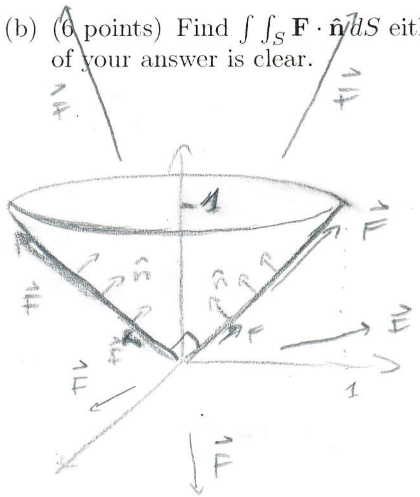
Annie's Survival Kit 7 - Math 324

1. (10 points) (a) (4 points) Let $\mathbf{F} = \langle x, y, z\rangle$ and let S be the part of the surface $z = \sqrt{x^2+y^2}$ lying underneath the plane $z = 1$, where $\hat{\mathbf{n}}$ is pointing generally upwards/inwards. Draw S and a few vectors for $\hat{\mathbf{n}}$ and $\mathbf{F}$.

   (b) (6 points) Find $\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ either by parametrizing S or in any other way. Make sure every part of your answer is clear.

a)



$\vec{F}$ is radially out

b) We can simply observe that, on S, $\vec{F}$ and $\hat{n}$ are perpendicular

$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S 0\,dS = 0$ (this would be true for any cone with its vertex at the origin in this vector field)

Else: $S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases}$ $\quad \theta\in[0,2\pi]$, $r\in[0,3]$

$\vec{r}_r = \langle\cos\theta, \sin\theta, 1\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$\pm\,\vec{r}_r\times\vec{r}_\theta = \pm\langle -r\cos\theta, -r\sin\theta, r\rangle$

↳ to point in the same direction, choose $\oplus$ since z component should be ≥ 0

$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_D \vec{F}\cdot(\vec{r}_r\times\vec{r}_\theta)\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \underbrace{\langle r\cos\theta, r\sin\theta, r\rangle}_{\vec{F}}\cdot\langle -r\cos\theta, -r\sin\theta, r\rangle\,dr\,d\theta$

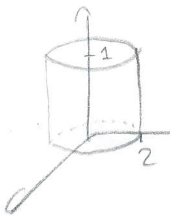
$= \int_0^{2\pi}\int_0^3 -r^2\cos^2\theta + (-r^2\sin^2\theta) + r^2\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^3 -r^2 + r^2\,dr\,d\theta = \int_0^{2\pi}\int_0^3 0\,dr\,d\theta = 0$

2. (10 points) (a) (8 points) Find the moment of inertia around the z-axis for the surface $x^2 + y^2 = 4$ with $0 \leq z \leq 1$ and with density equal to the square of the distance to the z-axis.

(b) (2 points) Without doing further calculations, determine whether or not the moment of inertia around the z-axis for the surface $x^2 + y^2 = 4$ with $1 \leq z \leq 2$ is the same as in part (a). Explain your answer.

a)



cylinder $r=2$ with open top + bottom

$$\delta = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$$

$$S: \begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \\ z = z \end{cases}$$

$0 \leq z \leq 1$

$0 \leq \theta \leq 2\pi$

$$I_z = \iint_S (x^2+y^2)\cdot\delta \, dS$$

$$= \int_0^{2\pi}\int_0^1 (x^2+y^2)^2 \, |\vec{r}_z \times \vec{r}_\theta| \, dz\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (x^2+y^2)^2 \cdot 2 \, dz\, d\theta$$

4 on cylinder or do $((2\cos\theta)^2+(2\sin\theta)^2)^2$

$$= 2\int_0^{2\pi}\int_0^1 4^2 \, dz\, d\theta$$

$$= 32 \cdot \int_0^{2\pi}\int_0^1 dz\, d\theta = 32\cdot 2\pi = 64\pi$$

$\vec{r}_z = \langle 0, 0, 1\rangle$

$\vec{r}_\theta = \langle -2\sin\theta, 2\cos\theta, 0\rangle$

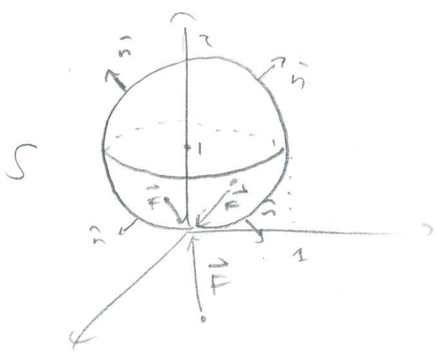
$\vec{r}_z \times \vec{r}_\theta = \langle -2\cos\theta, -2\sin\theta, 0\rangle$

$|\vec{r}_z \times \vec{r}_\theta| = \sqrt{4\cos^2\theta + 4\sin^2\theta + 0}$

$= 2$

b) The surface is the same, just translated up by one unit i.e., for every point $(x', y', z') \in S$, $(x', y', z'+1)$ is in our new surface and viceversa. The contribution of both of these points to the integral is the same: $(x'^2 + y'^2)^2$. Therefore, the integrals yield the same answer.

3. (10 points) Find the flux through the surface $x^2 + y^2 + (z-1)^2 = 1$ oriented outwards/upwards for $\mathbf{F} = \langle -x, -y, -z \rangle$.



$$S: \begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi + 1 \end{cases}$$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$\vec{r}_\varphi = \langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi \rangle$

$\vec{r}_\theta = \langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0 \rangle$

$\pm \vec{r}_\varphi \times \vec{r}_\theta = \pm\langle +\sin^2\varphi\cos\theta, +\sin^2\varphi\sin\theta, \sin\varphi\cos\varphi \rangle$

to point in same direction as $\hat{n}$, notice that $\hat{n}$ at $(1,0,1)$ is $(1,0,0)$. Since point $(1,0,1)$ is equivalent to $\varphi = \frac{\pi}{2}$ and $\theta = 0$, and since $\sin^2\frac{\pi}{2}\cos 0 = 1$, we take $+$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_D \vec{F}\cdot \vec{r}_\varphi \times \vec{r}_\theta \, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \langle -\sin\varphi\cos\theta, -\sin\varphi\sin\theta, -\cos\varphi - 1 \rangle \cdot \langle \sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi \rangle \, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \underbrace{-\sin^3\varphi\cos^2\theta - \sin^3\varphi\sin^2\theta}_{=} - \sin\varphi\cos^2\varphi - \sin\varphi\cos\varphi \, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \underbrace{-\sin^3\varphi - \sin\varphi\cos^2\varphi} - \sin\varphi\cos\varphi \, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} -\sin\varphi - \sin\varphi\cos\varphi \, d\varphi\, d\theta$$

need to use identity $\sin(2t) = 2\sin(t)\cos(t)$

$$= \int_0^{2\pi}\int_0^{\pi} -\sin\varphi - \tfrac{1}{2}\sin(2\varphi)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \left[\cos\varphi\right]_0^{\pi} + \tfrac{1}{4}\left[\cos(2\varphi)\right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} (-1-1) + \tfrac{1}{4}(1-1)\, d\theta$$

$$= -2 \cdot 2\pi = -4\pi$$