$ and $M_2$?

   Then, assume that $M_1 \Delta M_2$ has exactly one connected component. Note that two vertices are adjacent on $P$ if and only if there exists an objective function $c$ such that these two vertices are the only ones minimizing $c^\top x$ over $P$. Find such a function $c$ for the vertices corresponding to $M_1$ and $M_2$ to show that they form an edge.